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CLOSED-FORM FORMULAS, DETERMINANTAL EXPRESSIONS, RECURSIVE RELATIONS, POWER SERIES, AND SPECIAL VALUES OF SEVERAL FUNCTIONS USED IN CLARK–ISMAIL'S TWO CONJECTURES

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Dedicated to Mr. Lin-Quan Yu, former Head of Department of Fundamental Courses at Jiaozuo Mining Institute in China

Abstract. In the paper, by virtue of the famous formula of Faà di Bruno, with the aid of several identities of partial Bell polynomials, by means of a formula for derivatives of the ratio of two differentiable functions, and with availability of other techniques, the authors establish closed-form formulas in terms of the Bernoulli numbers and the second kind Stirling numbers, present determinantal expressions, derive recursive relations, obtain power series, and compute special values of the function $\frac{v^j}{1-\mathrm{e}^{-v}}$, its derivatives, and related ones used in Clark–Ismail's two conjectures. By these results, the authors also discover a formula for the determinant of a Hessenberg matrix and derive logarithmic convexity of a sequence related to the function and its derivatives.

Keywords: Conjecture, Bell Polynomial of the Second Kind, Faà di Bruno Formula, Closed-Form Formula, Recursive Relation, Special Value, Exponential Function, Positivity, Stirling Number of the Second Kind, Power Series, Hessenberg Determinant, Bernoulli Number, Logarithmic Convexity.

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1. Introduction

The classical Gamma function $\Gamma(v)$, firstly defined by Euler's integral, can also be defined by

$$\Gamma(v)=\lim_{m\to\infty}\frac{m!m^v}{\prod\limits_{j=0}^{m}(v+j)},\quad v\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$$

(see [1, p. 255, 6.1.2] and [40, Theorem 3.1]). Its logarithmic derivative $\psi(v)=[\ln\Gamma(v)]'=\frac{\Gamma'(v)}{\Gamma(v)}$ is called the Digamma function. The Gamma function $\Gamma(v)$ and Polygamma functions $\psi^{(j)}(v)$ for $j\ge0$ have much extensive applications in many branches such as statistics, probability, number theory, theory of 0-1 matrices, graph theory, combinatorics, physics, engineering, and other mathematical sciences.

Let $I\subseteq\mathbb{R}$ stand for an interval and $\mathbb{N}$ denote the set of all natural numbers (not including zero). If a smooth function $f(v)\in C^\infty(I)$ satisfies $(-1)^jf^{(j)}(v)\ge0$ on I for $j\in\{0\}\cup\mathbb{N}$,

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then we say that $f(v)$ is of complete monotonicity on I (see [27, Chapter XIII], [38, Chapter 1], and [43, Chapter IV]). A function $f(v)$ is of complete monotonicity on $I=(0,\infty)$ if and only if

$$f(v)=\int_0^\infty \mathrm{e}^{-vs}\,\mathrm{d}\,\sigma(s),\quad s\in(0,\infty), \tag{1}$$

where $\sigma(s)$ is non-decreasing and the integral (1) converges for $s\in(0,\infty)$. See [43, p. 161, Theorem 12b].

In [3, Lemmas 1 and 3], the author proved that the second derivatives

$$[v\psi(v)]''=2\psi'(v)+v\psi''(v)\quad\text{and}\quad\left[v^2\psi(v)\right]''=v^2\psi''(v)+4v\psi'(v)+2\psi(v)$$

are both completely monotonic on $(0,\infty)$. In the paper [14], among other things, Clark and Ismail proved that all the derivatives $[v^j\psi(v)]^{(j+1)}$ for $j\in\mathbb{N}$ are completely monotonic on $(0,\infty)$ and that, with the help of the Maple V Release 5, the jth derivatives

$$\Phi_j^{(j)}(v)=-\left[v^j\psi^{(j)}(v)\right]^{(j)}$$

for $2\le j\le 16$ are completely monotonic on $(0,\infty)$. In the paper [14], the proof of the complete monotonicity of $\Phi_j^{(j)}(v)$ relies on the positivity of

$$f_j(v)=\frac{\mathrm{d}^j}{\mathrm{d}\,v^j}\left(\frac{v^j}{1-\mathrm{e}^{-v}}\right)\quad\text{or}\quad g_j(v)=\left(\frac{1-\mathrm{e}^{-v}}{v}\right)^{j+1}\mathrm{e}^{jv}\,f_j(v)$$

on $(0,\infty)$ for $2\le j\le 16$. For $j\in\mathbb{N}$, there exists the relation

$$\Phi_j^{(j)}(v)=\int_0^\infty t^j f_j(t)\,\mathrm{e}^{-vt}\,\mathrm{d}\,t$$

(see [5, p. 108] and [14]). Thereafter, Clark and Ismail posed in [14] the following two conjectures.

Conjecture 1. *The functions $f_j(v)$, or equivalently $g_j(v)$, are positive on $(0,\infty)$ for all $j>1$.*

Conjecture 2. *Let*

$$g_j(v)=\sum_{k=0}^\infty \gamma(j,k)v^k. \tag{2}$$

Then, $\gamma(j,k)\ge 0$ for all $j>1$ and $k\ge 0$.

Conjecture 2 implies Conjecture 1. In [2, p. 400], Conjecture 2 was restated as follows. The Maclaurin power series expansion of the function

$$\mathfrak{g}_j(v)=\left(\mathrm{e}^v-1\right)^{j+1}\left(\frac{v^j}{1-\mathrm{e}^{-v}}\right)^{(j)},\quad j\ge 0 \tag{3}$$

has non-negative coefficients. It is clear that $\mathfrak{g}_j(v)=v^{j+1}\,\mathrm{e}^v\,g_j(v)$.

Theorem 1.1 in [5] reads that there is an integer j_0 such that for all $j\ge j_0$ the derivatives $\Phi_j^{(j)}(v)$ are not completely monotonic on $(0,\infty)$. In the proof of [5, Theorem 1.1], the function

$$H(v)=\sum_{k=1}^\infty\frac{1}{k}\sin\frac{v}{k},\quad v\in\mathbb{C},$$

which was investigated by Hardy and Littlewood in 1936, plays a key role, because Conjecture 1 is equivalent to the nonnegativity of the function $\frac{1}{2}+\frac{1}{\pi}H\left(\frac{v}{2\pi}\right)$ for $v>0$. To verify this equivalence, the authors proved in [5, pp. 112–113, Appendix] that the $f_j(v)$ is nonnegative on $[2\ln 2,\infty)$. In [5, p. 113], the authors wrote the following observation.

An examination of the function $f_j(v)$ shows that $f_j(v)$ for $j > 1$ starts as an increasing function at $v = 0$ with

$$f_j(0) = \frac{j!}{2}, \quad f_j'(0) = \frac{(j+1)!}{12}, \tag{4}$$

and it oscillates crossing the line $y = j!$ a number of times. It approaches $j!$ from above or below depending on the parity of j for $v \to \infty$. Computer experiments suggest that it crosses $j - 1$ times. The oscillation close to $v = 0$ becomes very wild as j becomes very large.

In [2, Theorem 2.1], the positivity of $f_j(v)$ for $j \ge 0$ on $(2\ln 2, \infty)$ was simply recovered by expressing the function $f_j(v)$ in terms of the Laguerre polynomials.

In [12], the interval $(2\ln 2, \infty)$, where the function $f_j(v)$ for $j \ge 0$ is positive, was extended to the longer interval $(\ln 2, \infty)$.

In [41], among other things, the following conclusions were obtained.

(1) For $j \in \{0\} \cup \mathbb{N}$, the function $(-1)^j f_0^{(j)}(v)$ is of complete monotonicity on $(0, \infty)$. More strongly, the function $f_0(v)$ is of logarithmically complete monotonicity on $(0, \infty)$. For the notion of logarithmically complete monotonicity, please look up in the paper [41] or in the monograph [38].

(2) For $j \in \{0\} \cup \mathbb{N}$, the functions

$$1 - f_0'(v) - f_0(v) \quad \text{and} \quad (-1)^j \left[f_0^{(j+2)}(v) + f_0^{(j+1)}(v)\right]$$

are of complete monotonicity on $(0, \infty)$. In particular, the sequence $(-1)^k f_0^{(j)}(v)$ is increasing with respect to j, that is, the inequality

$$(-1)^k f_0^{(j)}(v) < (-1)^{j+1} f_0^{(j+1)}(v)$$

is valid for all $j \in \{0\} \cup \mathbb{N}$ and $v \in (0, \infty)$.

(3) For $j \in \{0\} \cup \mathbb{N}$, the ratio

$$\mathcal{F}_j(v) = -\frac{f_0^{(j+1)}(v)}{f_0^{(j)}(v)}$$

is decreasing on $(0, \infty)$, with

$$\lim_{v\to\infty} \mathcal{F}_0(v) = 0 \quad \text{and} \quad \lim_{v\to\infty} \mathcal{F}_{j+1}(v) = 1.$$

On applications of derivatives of the function $f_0(v)$, please refer to the paper [37] and closely related references therein.

The function $f_1(v)$ has been investigated in [29, Lemma 2.3] for applying therein. The function $f_2(v)$ has been studied in [28, Lemma 2.3] and [30, Lemma 2.1] for applying therein.

The simple function

$$F_1(v) = \frac{1 - \mathrm{e}^{-v}}{v} = \int_0^1 \mathrm{e}^{-vu}\,\mathrm{d}\,u, \tag{5}$$

which is of completely monotonic property on $(-\infty, \infty)$, has been deeply investigated, extensively applied, and expositorily reviewed in the paper [11] and closely related references therein. It is not hard to see that

$$F_j(v) = \frac{1 - \mathrm{e}^{-v}}{v^j} = \frac{1}{v^{j-1}} \int_0^1 \mathrm{e}^{-vu}\,\mathrm{d}\,u. \tag{6}$$

This implies that $F_j(v)$ for $j \ge 2$ is of complete monotonicity on $(0, \infty)$, while $F_0(v)$ is a Bernstein function on $(0, \infty)$. For information on the Bernstein functions, please refer to the monograph [38].

For $j \in \mathbb{N}$ and $v \in (-\infty, \infty)$, let

$$G_j(v) = \frac{1}{F_j(v)} = \begin{cases} \frac{v^j}{1-\mathrm{e}^{-v}}, & v \neq 0; \\ 1, & v = 0, \quad j = 1; \\ 0, & v = 0, \quad j > 1. \end{cases} \tag{7}$$

It is clear that $f_j(v) = G_j^{(j)}(v)$ for $j \in \mathbb{N}$. In [33, Theorem 1.2 and Remark 5.5], it was proved that, when $1 \le j \le 5$, the inequalities

$$G_1^{(j-1)}(v) \le \frac{I_j\left(2\sqrt{v}\right)}{v^{j/2}} \quad \text{and} \quad G_1^{(4)}(v) \le \frac{v+6}{720}$$

are valid on $(0, \infty)$, where

$$I_\nu(v) = \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(\nu+j+1)} \left(\frac{v}{2}\right)^{2j+\nu}$$

for $\nu \in \mathbb{R}$ and $v \in \mathbb{C}$ is the first kind modified Bessel function.

The aim of this paper is to further discuss the functions $f_j(v)$, $g_j(v)$, and $\mathfrak{g}_j(v)$ via the functions $F_j(v)$, $G_j(v)$, and their derivatives. Our main results include closed-form formulas in terms of the Bernoulli numbers and the second kind Stirling numbers, determinantal expressions, recursive relations, power series, logarithmic convexity of these functions or their special values, as well as a formula for a Hessenberg determinant in terms of the Bernoulli numbers.

2. Preliminaries

In [13, Definition 11.2] and [15, p. 134, Theorem A], partial Bell polynomials, denoted by $\mathrm{B}_{j,k}(v_1, v_2, \dots, v_{j-k+1})$ for $j \ge k \ge 0$, are defined by

$$\mathrm{B}_{j,k}(v_1, v_2, \dots, v_{j-k+1}) = \sum_{\substack{1 \le i \le j-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{j-k+1} i\ell_i = j \\ \sum_{i=1}^{j-k+1} \ell_i = k}} \frac{j!}{\prod_{i=1}^{j-k+1} \ell_i!} \prod_{i=1}^{j-k+1} \left(\frac{v_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of $\mathrm{B}_{j,k}(v_1, v_2, \dots, v_{j-k+1})$ by

$$\frac{\mathrm{d}^j}{\mathrm{d}\, v^j} f \circ h(v) = \sum_{k=0}^{n} f^{(k)}(h(v))\, \mathrm{B}_{j,k}\left(h'(v), h''(v), \dots, h^{(j-k+1)}(v)\right), \quad j \ge 0 \tag{8}$$

(see [13, Theorem 11.4] and [15, p. 139, Theorem C]). The identities

$$\mathrm{B}_{j,k}\left(abv_1, ab^2v_2, \dots, ab^{j-k+1}v_{j-k+1}\right) = a^k b^j\, \mathrm{B}_{j,k}(v_1, v_2, \dots, v_{j-k+1}), \tag{9}$$

$$\mathrm{B}_{j,k}(1, 1, \dots, 1) = S(j,k) \tag{10}$$

for $j \ge k \ge 0$ and $a, b \in \mathbb{C}$ can be found in [13, p. 412] and [15, p. 135]. The identity

$$\mathrm{B}_{j,k}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{j-k+2}\right) = \frac{j!}{(j+k)!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{j+k}{k-\ell} S(j+\ell, \ell) \tag{11}$$

can be found in [35, Section 1.8, (1.16)] and the paper [47] respectively, where the second kind Stirling numbers $S(j,k)$ can be generated by

$$\frac{(\mathrm{e}^v - 1)^k}{k!} = \sum_{j=k}^{\infty} S(j,k) \frac{v^j}{j!}.$$

From [20, Theorem 2.1], [21, Theorem 2.1], and [45, Theorem 3.1], we can derive that

$$\frac{\mathrm{d}^j}{\mathrm{d}\,v^j}\left(\frac{1}{1-\mathrm{e}^{-v}}\right)=\sum_{\ell=0}^{j}(-1)^\ell \ell! S(j+1,\ell+1)\left(\frac{1}{1-\mathrm{e}^{-v}}\right)^{\ell+1},\quad j\ge 0. \tag{12}$$

Let $g(v)$ and $h(v)\ne 0$ be two differentiable functions. Let $U_{(j+1)\times 1}(v)$ be an $(j+1)\times 1$ matrix whose elements $g_{\ell,1}(v)=g^{(\ell-1)}(v)$ for $1\le \ell\le j+1$, let $V_{(j+1)\times j}(v)$ be an $(j+1)\times j$ matrix whose elements

$$h_{i,\ell}(v)=\begin{cases}\binom{i-1}{\ell-1}h^{(i-\ell)}(v), & i-\ell\ge 0\\ 0, & i-\ell<0\end{cases}$$

for $1\le i\le j+1$ and $1\le \ell\le j$, and let $|W_{(j+1)\times(j+1)}(v)|$ denote the determinant of the $(j+1)\times(j+1)$ matrix

$$W_{(j+1)\times(j+1)}(v)=\begin{pmatrix}U_{(j+1)\times 1}(v) & V_{(j+1)\times j}(v)\end{pmatrix}.$$

Then, the jth derivative of the ratio $\frac{g(v)}{h(v)}$ can be computed by

$$\frac{\mathrm{d}^j}{\mathrm{d}\,v^j}\left[\frac{g(v)}{h(v)}\right]=\frac{(-1)^j}{h^{j+1}(v)}\left|W_{(j+1)\times(j+1)}(v)\right|. \tag{13}$$

This formula is a reformulation of [8, p. 40, Exercise 5)].

The jth falling factorial of $v\in\mathbb{C}$ can be defined by

$$\langle v\rangle_j=\prod_{\ell=0}^{j-1}(v-\ell)=\begin{cases}v(v-1)\cdots(v-j+1), & j\ge 1;\\ 1, & j=0.\end{cases} \tag{14}$$

Let $H_0=1$ and

$$H_j=\begin{vmatrix} h_{1,1} & h_{1,2} & 0 & \dots & 0 & 0\\ h_{2,1} & h_{2,2} & h_{2,3} & \dots & 0 & 0\\ h_{3,1} & h_{3,2} & h_{3,3} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ h_{j-2,1} & h_{j-2,2} & h_{j-2,3} & \dots & h_{j-2,j-1} & 0\\ h_{j-1,1} & h_{j-1,2} & h_{j-1,3} & \dots & h_{j-1,j-1} & h_{j-1,j}\\ h_{j,1} & h_{j,2} & h_{j,3} & \dots & h_{j,j-1} & h_{j,j}\end{vmatrix}$$

for $j\in\mathbb{N}$. Then, the sequence H_j for $j\ge 0$ satisfies $H_1=h_{1,1}$ and

$$H_j=\sum_{r=1}^{j}(-1)^{j-r}h_{j,r}\left(\prod_{\ell=r}^{j-1}h_{\ell,\ell+1}\right)H_{r-1} \tag{15}$$

for $j\ge 2$, where any empty product is assumed to be 1. See [9, p. 222, Theorem].

3. The power series of $f_j(v)$ and more

In this section, via computing $G_j^{(k)}(0)$ for $k\ge 0$ and $j\in\mathbb{N}$, we obtain power series of $f_j(v)$.

Theorem 1. *The following conclusions are valid.*

(1) *The derivatives of the completely monotonic function $F_1(v)$ defined on $(-\infty,\infty)$ by (5) satisfy*

$$F_1^{(k)}(0)=\frac{(-1)^k}{k+1},\quad k\ge 0. \tag{16}$$

(2) *For $k \ge 0$ and $j \in \mathbb{N}$, we have*

$$G_j^{(k)}(0) = \begin{cases} 0, & k < j-1; \\ (-1)^{k-j+1} k!(k-j+2)! \sum\limits_{q=0}^{k-j+1} \frac{(-1)^q}{q+1} \frac{S(k-j+q+1,q)}{(k-j-q+1)!(k-j+q+1)!}, & k \ge j-1. \end{cases} \tag{17}$$

In particular, we have

$$G_j^{(j-1)}(0) = (j-1)!, \qquad G_j^{(j)}(0) = f_j(0) = \frac{j!}{2}, \tag{18}$$

$$G_j^{(j+1)}(0) = f_j'(0) = \frac{(j+1)!}{12}, \qquad G_j^{(j+2)}(0) = 0 \tag{19}$$

for $j \in \mathbb{N}$.

(3) *For $j \in \mathbb{N}$, the power series of $f_j(v)$ is*

$$f_j(v) = \sum_{i=0}^{\infty} (-1)^{i+1}(i+1)(i+2)(j+i)! \left[\sum_{q=0}^{i+1} \frac{(-1)^q}{q+1} \frac{S(i+q+1,q)}{(i-q+1)!(i+q+1)!} \right] v^i. \tag{20}$$

Proof. For $k \ge 0$, by the integral representation (5), it follows that the derivatives of $F_1(v)$ are

$$F_1^{(k)}(v) = (-1)^k \int_0^1 u^k \, \mathrm{e}^{-vu} \, \mathrm{d}\, u$$

and satisfy

$$F_1^{(k)}(0) = (-1)^k \int_0^1 u^k \, \mathrm{d}\, u = \frac{(-1)^k}{k+1}.$$

The function $G_j(v)$ defined by (7) can be rewritten as

$$G_j(v) = \frac{v^{j-1}}{(1-\mathrm{e}^{-v})/v} = \frac{v^{j-1}}{F_1(v)}. \tag{21}$$

By virtue of the Leibnz rule for differentiation and the Faà di Bruno formula (8), we acquire

$$\begin{aligned} G_j^{(k)}(v) &= \sum_{\ell=0}^{k} \binom{k}{\ell} \langle j-1 \rangle_\ell v^{j-\ell-1} \left[\frac{1}{F_1(v)} \right]^{(k-\ell)} \\ &= \sum_{\ell=0}^{k} \binom{k}{\ell} \langle j-1 \rangle_\ell v^{j-\ell-1} \sum_{q=0}^{k-\ell} \frac{(-1)^q q!}{[F_1(v)]^{q+1}} \mathrm{B}_{k-\ell,q}\big(F_1'(v), F_1''(v), \dots, F_1^{(k-\ell-q+1)}(v)\big), \end{aligned}$$

where $\langle z \rangle_\ell$ is the falling factorial defined by (14). Making use of the formula (16) and the identities (9) and (11) gives

$$\begin{aligned} &\lim_{v \to 0} \mathrm{B}_{k-\ell,q}\big(F_1'(v), F_1''(v), \dots, F_1^{(k-\ell-q+1)}(v)\big) \\ &= \mathrm{B}_{k-\ell,q}\big(F_1'(0), F_1''(0), \dots, F_1^{(k-\ell-q+1)}(0)\big) \\ &= \mathrm{B}_{k-\ell,q}\left(-\frac{1}{2}, \frac{1}{3}, \dots, \frac{(-1)^{k-\ell-q+1}}{k-\ell-q+2} \right) \\ &= (-1)^{k-\ell} \mathrm{B}_{k-\ell,q}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k-\ell-q+2} \right) \\ &= (-1)^{k-\ell+q} \frac{(k-\ell)!}{(k-\ell+q)!} \sum_{p=0}^{q} (-1)^p \binom{k-\ell+q}{q-p} S(k-\ell+p,p), \end{aligned} \tag{22}$$

where we used the identity (9) and the formula (11). Consequently, it follows that

(1) when $k < j-1$, we have

$$\begin{aligned}\lim_{v\to 0} G_j^{(k)}(v) &= \sum_{\ell=0}^{k}\binom{k}{\ell}\lim_{v\to 0}\big[\langle j-1\rangle_\ell v^{j-\ell-1}\big]\sum_{m=0}^{k-\ell}(-1)^m m!\,\mathrm{B}_{k-\ell,m}\big(F_1'(0),F_1''(0),\dots,F_1^{(k-\ell-m+1)}(0)\big)\\ &= 0;\end{aligned}$$

(2) when $k \ge j-1$, we have

$$\begin{aligned}\lim_{v\to 0} G_j^{(k)}(v) &= \sum_{\ell=0}^{k}\binom{k}{\ell}\lim_{v\to 0}\big[\langle j-1\rangle_\ell v^{j-\ell-1}\big]\sum_{m=0}^{k-\ell}(-1)^m m!\,\mathrm{B}_{k-\ell,m}\big(F_1'(0),F_1''(0),\dots,F_1^{(k-\ell-m+1)}(0)\big)\\
&= \binom{k}{j-1}(j-1)!\sum_{m=0}^{k-j+1}(-1)^m m!\,\mathrm{B}_{k-j+1,m}\big(F_1'(0),F_1''(0),\dots,F_1^{(k-j-m+2)}(0)\big)\\
&= \frac{k!}{(k-j+1)!}\sum_{m=0}^{k-j+1}(-1)^m m!(-1)^{k-j+1+m}\frac{(k-j+1)!}{(k-j+1+m)!}\\
&\quad\times\sum_{q=0}^{m}(-1)^q\binom{k-j+1+m}{m-q}S(k-j+1+q,q)\\
&= (-1)^{k-j+1}k!\sum_{m=0}^{k-j+1}\frac{m!}{(k-j+1+m)!}\sum_{q=0}^{m}(-1)^q\binom{k-j+1+m}{k-j+1+q}S(k-j+1+q,q)\\
&= (-1)^{k-j+1}k!\sum_{m=0}^{k-j+1}m!\sum_{q=0}^{m}\frac{(-1)^q}{(m-q)!}\frac{S(k-j+q+1,q)}{(k-j+q+1)!}\\
&= (-1)^{k-j+1}k!\sum_{q=0}^{k-j+1}(-1)^q\frac{S(k-j+q+1,q)}{(k-j+q+1)!}\sum_{m=q}^{k-j+1}\frac{m!}{(m-q)!}\\
&= (-1)^{k-j+1}k!(k-j+2)!\sum_{q=0}^{k-j+1}\frac{(-1)^q}{q+1}\frac{S(k-j+q+1,q)}{(k-j-q+1)!(k-j+q+1)!}.\end{aligned}$$

The power series (20) comes from the relation $G_j^{(j+i)}(v) = f_j^{(i)}(v)$ and the formula (17). The proof of Theorem 1 is completed. □

4. A closed-form formula of $f_j(v)$ and more

In this section, via computing $G_j^{(k)}(v)$ for $j \in \mathbb{N}$ and $k \in \{0\}\cup\mathbb{N}$, we establish a closed-form formula of the function $f_j(v)$.

Theorem 2. *For $j \in \mathbb{N}$ and $k \in \{0\}\cup\mathbb{N}$, the derivatives $G_j^{(k)}(v)$ can be computed by*

$$G_j^{(k)}(v) = v^{j-k-1}\sum_{q=0}^{k}\left[\sum_{\ell=q}^{k}(-1)^{\ell-q}\binom{k}{\ell}\langle j\rangle_{k-\ell}(\ell-q)!S(\ell+1,\ell-q+1)\left(\frac{v}{1-\mathrm{e}^{-v}}\right)^{\ell-q+1}\right]v^q.$$

In particular, we have

$$f_j(v) = \frac{1}{v}\sum_{q=0}^{j}\left[\sum_{\ell=q}^{j}(-1)^{\ell-q}\binom{j}{\ell}\langle j\rangle_{j-\ell}(\ell-q)!S(\ell+1,\ell-q+1)\left(\frac{v}{1-\mathrm{e}^{-v}}\right)^{\ell-q+1}\right]v^q,$$

$$G_j^{(k)}(0) = 0,\ \ k < j-1,$$

$$\sum_{q=0}^{j-1}(-1)^q q!\binom{k}{q+k-j+1}\langle j\rangle_{j-q-1}S(q+k-j+2,q+1), k \ge j-1,$$

and those two equalities in (18) *and* (19).

Proof. By the Leibniz rule for differentiation of a product and with the aid of the derivative formula (12), we have

$$\begin{aligned}
G_j^{(k)}(v) &= \sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{1-\mathrm{e}^{-v}}\right)^{(\ell)}(v^j)^{(k-\ell)} \\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}\langle j\rangle_{k-\ell}\left[\sum_{m=0}^{\ell}(-1)^m m!S(\ell+1,m+1)\left(\frac{1}{1-\mathrm{e}^{-v}}\right)^{m+1}\right]v^{j-k+\ell} \\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}\langle j\rangle_{k-\ell}\left[\sum_{m=0}^{\ell}(-1)^m m!S(\ell+1,m+1)\left(\frac{v}{1-\mathrm{e}^{-v}}\right)^{m+1}\frac{1}{v^{m+1}}\right]v^{j-k+\ell} \\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}\langle j\rangle_{k-\ell}\left[\sum_{m=0}^{\ell}(-1)^m m!S(\ell+1,m+1)\left(\frac{v}{1-\mathrm{e}^{-v}}\right)^{m+1}v^{\ell-m}\right]v^{j-k-1} \\
&= v^{j-k-1}\sum_{\ell=0}^{k}\binom{k}{\ell}\langle j\rangle_{k-\ell}\left[\sum_{q=0}^{\ell}(-1)^{\ell-q}(\ell-q)!S(\ell+1,\ell-q+1)\left(\frac{v}{1-\mathrm{e}^{-v}}\right)^{\ell-q+1}v^q\right] \\
&= v^{j-k-1}\sum_{q=0}^{k}\left[\sum_{\ell=q}^{k}(-1)^{\ell-q}\binom{k}{\ell}\langle j\rangle_{k-\ell}(\ell-q)!S(\ell+1,\ell-q+1)\left(\frac{v}{1-\mathrm{e}^{-v}}\right)^{\ell-q+1}\right]v^q.
\end{aligned}$$

If $0 \le k < j-1$, then it is clear that $G_j^{(k)}(0) = 0$. If $k = j-1$, then

$$\begin{aligned}
G_j^{(j-1)}(v) &= \sum_{q=0}^{j-1}\left[\sum_{\ell=q}^{j-1}(-1)^{\ell-q}\binom{j-1}{\ell}\langle j\rangle_{j-\ell-1}(\ell-q)!S(\ell+1,\ell-q+1)\left(\frac{v}{1-\mathrm{e}^{-v}}\right)^{\ell-q+1}\right]v^q \\
&\to \sum_{\ell=0}^{j-1}(-1)^{\ell}\binom{j-1}{\ell}\langle j\rangle_{j-\ell-1}\ell!, \quad v\to 0 \\
&= (j-1)!\sum_{\ell=0}^{j-1}(-1)^{\ell}\binom{j}{\ell+1} \\
&= (j-1)!.
\end{aligned}$$

If $k > j-1$, then

$$\begin{aligned}
\sum_{\ell=q}^{k}(-1)^{\ell-q}\binom{k}{\ell}\langle j\rangle_{k-\ell}(\ell-q)!S(\ell+1,\ell-q+1) &= \sum_{\ell=0}^{k-q}(-1)^{\ell}\ell!\binom{k}{\ell+q}\langle j\rangle_{k-\ell-q}S(\ell+q+1,\ell+1) \\
&= 0, \quad 0\le q < k-j+1
\end{aligned}$$

and

$$\begin{aligned}
G_j^{(k)}(0) &= \sum_{\ell=k-j+1}^{k}(-1)^{\ell-(k-j+1)}\binom{k}{\ell}\langle j\rangle_{k-\ell}[\ell-(k-j+1)]!S(\ell+1,\ell-(k-j+1)+1) \\
&= \sum_{q=0}^{j-1}(-1)^q q!\binom{k}{q+k-j+1}\langle j\rangle_{j-q-1}S(q+k-j+2,q+1),
\end{aligned}$$

where we used the finity of $G_j^{(k)}(0)$ for $k \ge 0$ and $j \in \mathbb{N}$. The finity of $G_j^{(k)}(0)$ for $k \ge 0$ and $j \in \mathbb{N}$ can be seen from the formula (17). The proof of Theorem 2 is completed. □

5. A determinantal expression of $f_j(v)$ and more

In this section, with the help of the formula (13) for derivatives of the ratio of two differentiable functions, we simply establish determinantal formulas for the functions $G_j^{(k)}(v)$ and $f_j(v)$.

Theorem 3. *For $j \in \mathbb{N}$ and $k \ge 0$, we have*

$$G_j^{(k)}(v) = \frac{(-1)^k}{F_1^{k+1}(v)} \times \begin{vmatrix} \langle j-1\rangle_0 v^{j-1} & F_1(v) & 0 & \cdots & 0 & 0 \\ \langle j-1\rangle_1 v^{j-2} & F_1'(v) & F_1(v) & \cdots & 0 & 0 \\ \langle j-1\rangle_2 v^{j-3} & F_1''(v) & \binom{2}{1}F_1'(v) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle j-1\rangle_{k-2} v^{j-k+1} & F_1^{(k-2)}(v) & \binom{k-2}{1}F_1^{(k-3)}(v) & \cdots & F_1(v) & 0 \\ \langle j-1\rangle_{k-1} v^{j-k} & F_1^{(k-1)}(v) & \binom{k-1}{1}F_1^{(k-2)}(v) & \cdots & \binom{k-1}{k-2}F_1'(v) & F_1(v) \\ \langle j-1\rangle_k v^{j-k-1} & F_1^{(k)}(v) & \binom{k}{1}F_1^{(k-1)}(v) & \cdots & \binom{k}{k-2}F_1''(v) & \binom{k}{k-1}F_1'(v) \end{vmatrix}. \tag{23}$$

In particular, we obtain

$$f_j(v) = \frac{(-1)^j}{F_1^{j+1}(v)} \begin{vmatrix} \langle j-1\rangle_0 v^{j-1} & F_1(v) & 0 & \cdots & 0 & 0 \\ \langle j-1\rangle_1 v^{j-2} & F_1'(v) & F_1(v) & \cdots & 0 & 0 \\ \langle j-1\rangle_2 v^{j-3} & F_1''(v) & \binom{2}{1}F_1'(v) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle j-1\rangle_{j-2} v & F_1^{(j-2)}(v) & \binom{j-2}{1}F_1^{(j-3)}(v) & \cdots & F_1(v) & 0 \\ (j-1)! & F_1^{(j-1)}(v) & \binom{j-1}{1}F_1^{(j-2)}(v) & \cdots & \binom{j-1}{j-2}F_1'(v) & F_1(v) \\ 0 & F_1^{(j)}(v) & \binom{j}{1}F_1^{(j-1)}(v) & \cdots & \binom{j}{j-2}F_1''(v) & \binom{j}{j-1}F_1'(v) \end{vmatrix}, \tag{24}$$

$$G_j^{(k)}(0) = 0, \quad 0 \le k < j-1, \tag{25}$$

and those two equalities in (18) *and* (19).

Proof. From the integral representation (5) and the equation (21) or from the reciprocal of the equation (6), we obtain

$$G_j^{(k)}(v) = \frac{\mathrm{d}^k}{\mathrm{d}\, v^k}\left[\frac{v^{j-1}}{(1-\mathrm{e}^{-v})/v}\right] = \frac{\mathrm{d}^k}{\mathrm{d}\, v^k}\left[\frac{v^{j-1}}{F_1(v)}\right].$$

Applying the formula (13) to the functions $u(v) = v^{j-1}$ and $v(v) = F_1(v)$ directly results in the determinantal expression (23).

Setting $k = j$ in (23) gives (24).

Taking $v = 0$ in (23) yields (25). The proof of Theorem 3 is completed. □

6. A recursive relation of $f_j(v)$ and more

In this section, by virtue of the recursive relation (15), we simply derive recursive relations for the functions $G_j^{(k)}(v)$ and $f_j(v)$.

Theorem 4. *For $j \in \mathbb{N}$ and $k \ge 0$, we have*

$$G_j^{(k)}(v) = \frac{1}{F_1(v)}\left[\langle j-1\rangle_k v^{j-k-1} - \sum_{r=0}^{k-1}\binom{k}{r}F_1^{(k-r)}(v)G_j^{(r)}(v)\right]. \tag{26}$$

In particular, we have

$$f_j(v) = -\frac{1}{F_1(v)}\sum_{r=0}^{j-1}\binom{j}{r}F_1^{(j-r)}(v)G_j^{(r)}(v), \tag{27}$$

the first equality in (18)*, and those equalities in* (25) *for $0 \le k < j-1$ and $j \in \mathbb{N}$.*

Proof. In the recursive relation (15), replacing the quantities k, H_{k+1}, H_{r-1}, $h_{i,1}$, and $h_{i,\ell}(v)$ by

$$k+1, \quad (-1)^k F_1^{k+1}(v) G_j^{(k)}(v), \quad (-1)^{r-2} F_1^{r-1}(v) G_j^{(r-2)}(v), \quad \langle j-1\rangle_{i-1} v^{j-i}$$

for $1 \le i \le k+1$, and $\binom{i-1}{\ell-2} F_1^{(i-\ell+1)}(v)$ for $1 \le i \le k+1$ and $2 \le \ell \le k+1$ in (15) respectively arrives at

$$\begin{aligned}
(-1)^k F_1^{k+1}(v) G_j^{(k)}(v) &= (-1)^k h_{k+1,1} \left(\prod_{\ell=1}^{k} h_{\ell,\ell+1}\right) H_0 + \sum_{r=2}^{k+1} (-1)^{k-r+1} h_{k+1,r} \left(\prod_{\ell=r}^{k} h_{\ell,\ell+1}\right) H_{r-1} \\
&= (-1)^k \langle j-1\rangle_k v^{j-k-1} F_1^k(v) + \sum_{r=2}^{k+1} (-1)^{k-r+1} \binom{k}{r-2} F_1^{(k-r+2)}(v) F_1^{k-r+1}(v) (-1)^{r-2} F_1^{r-1}(v) G_j^{(r-2)}(v) \\
&= (-1)^k F_1^k(v) \left[\langle j-1\rangle_k v^{j-k-1} - \sum_{r=2}^{k+1} \binom{k}{r-2} F_1^{(k-r+2)}(v) G_j^{(r-2)}(v)\right] \\
&= (-1)^k F_1^k(v) \left[\langle j-1\rangle_k v^{j-k-1} - \sum_{r=0}^{k-1} \binom{k}{r} F_1^{(k-r)}(v) G_j^{(r)}(v)\right].
\end{aligned}$$

The recursive relation (26) is thus proved.

Taking $k = j$ in (26) and simplifying reduce to (27).

Letting $v = 0$ in (26) leads to the first equality in (18) and those equalities in (25). The proof of Theorem 4 is completed. □

7. A closed-form formula of $\gamma(j,k)$

In this section, we give a closed-form formula of the coefficients $\gamma(j,k)$ defined in Conjecture 2.

Theorem 5. *For $j \in \mathbb{N}$ and $k \ge 0$, the coefficients $\gamma(j,k)$ can be computed by*

$$\begin{aligned}
\gamma(j,k) = k! \sum_{r=0}^{k} (-1)^{r+1} (r+1)(r+2)(r+j)! \left[\sum_{q=0}^{r+1} \frac{(-1)^q}{q+1} \frac{S(r+q+1,q)}{(r-q+1)!(r+q+1)!}\right] \\
\times \left[\sum_{m=0}^{k-r} (-1)^m \frac{j^{k-r-m}}{(k-r-m)!} \sum_{p=0}^{m} (-1)^p \frac{\langle j+1\rangle_p}{(m+p)!} \sum_{q=0}^{p} (-1)^q \binom{m+p}{p-q} S(m+q,q)\right].
\end{aligned} \tag{28}$$

Proof. The equation (2) means that

$$\begin{aligned}
k!\gamma(j,k) &= \lim_{t\to 0} \frac{\mathrm{d}^k\, g_j(t)}{\mathrm{d}\, t^k} = \lim_{t\to 0} \frac{\mathrm{d}^k}{\mathrm{d}\, t^k} \left[F_1^{j+1}(t)\, \mathrm{e}^{nt} f_j(t)\right] \\
&= \lim_{t\to 0} \sum_{\ell=0}^{k} \binom{k}{\ell} \left[F_1^{j+1}(t)\, \mathrm{e}^{nt}\right]^{(\ell)} f_j^{(k-\ell)}(t) \\
&= \lim_{t\to 0} \sum_{\ell=0}^{k} \binom{k}{\ell} \left[\sum_{m=0}^{\ell} \binom{\ell}{m} \left(F_1^{j+1}(t)\right)^{(m)} j^{\ell-m}\, \mathrm{e}^{nt}\right] f_j^{(k-\ell)}(t) \\
&= \lim_{t\to 0} \sum_{\ell=0}^{k} \binom{k}{\ell} G_j^{(j+k-\ell)}(t) \sum_{m=0}^{\ell} \binom{\ell}{m} j^{\ell-m}\, \mathrm{e}^{nt} \\
&\quad \times \sum_{p=0}^{m} \langle j+1\rangle_p F_1^{j-p+1}(t)\, \mathrm{B}_{m,p}\left(F_1'(t), F_1''(t), \dots, F_1^{(m-p+1)}(t)\right) \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} G_j^{(j+k-\ell)}(0) \sum_{m=0}^{\ell} \binom{\ell}{m} j^{\ell-m} \sum_{p=0}^{m} \langle j+1\rangle_p F_1^{j-p+1}(0)
\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{B}_{m,p}\big(F_1'(0), F_1''(0), \ldots, F_1^{(m-p+1)}(0)\big) \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell+1} (j+k-\ell)!(k-\ell+2)! \\
&\quad \times \left[\sum_{q=0}^{k-\ell+1} \frac{(-1)^q}{q+1} \frac{S(k-\ell+q+1,q)}{(k-\ell-q+1)!(k-\ell+q+1)!} \right] \\
&\quad \times \left[\sum_{m=0}^{\ell} \binom{\ell}{m} j^{\ell-m} \sum_{p=0}^{m} (-1)^{m+p} \langle j+1 \rangle_p \frac{m!}{(m+p)!} \sum_{q=0}^{p} (-1)^q \binom{m+p}{p-q} S(m+q,q) \right] \\
&= (-1)^{k+1} k! \sum_{\ell=0}^{k} (-1)^{\ell} (k-\ell+2)(k-\ell+1)(k-\ell+j)! \\
&\quad \times \left[\sum_{q=0}^{k-\ell+1} \frac{(-1)^q}{q+1} \frac{S(k-\ell+q+1,q)}{(k-\ell-q+1)!(k-\ell+q+1)!} \right] \\
&\quad \times \left[\sum_{m=0}^{\ell} (-1)^m \frac{j^{\ell-m}}{(\ell-m)!} \sum_{p=0}^{m} (-1)^p \frac{\langle j+1 \rangle_p}{(m+p)!} \sum_{q=0}^{p} (-1)^q \binom{m+p}{p-q} S(m+q,q) \right] \\
&= k! \sum_{r=0}^{k} (-1)^{r+1} (r+1)(r+2)(r+j)! \left[\sum_{q=0}^{r+1} \frac{(-1)^q}{q+1} \frac{S(r+q+1,q)}{(r-q+1)!(r+q+1)!} \right] \\
&\quad \times \left[\sum_{m=0}^{k-r} (-1)^m \frac{j^{k-r-m}}{(k-r-m)!} \sum_{p=0}^{m} (-1)^p \frac{\langle j+1 \rangle_p}{(m+p)!} \sum_{q=0}^{p} (-1)^q \binom{m+p}{p-q} S(m+q,q) \right],
\end{aligned}
$$

where we used the Leibniz rule for differentiation of a product, the Faà di Bruno formula (8), the formula (17) in Theorem 1, and the identity expressed by (22), Theorem 5 is thus proved. □

8. The power series expansion of $\mathfrak{g}_j(v)$

In this section, we derive power series expansion of the function $\mathfrak{g}_j(v)$ defined by (3).

Theorem 6. *For $j \in \mathbb{N}$, the power series of the function $\mathfrak{g}_j(v)$ is*

$$
\mathfrak{g}_j(v) = (j+1)! \sum_{k=0}^{\infty} C(j,k) \frac{v^k}{k!}, \tag{29}
$$

where

$$
C(j,k) = k! \sum_{r=0}^{k} (-1)^{r+1} (r+1)(r+2)(r+j)! \frac{S(k-r,j+1)}{(k-r)!} \sum_{q=0}^{r+1} \frac{(-1)^q}{q+1} \frac{S(r+q+1,q)}{(r-q+1)!(r+q+1)!}. \tag{30}
$$

Proof. It is easy to see that

$$
\begin{aligned}
\lim_{v\to 0} \frac{\mathrm{d}^k\, \mathfrak{g}_j(v)}{\mathrm{d}\, v^k} &= \lim_{v\to 0} \big[\big(\mathrm{e}^v - 1\big)^{j+1} G_j^{(j)}(v) \big]^{(k)} \\
&= \lim_{v\to 0} \sum_{\ell=0}^{k} \binom{k}{\ell} \big[\big(\mathrm{e}^v - 1\big)^{j+1} \big]^{(\ell)} G_j^{(j+k-\ell)}(v) \\
&= \lim_{v\to 0} \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{p=0}^{\ell} \langle j+1 \rangle_p \big(\mathrm{e}^v - 1\big)^{j-p+1} \mathrm{B}_{\ell,p}\big(\mathrm{e}^v, \mathrm{e}^v, \ldots, \mathrm{e}^v\big) G_j^{(j+k-\ell)}(v)
\end{aligned}
$$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} \langle j+1 \rangle_{j+1} \mathrm{B}_{\ell,j+1}(1,1,\dots,1) G_j^{(j+k-\ell)}(0)$$

$$= (j+1)!k! \sum_{r=0}^{k} (-1)^{r+1}(r+1)(r+2)(r+j)! \frac{S(k-r,j+1)}{(k-r)!}$$

$$\times \sum_{q=0}^{r+1} \frac{(-1)^q}{q+1} \frac{S(r+q+1,q)}{(r-q+1)!(r+q+1)!},$$

where we used the Leibniz rule for differentiation of a product, the Faà di Bruno formula (8), the identity (10), and the formula (17) in Theorem 1. The power series (29) is thus proved. The proof of Theorem 6 is completed. □

9. Computation of a Hessenberg determinant

We call a matrix $H = (h_{\ell,m})_{n\times n}$ a lower or upper Hessenberg matrix if $h_{\ell,m} = 0$ for all pairs (ℓ, m) such that $\ell + 1 < m$ or $h_{\ell,m} = 0$ for all pairs (ℓ, m) such that $m + 1 < \ell$. We call the determinant of a lower or upper Hessenberg matrix the Hessenberg determinant.

In this section, utilizing the formulas (16) and (17) in Theorem 1, utilizing the determinantal expression (23) in Theorem 3, and employing some basic properties of determinants, we find the following formula for computing a Hessenberg determinant and supply an alternative proof.

Theorem 7. *For $j \in \{0\} \cup \mathbb{N}$, we have*

$$\begin{vmatrix} \frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{j!} & \frac{1}{(j-1)!} & \frac{1}{(j-2)!} & \cdots & \frac{1}{2!} & 1 & 0 \\ \frac{1}{(j+1)!} & \frac{1}{j!} & \frac{1}{(j-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{(j+2)!} & \frac{1}{(j+1)!} & \frac{1}{j!} & \cdots & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} = (-1)^{j+1} \frac{B_{j+1}}{(j+1)!}, \tag{31}$$

where B_j denotes the Bernoulli numbers which can be generated by

$$\frac{v}{\mathrm{e}^v - 1} = \sum_{j=0}^{\infty} B_j \frac{v^j}{j!} = 1 - \frac{v}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{v^{2j}}{(2j)!}, \quad |v| < 2\pi. \tag{32}$$

First proof. Let $f(v) = 1 + \sum_{k=1}^{\infty} a_k v^k$ and $g(v) = 1 + \sum_{k=1}^{\infty} b_k v^k$ be two formal power series such that $f(v)g(v) = 1$. Then

$$b_j = (-1)^j \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{j-1} & a_{j-2} & a_{j-3} & a_{j-4} & \cdots & 1 \\ a_j & a_{j-1} & a_{j-2} & a_{j-3} & \cdots & a_1 \end{vmatrix}. \tag{33}$$

The formula (33) can be found in either [22, p. 17, Theorem 1.3] or [23, p. 347]. Letting $a_k = \frac{1}{(k+1)!}$ yields

$$f(v) = 1 + \sum_{k=1}^{\infty} \frac{v^k}{(k+1)!} = \frac{\mathrm{e}^v - 1}{v} \quad \text{and} \quad g(v) = \frac{1}{f(v)} = \frac{v}{\mathrm{e}^v - 1} = 1 + \sum_{k=1}^{\infty} B_k \frac{v^k}{k!}.$$

This means that $b_k = \frac{B_k}{k!}$. Making use of these concrete values of a_k and b_k in (33) arrives at (31). The first proof of Theorem 7 is completed. □

Second proof. When $k > j-1$ and $v = 0$, the determinantal expression (23) in Theorem 3 can be re-arranged as

$$
\begin{aligned}
G_j^{(k)}(0) &= (-1)^k \begin{vmatrix}
0 & F_1(0) & 0 & \cdots & 0 & 0 \\
0 & F_1'(0) & F_1(0) & \cdots & 0 & 0 \\
0 & F_1''(0) & \binom{2}{1}F_1'(0) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & F_1^{(j-2)}(0) & \binom{j-2}{1}F_1^{(j-3)}(0) & \cdots & 0 & 0 \\
\langle j-1\rangle_{j-1} & F_1^{(j-1)}(0) & \binom{j-1}{1}F_1^{(j-2)}(0) & \cdots & 0 & 0 \\
0 & F_1^{(j)}(0) & \binom{j}{1}F_1^{(j-1)}(0) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & F_1^{(k-2)}(0) & \binom{k-2}{1}F_1^{(k-3)}(0) & \cdots & F_1(0) & 0 \\
0 & F_1^{(k-1)}(0) & \binom{k-1}{1}F_1^{(k-2)}(0) & \cdots & \binom{k-1}{k-2}F_1'(0) & F_1(0) \\
0 & F_1^{(k)}(0) & \binom{k}{1}F_1^{(k-1)}(0) & \cdots & \binom{k}{k-2}F_1''(0) & \binom{k}{k-1}F_1'(0)
\end{vmatrix} \\
&= (-1)^{k+j+1}\langle j-1\rangle_{j-1} \begin{vmatrix} A_{(j-1)\times(j-1)} & O_{(j-1)\times(k-j+1)} \\ C_{(k-j+1)\times(j-1)} & B_{(k-j+1)\times(k-j+1)} \end{vmatrix},
\end{aligned}
$$

where, by suitable operations in linear algebra,

$$
\begin{aligned}
\left|A_{(j-1)\times(j-1)}\right| &= \begin{vmatrix}
F_1(0) & 0 & 0 & \cdots & 0 & 0 \\
F_1'(0) & F_1(0) & 0 & \cdots & 0 & 0 \\
F_1''(0) & \binom{2}{1}F_1'(0) & F_1(0) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
F_1^{(j-4)}(0) & \binom{j-4}{1}F_1^{(j-5)}(0) & \binom{j-4}{2}F_1^{(j-6)}(0) & \cdots & 0 & 0 \\
F_1^{(j-3)}(0) & \binom{j-3}{1}F_1^{(j-4)}(0) & \binom{j-3}{2}F_1^{(j-5)}(0) & \cdots & F_1(0) & 0 \\
F_1^{(j-2)}(0) & \binom{j-2}{1}F_1^{(j-3)}(0) & \binom{j-2}{2}F_1^{(j-4)}(0) & \cdots & \binom{j-2}{j-3}F_1'(0) & F_1(0)
\end{vmatrix} \\
&= F_1^{j-1}(0) \\
&= 1,
\end{aligned}
$$

$$
\left|B_{(k-j+1)\times(k-j+1)}\right|
$$

$$
= \begin{vmatrix}
\binom{j}{j-1}F_1'(0) & F_1(0) & 0 & \cdots & 0 & 0 \\
\binom{j+1}{j-1}F_1''(0) & \binom{j+1}{j}F_1'(0) & F_1(0) & \cdots & 0 & 0 \\
\binom{j+2}{j-1}F_1^{(3)}(0) & \binom{j+2}{j}F_1''(0) & \binom{j+2}{j+1}F_1'(0) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\binom{k-2}{j-1}F_1^{(k-j-1)}(0) & \binom{k-2}{j}F_1^{(k-j-2)}(0) & \binom{k-2}{j+1}F_1^{(k-j-3)}(0) & \cdots & F_1(0) & 0 \\
\binom{k-1}{j-1}F_1^{(k-j)}(0) & \binom{k-1}{j}F_1^{(k-j-1)}(0) & \binom{k-1}{j+1}F_1^{(k-j-2)}(0) & \cdots & \binom{k-1}{k-2}F_1'(0) & F_1(0) \\
\binom{k}{j-1}F_1^{(k-j+1)}(0) & \binom{k}{j}F_1^{(k-j)}(0) & \binom{k}{j+1}F_1^{(k-j-1)}(0) & \cdots & \binom{k}{k-2}F_1''(0) & \binom{k}{k-1}F_1'(0)
\end{vmatrix}
$$

$$= \begin{vmatrix} \frac{-j!}{(j-1)!2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{(j+1)!}{(j-1)!3!} & \frac{-(j+1)!}{j!2!} & 1 & \cdots & 0 & 0 & 0 \\ \frac{-(j+2)!}{(j-1)!4!} & \frac{(j+2)!}{j!3!} & \frac{-(j+2)!}{(j+1)!2!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(-1)^{k-j-1}(k-2)!}{(j-1)!(k-j)!} & \frac{(-1)^{k-j-2}(k-2)!}{j!(k-j-1)!} & \frac{(-1)^{k-j-3}(k-2)!}{(j+1)!(k-j-2)!} & \cdots & \frac{-(k-2)!}{(k-3)!2!} & 1 & 0 \\ \frac{(-1)^{k-j}(k-1)!}{(j-1)!(k-j+1)!} & \frac{(-1)^{k-j-1}(k-1)!}{j!(k-j)!} & \frac{(-1)^{k-j-2}(k-1)!}{(j+1)!(k-j-1)!} & \cdots & \frac{(k-1)!}{(k-3)!3!} & \frac{-(k-1)!}{(k-2)!2!} & 1 \\ \frac{(-1)^{k-j+1}k!}{(j-1)!(k-j+2)!} & \frac{(-1)^{k-j}k!}{j!(k-j+1)!} & \frac{(-1)^{k-j-1}k!}{(j+1)!(k-j)!} & \cdots & \frac{-k!}{(k-3)!4!} & \frac{k!}{(k-2)!3!} & \frac{-k!}{(k-1)!2!} \end{vmatrix}$$

$$= \frac{k!}{(j-1)!} \begin{vmatrix} \frac{-1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{-1}{2!} & 1 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(-1)^{k-j-1}}{((k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \frac{(-1)^{k-j-3}}{(k-j-2)!} & \cdots & \frac{-1}{2!} & 1 & 0 \\ \frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \cdots & \frac{1}{3!} & \frac{-1}{2!} & 1 \\ \frac{(-1)^{k-j+1}}{(k-j+2)!} & \frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \cdots & \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} \end{vmatrix},$$

and $O_{(j-1)\times(k-j+1)}$ is a matrix whose elements are all zero, where we used the formula (16) in Theorem 1. Employing the formula

$$\begin{vmatrix} A & O \\ C & B \end{vmatrix} = |A||B|$$

for square matrices A and B, see [6, p. 103, eq. (2.7.6)], we obtain

$$\frac{k!}{(j-1)!} \begin{vmatrix} \frac{-1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{-1}{2!} & 1 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(-1)^{k-j-1}}{((k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \frac{(-1)^{k-j-3}}{(k-j-2)!} & \cdots & \frac{-1}{2!} & 1 & 0 \\ \frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \cdots & \frac{1}{3!} & \frac{-1}{2!} & 1 \\ \frac{(-1)^{k-j+1}}{(k-j+2)!} & \frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \cdots & \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} \end{vmatrix}$$

$$= (-1)^{k+j+1} \frac{G_j^{(k)}(0)}{\langle j-1 \rangle_{j-1}}$$

$$= \frac{k!(k-j+2)!}{(j-1)!} \sum_{q=0}^{k-j+1} \frac{(-1)^q}{q+1} \frac{S(k-j+q+1,q)}{(k-j-q+1)!(k-j+q+1)!},$$

where we used the formula (17) in Theorem 1. Consequently, it follows that

$$\begin{vmatrix} \frac{-1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{-1}{2!} & 1 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(-1)^{k-j-1}}{((k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \frac{(-1)^{k-j-3}}{(k-j-2)!} & \cdots & \frac{-1}{2!} & 1 & 0 \\ \frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \cdots & \frac{1}{3!} & \frac{-1}{2!} & 1 \\ \frac{(-1)^{k-j+1}}{(k-j+2)!} & \frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \cdots & \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} \end{vmatrix}$$

$$= (k-j+2)! \sum_{q=0}^{k-j+1} \frac{(-1)^q}{q+1} \frac{S(k-j+q+1,q)}{(k-j-q+1)!(k-j+q+1)!}.$$

Further replacing $k-j$ by ℓ and utilizing related properties in linear algebra yield

$$\begin{aligned}
&\begin{vmatrix}
\frac{-1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{3!} & \frac{-1}{2!} & 1 & \cdots & 0 & 0 & 0 \\
\frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{(-1)^{\ell-1}}{\ell!} & \frac{(-1)^{\ell-2}}{(\ell-1)!} & \frac{(-1)^{\ell-3}}{(\ell-2)!} & \cdots & \frac{-1}{2!} & 1 & 0 \\
\frac{(-1)^{\ell}}{(\ell+1)!} & \frac{(-1)^{\ell-1}}{\ell!} & \frac{(-1)^{\ell-2}}{(\ell-1)!} & \cdots & \frac{1}{3!} & \frac{-1}{2!} & 1 \\
\frac{(-1)^{\ell+1}}{(\ell+2)!} & \frac{(-1)^{\ell}}{(\ell+1)!} & \frac{(-1)^{\ell-1}}{\ell!} & \cdots & \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!}
\end{vmatrix} \\
&= (-1)^{\ell+1} \begin{vmatrix}
\frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 & 0 \\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{1}{\ell!} & \frac{1}{(\ell-1)!} & \frac{1}{(\ell-2)!} & \cdots & \frac{1}{2!} & 1 & 0 \\
\frac{1}{(\ell+1)!} & \frac{1}{\ell!} & \frac{1}{(\ell-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & 1 \\
\frac{1}{(\ell+2)!} & \frac{1}{(\ell+1)!} & \frac{1}{\ell!} & \cdots & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!}
\end{vmatrix} \\
&= (\ell+2)! \sum_{q=0}^{\ell+1} \frac{(-1)^q}{q+1} \frac{S(\ell+q+1,q)}{(\ell-q+1)!(\ell+q+1)!}.
\end{aligned} \tag{34}$$

In [19, Theorem 1], among other things, it was obtained that

$$B_j = \sum_{i=0}^{j} (-1)^i \frac{\binom{j+1}{i+1}}{\binom{j+i}{i}} S(j+i,i), \quad j \ge 0. \tag{35}$$

The formula (35) can be re-arranged as

$$\begin{aligned}
B_{j+1} &= \sum_{q=0}^{j+1} (-1)^q \frac{\binom{j+2}{q+1}}{\binom{j+q+1}{q}} S(j+q+1,q) \\
&= \sum_{q=0}^{j+1} (-1)^q \frac{(j+2)!}{(q+1)!(j-q+1)!} \frac{q!(j+1)!}{(j+q+1)!} S(j+q+1,q) \\
&= (j+1)!(j+2)! \sum_{q=0}^{j+1} \frac{(-1)^q}{q+1} \frac{S(j+q+1,q)}{(j-q+1)!(j+q+1)!}
\end{aligned} \tag{36}$$

for $j \ge 0$. Substituting (36) into (34) results in the formula (31). The second proof of Theorem 7 is completed. □

10. Coefficients in power series in terms of Bernoulli numbers

In this section, we derive coefficients in power series of the functions $f_j(v)$, $g_j(v)$, and $\mathfrak{g}_j(v)$ in terms of the Bernoulli numbers B_j generated by (32).

Theorem 8. *For $k \ge 0$ and $j \in \mathbb{N}$, we have*

$$G_j^{(k)}(0) = \begin{cases} 0, & 0 \le k < j-1; \\ k! \frac{(-1)^{k-j+1} B_{k-j+1}}{(k-j+1)!}, & k \ge j-1 \ge 0. \end{cases} \tag{37}$$

First proof. From the first proof of Theorem 7, we obtain

$$
\begin{aligned}
G_j^{(k)}(0) &= (-1)^{k+j+1}k!\begin{vmatrix}
\frac{-1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0\\
\frac{1}{3!} & \frac{-1}{2!} & 1 & \cdots & 0 & 0 & 0\\
\frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\frac{(-1)^{k-j-1}}{((k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \frac{(-1)^{k-j-3}}{(k-j-2)!} & \cdots & \frac{-1}{2!} & 1 & 0\\
\frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \frac{(-1)^{k-j-2}}{(k-j-1)!} & \cdots & \frac{1}{3!} & \frac{-1}{2!} & 1\\
\frac{(-1)^{k-j+1}}{(k-j+2)!} & \frac{(-1)^{k-j}}{(k-j+1)!} & \frac{(-1)^{k-j-1}}{(k-j)!} & \cdots & \frac{-1}{4!} & \frac{1}{3!} & \frac{-1}{2!}
\end{vmatrix}\\
&= k!\begin{vmatrix}
\frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0\\
\frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 & 0\\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\frac{1}{((k-j)!} & \frac{1}{(k-j-1)!} & \frac{1}{(k-j-2)!} & \cdots & \frac{1}{2!} & 1 & 0\\
\frac{1}{(k-j+1)!} & \frac{1}{(k-j)!} & \frac{1}{(k-j-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & 1\\
\frac{1}{(k-j+2)!} & \frac{1}{(k-j+1)!} & \frac{1}{(k-j)!} & \cdots & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!}
\end{vmatrix}\\
&= k!\frac{(-1)^{k-j+1}B_{k-j+1}}{(k-j+1)!}.
\end{aligned}
$$

The first proof of Theorem 8 is completed. □

Second proof. For $j \in \mathbb{N}$, the function $G_j(v)$ can be rewritten as

$$G_j(v) = v^{j-1}\frac{-v}{\mathrm{e}^{-v}-1} = v^{j-1}\sum_{\ell=0}^{\infty}\frac{B_\ell}{\ell!}(-v)^\ell = \sum_{\ell=0}^{\infty}(-1)^\ell\frac{B_\ell}{\ell!}v^{j+\ell-1}, \quad |v| < 2\pi.$$

Hence, it follows that

$$G_j^{(k)}(v) = \sum_{\ell=0}^{\infty}(-1)^\ell\frac{B_\ell}{\ell!}\langle j+\ell-1\rangle_k t^{j+\ell-k-1}, \quad |v| < 2\pi.$$

Therefore, we arrive at

$$G_j^{(k)}(0) = \begin{cases} 0, & 0 \le k < j-1;\\ (-1)^{k-j+1}\frac{B_{k-j+1}}{(k-j+1)!}\langle k\rangle_k, & k \ge j-1 \ge 0. \end{cases}$$

The second proof of Theorem 8 is completed. □

Third proof. This follows from combining (17) and (36). The third proof of Theorem 8 is completed. □

Theorem 9. *For $j \in \mathbb{N}$ and $k \ge 0$, we have*

$$\frac{f_j(v)}{j!} = \frac{1}{2} + \sum_{r=1}^{\infty}\binom{j+2r-1}{j}\frac{B_{2r}}{(2r)!}v^{2r-1}, \tag{38}$$

$$
\begin{aligned}
\frac{\gamma(j,k)}{j!k!} = \sum_{r=0}^{k}\binom{j+r}{j}\frac{(-1)^{r+1}B_{r+1}}{(r+1)!}\Bigg[\sum_{m=0}^{k-r}(-1)^m\frac{j^{k-r-m}}{(k-r-m)!}\\
\times\sum_{p=0}^{m}(-1)^p\frac{\langle j+1\rangle_p}{(m+p)!}\sum_{q=0}^{p}(-1)^q\binom{m+p}{p-q}S(m+q,q)\Bigg],
\end{aligned} \tag{39}
$$

and

$$\frac{C(j,k)}{j!k!}=\sum_{r=0}^{k}\binom{j+r}{j}\frac{(-1)^{r+1}B_{r+1}}{(r+1)!}\frac{S(k-r,j+1)}{(k-r)!}. \tag{40}$$

Proof. These follow from combination of (36) with (20) in Theorem 1, (28) in Theorem 5, and (30) in Theorem 6 respectively and from consideration of $B_{2k+1}=0$ for $k\in\mathbb{N}$. The proof of Theorem 9 is completed. □

11. Logarithmic convexity of the sequence $\left|f_j^{(k)}(0)\right|$

In this section, making use of the main results in [37], we derive logarithmic convexity of the sequence $\left|f_j^{(k)}(0)\right|$ and another one.

Theorem 10. *For $j,k\in\mathbb{N}$, we have*

$$f_j(0)=\frac{j!}{2},\quad f_j^{(2k-1)}(0)=(2k+j-1)!\frac{B_{2k}}{(2k)!},\quad f_j^{(2k)}(0)=0.$$

For fixed $j\in\mathbb{N}$, the sequence

$$\left|f_j'(0)\right|,\left|f_j^{(3)}(0)\right|,\dots,\left|f_j^{(2k-3)}(0)\right|,\left|f_j^{(2k-1)}(0)\right|,\left|f_j^{(2k+1)}(0)\right|,\dots \tag{41}$$

is logarithmically convex with respect to $k\in\mathbb{N}$.

For fixed $j\ge 6$, the extended sequence

$$|f_j(0)|,\left|f_j'(0)\right|,\left|f_j^{(3)}(0)\right|,\dots,\left|f_j^{(2k-3)}(0)\right|,\left|f_j^{(2k-1)}(0)\right|,\left|f_j^{(2k+1)}(0)\right|,\dots \tag{42}$$

is logarithmically convex with respect to $k\in\mathbb{N}$.

Proof. The power series expansion (38) can be rewritten as

$$\frac{f_j(v)}{j!}=\frac{1}{2}+\frac{1}{j!}\sum_{k=1}^{\infty}\left[(2k+j-1)!\frac{B_{2k}}{(2k)!}\right]\frac{v^{2k-1}}{(2k-1)!}.$$

This means the second equality (18) and

$$f_j^{(2k-1)}(0)=(2k+j-1)!\frac{B_{2k}}{(2k)!},\quad f_j^{(2k)}(0)=0$$

for $k\in\mathbb{N}$.

From [37, Theorems 1.1 and 1.2], we conclude that, for fixed $\ell\in\{0\}\cup\mathbb{N}$, the sequence

$$(2k+\ell)!\frac{|B_{2k}|}{(2k)!} \tag{43}$$

is logarithmically convex in $k\in\mathbb{N}$. Setting $\ell=j-1$ in (43) reveals that, for fixed $j\in\mathbb{N}$, the sequence $\left|f_j^{(2k-1)}(0)\right|$, that is, the sequence in (41), is logarithmically convex with respect to $k\in\mathbb{N}$.

It is easy to verify that the inequality

$$\left|f_j'(0)\right|^2=\left|(j+1)!\frac{B_2}{2!}\right|^2=\frac{[(j+1)!]^2}{12^2}\le|f_j(0)|\left|f_j^{(3)}(0)\right|=\frac{j!}{2}(j+3)!\frac{|B_4|}{4!}=\frac{j!(j+3)!}{1440},$$

which is equivalent to $j^2-5j-4\ge 0$, is valid for $j\ge 6$. Consequently, the extended sequence in (42) is logarithmically convex with respect to $k\in\mathbb{N}$. The proof of Theorem 10 is completed. □

12. Remarks

In this section, we give several remarks on our main results in this paper.

Remark 1. The first two values in (18) confirm the examination in (4).

When taking $j = 1, 2, \ldots, 9$ and $k = 0, 1, 2, \ldots, 9$ in the formula (17) or (37), we obtain the first few values of $G_j^{(k)}(0)$, which are listed in Table 1.

Table 1. The values of $G_j^{(k)}(0)$ for $1 \le j \le 9$ and $0 \le k \le 9$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=1$	1	$\mathbf{\frac{1}{2}}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0
$j=2$	0	1	**1**	$\frac{1}{2}$	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	$-\frac{3}{10}$
$j=3$	0	0	2	**3**	2	0	-1	0	$\frac{4}{3}$	0
$j=4$	0	0	0	6	**12**	10	0	-7	0	12
$j=5$	0	0	0	0	24	**60**	60	0	-56	0
$j=6$	0	0	0	0	0	120	**360**	420	0	-504
$j=7$	0	0	0	0	0	0	720	**2520**	3360	0
$j=8$	0	0	0	0	0	0	0	5040	**20160**	30240
$j=9$	0	0	0	0	0	0	0	0	40320	**181440**

Those numbers with bold fonts in Table 1 are values of $f_j(0)$ for $1 \le j \le 9$.

Remark 2. From (23), letting $v \to 0$, it is easy to see that $G_j^{(k)}(0) = 0$ for $0 \le k < j-1$.

When $k = j-1$, taking $v \to 0$ on both sides of (23) leads to

$$
\begin{aligned}
G_j^{(j-1)}(0) &= (-1)^{j-1}\begin{vmatrix}
0 & F_1(0) & 0 & \cdots & 0 & 0 \\
0 & F_1'(0) & F_1(0) & \cdots & 0 & 0 \\
0 & F_1''(0) & \binom{2}{1}F_1'(0) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & F_1^{(j-3)}(0) & \binom{j-3}{1}F_1^{(j-4)}(0) & \cdots & F_1(0) & 0 \\
0 & F_1^{(j-2)}(0) & \binom{j-2}{1}F_1^{(j-3)}(0) & \cdots & \binom{j-2}{j-3}F_1'(0) & F_1(0) \\
(j-1)! & F_1^{(j-1)}(0) & \binom{j-1}{1}F_1^{(j-2)}(0) & \cdots & \binom{j-1}{j-3}F_1''(0) & \binom{j-1}{j-2}F_1'(0)
\end{vmatrix} \\
&= (j-1)!\begin{vmatrix}
F_1(0) & 0 & \cdots & 0 & 0 \\
F_1'(0) & F_1(0) & \cdots & 0 & 0 \\
F_1''(0) & \binom{2}{1}F_1'(0) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_1^{(j-3)}(0) & \binom{j-3}{1}F_1^{(j-4)}(0) & \cdots & F_1(0) & 0 \\
F_1^{(j-2)}(0) & \binom{j-2}{1}F_1^{(j-3)}(0) & \cdots & \binom{j-2}{j-3}F_1'(0) & F_1(0)
\end{vmatrix} \\
&= (j-1)!F_1^{j-1}(0) \\
&= (j-1)!.
\end{aligned}
$$

This recovers the first equality in (18).

Similarly, we can recover two equalities in (4) and the left three equalities in (18) and (19).

Remark 3. When taking $j = 1, 2, \ldots, 9$ and $k = 0, 1, 2, \ldots, 7$ in the formula (28), we obtain the first few values of $\gamma(j, k)$, which are listed in Table 2.

Table 2. The values of $\gamma(j,k)$ for $1 \le j \le 9$ and $0 \le k \le 7$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=1$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$	$\frac{1}{5040}$	$\frac{1}{40320}$	$\frac{1}{362880}$
$j=2$	1	1	$\frac{1}{2}$	$\frac{13}{72}$	$\frac{19}{360}$	$\frac{19}{1440}$	$\frac{11}{3780}$	$\frac{349}{604800}$
$j=3$	3	5	4	$\frac{13}{6}$	$\frac{73}{80}$	$\frac{77}{240}$	$\frac{11}{112}$	$\frac{271}{10080}$
$j=4$	12	28	31	$\frac{68}{3}$	$\frac{151}{12}$	$\frac{103}{18}$	$\frac{563}{252}$	$\frac{7789}{10080}$
$j=5$	60	180	255	$\frac{707}{3}$	$\frac{1957}{12}$	$\frac{365}{4}$	$\frac{2731}{63}$	$\frac{22783}{1260}$
$j=6$	360	1320	2280	2551	$\frac{4237}{2}$	$\frac{5639}{4}$	$\frac{3159}{4}$	$\frac{485809}{1260}$
$j=7$	2520	10920	22260	29260	28378	$\frac{65734}{3}$	$\frac{42451}{3}$	$\frac{712363}{90}$
$j=8$	20160	100800	236880	357840	397152	349224	255543	161154
$j=9$	181440	1028160	2736720	4672080	5841360	5764752	4715343	3310319

Can one numerically find a negative value among the coefficients $\gamma(j,k)$ for $j > 9$ or $k > 7$?

Remark 4. When taking $j = 1, 2, \ldots, 8$ and $k = 0, 1, 2, \ldots, 9$ in the formula (30), we obtain the first few values of $C(j,k)$, which are listed in Table 3.

Table 3. The values of $C(j,k)$ for $1 \le j \le 8$ and $0 \le k \le 9$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=1$	0	0	$\frac{1}{2}$	2	$\frac{11}{2}$	13	$\frac{57}{2}$	60	$\frac{247}{2}$	251
$j=2$	0	0	0	1	8	40	$\frac{485}{3}$	581	1946	6238
$j=3$	0	0	0	0	3	40	315	1925	10143	48636
$j=4$	0	0	0	0	0	12	240	2730	23408	169092
$j=5$	0	0	0	0	0	0	60	1680	26040	297696
$j=6$	0	0	0	0	0	0	0	360	13440	272160
$j=7$	0	0	0	0	0	0	0	0	2520	120960
$j=8$	0	0	0	0	0	0	0	0	0	20160

Can one numerically find a negative value among the coefficients $C(j,k)$ for $j > 8$ or $k > 9$?

Remark 5. The formula (31) in Theorem 7 is equivalent to

$$B_j = j! \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{(j-1)!} & \frac{1}{(j-2)!} & \frac{1}{(j-3)!} & \cdots & 1 & 0 & 0 \\ \frac{1}{j!} & \frac{1}{(j-1)!} & \frac{1}{(j-2)!} & \cdots & \frac{1}{2!} & 1 & 0 \\ \frac{1}{(j+1)!} & \frac{1}{j!} & \frac{1}{(j-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & 0 \end{vmatrix}, \quad j \ge 0$$

which was listed in [25, Section 21.5] and [26, p. 1], can be deduced from a result in [7].

Remark 6. Motivated by the formula (31) in Theorem 7, we mention [34, Theorem 3.3] which reads that, for $\theta \in \mathbb{C}$ and $j \in \mathbb{N}$,

$$\begin{vmatrix} \binom{\theta}{1} & \binom{\theta}{0} & 0 & \cdots & 0 & 0 & 0 \\ \binom{\theta}{2} & \binom{\theta}{1} & \binom{\theta}{0} & \cdots & 0 & 0 & 0 \\ \binom{\theta}{3} & \binom{\theta}{2} & \binom{\theta}{1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \binom{\theta}{j-2} & \binom{\theta}{j-3} & \binom{\theta}{j-4} & \cdots & \binom{\theta}{1} & \binom{\theta}{0} & 0 \\ \binom{\theta}{j-1} & \binom{\theta}{j-2} & \binom{\theta}{j-3} & \cdots & \binom{\theta}{2} & \binom{\theta}{1} & \binom{\theta}{0} \\ \binom{\theta}{j} & \binom{\theta}{j-1} & \binom{\theta}{j-2} & \cdots & \binom{\theta}{3} & \binom{\theta}{2} & \binom{\theta}{1} \end{vmatrix} = \frac{(\theta)_j}{j!},$$

where the quantity

$$(\theta)_j=\prod_{\ell=0}^{j-1}(\theta+\ell)=\begin{cases}\theta(\theta+1)\cdots(\theta+j-1), & j\ge1\\ 1, & j=0\end{cases}$$

is called the rising factorial of $\theta\in\mathbb{C}$.

Remark 7. In 26 June 2021, Omran Kouba (Higher Institute for Applied Sciences and Technology, Syria) told the author, Feng Qi, via the ResearchGate that the formula (33) can also be found in [23, p. 347]. Hereafter, we found that the formula (33) also appeared in [22, p. 17, Theorem 1.3] and [36, Section 2] and was called the Wronski formula dated back to the French paper [44] in the year 1811.

In [42], applying the Wronski formula (33) from [44], Whittaker proved that the root of the equation

$$0=\lambda_0+\lambda_1v+\lambda_2v^2+\lambda_3v^3+\lambda_4v^4+\cdots,$$

which is the smallest in absolute value, is given by the series

$$-\frac{\lambda_0}{\lambda_1}-\frac{\lambda_0^2\lambda_2}{\lambda_1\begin{vmatrix}\lambda_1&\lambda_2\\ \lambda_0&\lambda_1\end{vmatrix}}-\frac{\lambda_0^3\begin{vmatrix}\lambda_2&\lambda_3\\ \lambda_1&\lambda_2\end{vmatrix}}{\begin{vmatrix}\lambda_1&\lambda_2\\ \lambda_0&\lambda_1\end{vmatrix}\begin{vmatrix}\lambda_1&\lambda_2&\lambda_3\\ \lambda_0&\lambda_1&\lambda_2\\ 0&\lambda_0&\lambda_1\end{vmatrix}}-\frac{\lambda_0^4\begin{vmatrix}\lambda_2&\lambda_3&\lambda_4\\ \lambda_1&\lambda_2&\lambda_3\\ \lambda_0&\lambda_1&\lambda_2\end{vmatrix}}{\begin{vmatrix}\lambda_1&\lambda_2&\lambda_3\\ \lambda_0&\lambda_1&\lambda_2\\ 0&\lambda_0&\lambda_1\end{vmatrix}\begin{vmatrix}\lambda_1&\lambda_2&\lambda_3&\lambda_4\\ \lambda_0&\lambda_1&\lambda_2&\lambda_3\\ 0&\lambda_0&\lambda_1&\lambda_2\\ 0&0&\lambda_0&\lambda_1\end{vmatrix}}-\cdots.$$

Let $P(v)=\lambda_0+\lambda_1v+\lambda_2v^2+\cdots$ be a formal power series over the field of real numbers such that $\lambda_n>0$ and $\lambda_{n+2}\lambda_n-\lambda_{n+1}^2>0$ for $n=0,1,2,\dots$. In [22, p. 13, Problem 6] and [24], it was obtained that, if $\frac{1}{P(v)}=\mu_0+\mu_1v+\mu_2v^2+\cdots$, then $\mu_n<0$ for $n=1,2,\dots$. It seems that the Wronski formula (33) from [44] was also applied in the paper [16]. This result and the Wronski formula (33) have been applied in [31, 32] to confirm the negativity or the positivity of several Hessenberg determinants whose elements involve the Bernoulli numbers B_{2j} for $j\in\{0\}\cup\mathbb{N}$ and the large Schröder numbers.

Remark 8. In [4, Section 3, Theorem, (3.1)], it is stated that

$$\frac{2(2j)!}{(2\pi)^{2j}}\frac{1}{1-2^{\alpha-2j}}\le(-1)^{j+1}B_{2j}\le\frac{2(2j)!}{(2\pi)^{2j}}\frac{1}{1-2^{\beta-2j}}\tag{44}$$

for $j\in\mathbb{N}$, where $\alpha=0$ and $\beta=2+\frac{\ln(1-6/\pi^2)}{\ln2}=0.6491\dots$ are the best possible in the sense that they can not be replaced by any bigger and smaller constants in the double inequality (44) respectively. See also [1, p. 805, 23.1.15] and [37]. By the double inequality (44), power series (38) in Theorem 9 can be re-formulated as

$$\frac{f_j(v)}{j!}>\frac12+\frac1\pi\left[\sum_{r=1}^{\infty}\frac{\binom{j+4r-3}{j}}{1-2^{\alpha-4r+2}}\left(\frac{v}{2\pi}\right)^{4r-3}-\sum_{r=1}^{\infty}\frac{\binom{j+4r-1}{j}}{1-2^{\beta-4r}}\left(\frac{v}{2\pi}\right)^{4r-1}\right].$$

This may be used to discover a simple and hand-computable lower bound for $f_j(v)$ for $j\in\mathbb{N}$ on the finite interval $(0,2\ln2)$ or on the infinite interval $(0,\infty)$.

Remark 9. Can one read out more useful information from main results, especially from power series (38) and the quantities (39) and (40), in this paper to find properties of three functions $f_j(v)$, $g_j(v)$, and $\mathfrak{g}_j(v)$?

Remark 10. Several new and motivating results related to number theory and abstract algebra can be found in the papers [10,17,18,39,46].

13. Conclusion

In this paper, by virtue of the Faà di Bruno formula (8), with the aid of three identities (9), (10), and (11) for partial Bell polynomials $B_{j,k}$, by means of the formula (13) for higher order derivatives of the ratio of two differentiable functions, and with availability of other techniques, the authors established closed-form formulas in Theorems 2, 5, and 9 in terms of the Bernoulli numbers B_{2j} and the second kind Stirling numbers $S(j,k)$, presented determinantal expressions in Theorems 3 and 7, derived recursive relations in Theorem 4, obtained power series expansions in Theorems 1 and 6, and computed special values of the function $\frac{v^j}{1-\mathrm{e}^{-v}}$, its derivatives, and related ones used in Clark–Ismail's two conjectures. By these results, the authors also found a formula for a Hessenberg determinant and derived logarithmic convexity in Theorems 8 and 10.

Some results reviewed or newly-obtained in this paper have been applied and will be potentially applied to other many fields, areas, subjects, directions, and disciplines. For example, the function defined in (5) and its reciprocal, the generating function of the classical Bernoulli numbers, have been applied in [19, 28–30, 37, 41, 45], for example.

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