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HOPF BIFURCATION CONTROL OF A FRACTIONAL-ORDER DELAYED TURBIDOSTAT MODEL VIA A NOVEL EXTENDED HYBRID CONTROLLER

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Abstract. Building delayed dynamical models to describe the inherent laws of different chemical matters has become a hot theme in recent years. In this current study, we set up a new fractional-order delayed turbidostat model. By using Laplace transform, we obtain the characteristic equation of established fractional-order delayed turbidostat model. By selecting the delay as bifurcation parameter and exploring the roots of the corresponding characteristic equation of the involved fractional-order delayed turbidostat model, a novel delay-dependent condition on stability and Hopf bifurcation is acquired. Taking advantage of a novel extended hybrid controller, the stability region and the time of Hopf bifurcation of the established fractional-order delayed turbidostat model are successfully controlled. The role of delay in stabilizing system and controlling Hopf bifurcation are revealed. Matlab experiments are carried out to check the rationality of the acquired key outcomes in this article. The acquired outcomes of this study are completely new and own great theoretical value in dominating concentrations of various chemical matters.

Keywords: Fractional-Order Turbidostat Model, Stability, Hopf Bifurcation, Extended Hybrid Controller, Delay.

AMS Subject Classification: 34C23, 34K18, 37GK15, 39A11, 92B20.

1. Introduction

During the past decades, delayed dynamical equation has witted great application in many natural science and social science. In particular, setting up some suitable delayed dynamical equations to reveal the intrinsic chemical reaction law has become a hotspot issue in chemistry. By investigating the dynamical behavior of the formulated delayed differential chemical reaction models, we can effectively grasp the variation law among different chemical substances and then better serve humanity. In recent years, a number of researchers are dedicated to exploring various dynamics of all kinds of chemical reaction models and a great deal of worthwhile results on chemical reaction models have been achieved. For instance, in [46] the authors dealt with Hopf bifurcation and the control issue of chaos of a chemical reaction system; the authors investigated the stability and bifurcation in a Gray-Scott chemical reaction system in [30]; the authors discussed the multiple stability switches and bifurcation driven by time delay for a Lengyel-Epstein

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chemical reaction model in [49]; in [5] the authors explored the Neimark-Sacker bifurcation of a discrete-time chemical reaction system; in 2018 the authors carried out the stability, Neimark-Sacker bifurcation and chaos control analysis on chlorine dioxide-iodine-malonic acid reaction model in [3]. In details, one can see [7, 8, 16, 19, 24, 28].

We know that chemostat is an important laboratory instrument utilized for the continuous culture of microorganisms [48]. It plays a vital role in population dynamics and microbiology. The chemostat can be regarded as the simplest idealization of biological model and it has measurable parameters, reasonable experiments and tractable trait in mathematics [25, 48]. During the past decades, many scholars pay great attention to the various chemostat models to maintain the coexistence of the organisms (see [18, 25, 48, 52]). Turbidostat is a special chemostat owning feedback control of the dilution rate [21, 26]. The optical sensor in turbidostat can effectively measure the turbidity of the fluid and this signal can be applied to adjust the rate of dilution. In the papers [2, 6], the authors dealt with the coexistence issue of both species in turbidostat. The turbidostat model takes the following form:

$$\begin{cases} \dfrac{dw_1(t)}{dt} = A(w)(w_0 - w_1) - \dfrac{w_2}{\gamma_1} g_1(w_1) - \dfrac{w_3}{\gamma_2} g_2(w_1), \\ \dfrac{dw_2(t)}{dt} = w_2[g_1(w_1) - A(w)], \\ \dfrac{dw_3(t)}{dt} = w_3[g_2(w_1) - A(w)], \end{cases}$$

where $w_1(t)$ stands for the limiting nutrient concentration and $w_2(t)$ stands for the concentration of the first competitor at time t and $w_3(t)$ stands for the concentration of the second competitor at time t; w_0 represents the input concentration of the limiting nutrient; $\gamma_j\,(j = 1, 2)$ denote yield constants. g_i is called uptakes function and is a continuously differentiable function and $g_i(0) = 0$ and $g_i^{'}(w_1) > 0$ for arbitrary $w_1 \in R_+$. In general, g_i takes the following form:

$$g_i(w_1) = \frac{w_j}{\alpha_j + w_1}, j = 1, 2,$$

where $\alpha_j, w_j\,(j = 1, 2)$ stand for the halfsaturation constant of the jth competitor (or Michaelis-Menten constant) and the maximal growth rate, respectively; $A(w)$ denotes the dilution of the turbidostat and takes the following form:

$$A(w) = a + \kappa_1 w_2(t) + \kappa_2 w_3(t),$$

where $\kappa_1, \kappa_2, a > 0$ are control parameters for dilution of the turbidostat. In details, one can consult [2, 6]. In practice, although the sensor is very sensitive, there usually exist delay during the process of measurement for fluid turbidity. The delayed signal has a vital effect on controlling the dilution rate. Stimulated by this viewpoint, the authors set up the following delayed turbidostat model owning delayed feedback control in [48]:

$$\begin{cases} \dfrac{dw_1(t)}{dt} = [a + \kappa_1 w_2(t-\vartheta) + \kappa_2 w_3(t-\vartheta)](w_0 - w_1) - \dfrac{w_2}{\gamma_1} g_1(w_1) - \dfrac{w_3}{\gamma_2} g_2(w_1), \\ \dfrac{dw_2(t)}{dt} = w_2[g_1(w_1) - (a + \kappa_1 w_2(t-\vartheta) + \kappa_2 w_3(t-\vartheta))], \\ \dfrac{dw_3(t)}{dt} = w_3[g_2(w_1) - (a + \kappa_1 w_2(t-\vartheta) + \kappa_2 w_3(t-\vartheta))], \end{cases} \tag{1}$$

where ϑ is a delay. By selecting the delay as bifurcation parameter, in the same paper, the authors studied the stability and the existence of Hopf bifurcation of model (1). In addition, the bifurcation peculiarities of model (1) have been explored via center manifold theory and normal form theorem.

It should be noted that all the involved publications above on turbidostat models (see [2, 3, 5–8, 16, 18, 19, 21, 24–26, 28, 30, 46, 48, 49, 52]) are merely concerned with the integer-order turbidostat models. For a long time, fractional-order differential equation has maintained a relatively slow development state due to the lack of basic theoretical tools and practical application background.

In recent years, a lot of research shows that fractional-order differential equation is a vital theoretical tool in describing the true laws of nature since it owns the strong memory peculiarity and advantage of hereditary for different substances and development processes [39, 40, 42, 47]. Recently, fractional differential equation has been extensively used in many area such as financial engineering, complex networks, cryptology, multifarious waves in physics, aeroelasticity, capacitor principle, bioengineering, automation and so on [13, 27, 32, 33]. At present, great accomplishments on fractional dynamical systems have been acquired. For example, in [23] the authors studied the Hopf bifurcation and stability of a fractional-order delay predator-prey model involving Holling-type III function; the authors investigated the bifurcation phenomenon and stability for a fractional-order delayed predator-prey model owning interspecific competition in [29]; the authors discussed controller design of finite-time synchronization in fractional memristive neural network models in [31]; in [14] the authors explored the event-triggered bipartite synchronization issue of coupled fractional-order neural networks. For more concrete contents, one can see [4, 10–12, 20, 34–38, 41, 43–45].

Considering that the delayed turbidostat model owning delayed feedback control can better describe the memory peculiarity and advantage of hereditary for the concentrations of three chemical reactants and inspired by the exploration above and depending on system (1), in this study, we will set up the following fractional-order delayed turbidostat model owning delayed feedback control:

$$\begin{cases} \dfrac{d^p w_1(t)}{dt^p} = [a + \kappa_1 w_2(t-\vartheta) + \kappa_2 w_3(t-\vartheta)](w_0 - w_1) - \dfrac{w_2}{\gamma_1} g_1(w_1) - \dfrac{w_3}{\gamma_2} g_2(w_1), \\ \dfrac{d^p w_2(t)}{dt^p} = w_2[g_1(w_1) - (a + \kappa_1 w_2(t-\vartheta) + \kappa_2 w_3(t-\vartheta))], \\ \dfrac{d^p w_3(t)}{dt^p} = w_3[g_2(w_1) - (a + \kappa_1 w_2(t-\vartheta) + \kappa_2 w_3(t-\vartheta))], \end{cases} \tag{2}$$

where $p \in (0, 1]$ stands for the order of fractional-order model (2). All other parameters own the identical connotation as those in system (1).

In this study, we will explore the following two aspects: **(a)** Investigate the stability behavior and the generation of Hopf bifurcation of model (2). **(b)** Adjust the stability region and the time of appearance of Hopf bifurcation of model (2) via extended hybrid controller.

The elementary structure of this study is organized as follows. Some rudimentary principles about fractional-order differential equation are provided in Section 2. Section 3 handles the stability behavior and the existence of Hopf bifurcation of model (2) and a new delay-independent stability and bifurcation condition of model (2) is acquired. Section 4 deals with the control problem of stability region and the time of onset of Hopf bifurcation for model (2) via a novel extended hybrid controller. Section 5 carries out computer experiments to validate the validity of the gained key conclusions. Section 6 ends this study with a succinct conclusion.

2. Rudimentary knowledge

In this section, some essential basic theories about fractional-order differential equation are presented.

Definition 1. *[22] The fractional integral of order p of the function $\omega(\mu)$ is given by*

$$\mathcal{I}^p \omega(\mu) = \frac{1}{\Gamma(p)} \int_{\mu_0}^{\mu} (\mu - u)^{p-1} \omega(u) du,$$

where $\mu > \mu_0, p > 0, \Gamma(u) = \int_0^{\infty} s^{u-1} e^{-s} ds$ denotes the Gamma function.

Definition 2. *[17] Define the Caputo fractional-order derivative of order p of the function* $\omega(\mu) \in ([\mu_0, \infty), R)$ *as follows:*

$$\mathcal{D}^p\omega(\mu) = \frac{1}{\Gamma(m-p)} \int_{\mu_0}^{\mu} \frac{\omega^{(n)}(s)}{(\mu - s)^{p-n+1}} ds,$$

where $\mu \geq \mu_0$ *and n stands for a positive integer* $(n-1 \leq p < n)$*. Peculiarly, if* $p \in (0,1)$*, then*

$$\mathcal{D}^p\omega(\mu) = \frac{1}{\Gamma(1-p)} \int_{\mu_0}^{\mu} \frac{\omega^{'}(s)}{(\mu - s)^{p}} ds.$$

Lemma 1. *[9] Consider the system:* $\mathcal{D}^p v = \mathcal{H}v, v(0) = v_0$ *where* $p \in (0,1), v \in R^n, \mathcal{H} \in R^{n\times n}$*. Assume that* $\chi_l (l = 1, 2, \cdots, n)$ *is the root of the characteristic equation of* $\mathcal{D}^p v = \mathcal{H}v$*, then the system* $\mathcal{D}^p v = \mathcal{H}v$ *is locally asymptotically stable* $\Leftrightarrow |arg(\chi_l)| > \frac{p\pi}{2} (l = 1, 2, \cdots, n)$*. The system is stable* $\Leftrightarrow |arg(\chi_l)| > \frac{p\pi}{2} (l = 1, 2, \cdots, n)$ *and every critical eigenvalue satisfying* $|arg(\chi_l)| = \frac{p\pi}{2} (l = 1, 2, \cdots, n)$ *owns geometric multiplicity one.*

3. Bifurcation investigation

According to the paper [48], we know that if the following condition holds,

$$(Q_1)\ (i)\ a \in (0, A^*), (ii)\ \kappa_1 < \frac{A^* - a}{1 - w_1^*} < \kappa_2 \text{ or } \kappa_2 < \frac{A^* - a}{1 - w_1^*} < \kappa_1,$$

where A^* satisfies $g_1(w_1^*) = g_2(w_1^*) = A^*$, where $A^* \in (0,1), g_1^{'}(w_1^*) \neq g_2^{'}(w_1^*)$, then system (2) admits a unique positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*})$, where

$$\begin{cases} w_{2*} = \dfrac{A^* - \kappa_2(1 - w_{1*}) - a}{\kappa_1 - \kappa_2}, \\ \\ w_{3*} = \dfrac{\kappa_1(1 - w_{1*}) - A^* + a}{\kappa_1 - \kappa_2}. \end{cases}$$

The linear system of system (2) at $W(w_{1*}, w_{2*}, w_{3*})$ owns the following form:

$$\mathcal{D}^p w(t) = A_1 w(t) + A_2 w(t - \vartheta), \tag{3}$$

where

$$\begin{cases} w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, \\ \\ A_1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \end{bmatrix}, \\ \\ A_2 = \begin{bmatrix} 0 & \bar{b}_1 & \bar{b}_2 \\ 0 & \bar{b}_3 & \bar{b}_4 \\ 0 & \bar{b}_5 & \bar{b}_6 \end{bmatrix}, \end{cases}$$

and

$$\begin{cases} a_1 = -\left[a + \kappa_1 w_{2*} + \kappa_{3*} + \dfrac{g_1^{'}(w_{1*})}{\gamma_1} + \dfrac{g_2^{'}(w_{1*})}{\gamma_2}\right], \\ a_2 = -\dfrac{g_1(w_{1*}) + g_1^{'}(w_{1*})}{\gamma_1}, \\ a_3 = -\dfrac{g_2(w_{1*}) + g_2^{'}(w_{1*})}{\gamma_2}, \\ a_4 = w_{2*} g_1^{'}(w_{1*}), \\ a_5 = w_{3*} g_2^{'}(w_{1*}), \\ \bar{b}_1 = \kappa_1(w_0 - w_{1*}), \\ \bar{b}_2 = \kappa_2(w_0 - w_{1*}), \\ \bar{b}_3 = -\kappa_1 w_{2*}, \\ \bar{b}_4 = -\kappa_2 w_{2*}, \\ \bar{b}_5 = -\kappa_1 w_{3*}, \\ \bar{b}_6 = -\kappa_2 w_{3*}. \end{cases}$$

The characteristic equation of system (3) reads as

$$\det \begin{bmatrix} s^p - a_1 & -a_2 - \bar{b}_1 e^{-s\vartheta} & -a_3 - \bar{b}_2 e^{-s\vartheta} \\ -a_4 & s^p - \bar{b}_3 e^{-s\vartheta} & -\bar{b}_4 e^{-s\vartheta} \\ -a_5 & -\bar{b}_5 e^{-s\vartheta} & s^p - \bar{b}_6 e^{-s\vartheta} \end{bmatrix} = 0,$$

which results in

$$s^{3p} + b_1 s^{2p} + b_2 s^p + b_3 + (b_4 s^p + b_5) e^{-s\vartheta} + (b_6 s^p + b_7) e^{-2s\vartheta} = 0, \tag{4}$$

where

$$\begin{cases} b_1 = -a_1, \\ b_2 = -(a_2 a_4 + a_3 a_5), \\ b_3 = a_1 \bar{b}_4 \bar{b}_5, \\ b_4 = a_1 \bar{b}_3 + a_1 \bar{b}_6 - a_5 \bar{b}_2 - a_4 \bar{b}_1, \\ b_5 = a_3 a_5 \bar{b}_3 + a_2 a_4 a_6 \bar{b}_6 - a_2 a_5 \bar{b}_2 - a_3 a_4 \bar{b}_5, \\ b_6 = \bar{b}_3 \bar{b}_6 - \bar{b}_4 \bar{b}_5, \\ b_7 = a_1 \bar{b}_3 \bar{b}_6 - a_5 \bar{b}_1 \bar{b}_4 - a_4 \bar{b}_2 \bar{b}_5 + a_5 \bar{b}_2 \bar{b}_3 + a_4 \bar{b}_1 \bar{b}_6. \end{cases}$$

When $\vartheta = 0$, then Eq.(4) has the following form:

$$\lambda^3 + b_1 \lambda^2 + (b_2 + b_4 + b_6)\lambda + b_3 + b_5 + b_7 = 0. \tag{5}$$

If

$$(Q_2) \begin{cases} \Delta_1 = b_1 > 0, \\ \Delta_2 = \det \begin{bmatrix} b_1 & 1 \\ b_3 + b_5 + b_7 & b_2 + b_4 + b_6 \end{bmatrix}, \\ \Delta_3 = (b_3 + b_5 + b_7)\Delta_2 > 0 \end{cases}$$

is fulfilled, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq.(5) obey $|\arg(\lambda_1)| > \frac{p\pi}{2}, |\arg(\lambda_2)| > \frac{p\pi}{2}, |\arg(\lambda_3)| > \frac{p\pi}{2}$. In view of Lemma 1, we can understand that the positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*})$ of system (2) with the delay $\vartheta = 0$ maintains locally asymptotically stability.

By (4), we have

$$(s^{3p} + b_1 s^{2p} + b_2 s^p + b_3) e^{s\vartheta} + (b_4 s^p + b_5) + (b_6 s^p + b_7) e^{-s\vartheta} = 0. \tag{6}$$

Suppose that $s = i\varepsilon = \varepsilon\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ is the root of Eq.(6). Then, by Eq.(6), one gains

$$\left[\varepsilon^{3p}\left(\cos\frac{3p\pi}{2} + i\sin\frac{3p\pi}{2}\right) + b_1\varepsilon^{2p}(\cos p\pi + i\sin p\pi)\right.$$
$$\left. + b_2\varepsilon^p\left(\cos\frac{p\pi}{2} + i\sin\frac{p\pi}{2}\right) + b_3\right](\cos\varepsilon\vartheta + i\sin\varepsilon\vartheta)$$

$$+\left[b_4\varepsilon^p\left(\cos\frac{p\pi}{2}+i\sin\frac{p\pi}{2}\right)+b_5\right]+\left[b_6\varepsilon^p\left(\cos\frac{p\pi}{2}+i\sin\frac{p\pi}{2}\right)+b_7\right]$$

$$\times(\cos\varepsilon\vartheta-i\sin\varepsilon\vartheta)=0. \tag{7}$$

Then, it follows from (7) that

$$\begin{cases}\Phi_1(\varepsilon)\cos\varepsilon\vartheta-\Phi_2(\varepsilon)\sin\varepsilon\vartheta=-\Phi_3(\varepsilon),\\ \Phi_4(\varepsilon)\cos\varepsilon\vartheta+\Phi_5(\varepsilon)\sin\varepsilon\vartheta=-\Phi_6(\varepsilon),\end{cases} \tag{8}$$

where

$$\begin{cases}\Phi_1(\varepsilon)=c_1\varepsilon^{3p}+c_2\varepsilon^{2p}+c_3\varepsilon^p+c_4,\\ \Phi_2(\varepsilon)=c_5\varepsilon^{3p}+c_6\varepsilon^{2p}+c_7\varepsilon^p,\\ \Phi_3(\varepsilon)=c_8\varepsilon^p+c_9,\\ \Phi_4(\varepsilon)=c_{10}\varepsilon^{3p}+c_{11}\varepsilon^{2p}+c_{12}\varepsilon^p,\\ \Phi_5(\varepsilon)=c_{13}\varepsilon^{3p}+c_{14}\varepsilon^{2p}+c_{15}\varepsilon^p+c_{16},\\ \Phi_6(\varepsilon)=c_{17}\varepsilon^p,\end{cases}$$

and

$$\begin{cases}c_1=\cos\dfrac{3p\pi}{2}, c_2=b_1\cos p\pi,\\ c_3=(b_2+b_6)\cos\dfrac{p\pi}{2}, c_4=b_3+b_7,\\ c_5=\sin\dfrac{3p\pi}{2}, c_6=b_1\sin p\pi,\\ c_7=(b_2-b_6)\sin\dfrac{p\pi}{2}, c_8=b_4\cos\dfrac{p\pi}{2},\\ c_9=b_5, c_{10}=\sin\dfrac{3p\pi}{2},\\ c_{11}=b_1\sin p\pi, c_{12}=(b_2+b_6)\sin\dfrac{p\pi}{2},\\ c_{13}=\cos\dfrac{3p\pi}{2}, c_{14}=b_1\cos p\pi,\\ c_{15}=(b_2-b_6)\cos\dfrac{p\pi}{2}, c_{16}=b_3-b_7,\\ c_{17}=b_4\sin\dfrac{p\pi}{2}.\end{cases}$$

By (8), we have

$$\begin{cases}\cos\varepsilon\vartheta=-\dfrac{\Phi_3(\varepsilon)\Phi_5(\varepsilon)+\Phi_2(\varepsilon)\Phi_6(\varepsilon)}{\Phi_1(\varepsilon)\Phi_5(\varepsilon)+\Phi_2(\varepsilon)\Phi_4(\varepsilon)},\\ \sin\varepsilon\vartheta=\dfrac{\Phi_1(\varepsilon)\Phi_6(\varepsilon)-\Phi_3(\varepsilon)\Phi_4(\varepsilon)}{\Phi_1(\varepsilon)\Phi_5(\varepsilon)+\Phi_2(\varepsilon)\Phi_4(\varepsilon)}.\end{cases} \tag{9}$$

In view of $\cos^2\varepsilon\vartheta+\sin^2\varepsilon\vartheta=1$, it follows from (9) that

$$\begin{aligned}&[\Phi_3(\varepsilon)\Phi_5(\varepsilon)+\Phi_2(\varepsilon)\Phi_6(\varepsilon)]^2+[\Phi_1(\varepsilon)\Phi_6(\varepsilon)-\Phi_3(\varepsilon)\Phi_4(\varepsilon)]^2\\ &=[\Phi_1(\varepsilon)\Phi_5(\varepsilon)+\Phi_2(\varepsilon)\Phi_4(\varepsilon)]^2,\end{aligned}$$

which results in

$$d_1\varepsilon^{12p}+d_2\varepsilon^{11p}+d_3\varepsilon^{10p}+d_4\varepsilon^{9p}+d_5\varepsilon^{8p}+d_6\varepsilon^{7p}+d_7\varepsilon^{6p}$$

$$+d_8\varepsilon^{5p}+d_9\varepsilon^{4p}+d_{10}\varepsilon^{3p}+d_{11}\varepsilon^{2p}+d_{12}\varepsilon^p+d_{13}=0, \tag{10}$$

where

$$\begin{cases} d_1 = (c_1c_{13} + c_5c_{10})^2, \\ d_2 = 2(c_1c_{13} + c_5c_{10})(c_1c_{14} + c_2c_{13} + c_5c_{11} + c_6c_{10}) \\ d_3 = (c_1c_{14} + c_2c_{13} + c_5c_{11} + c_6c_{10})^2 - 2(c_1c_{13} + c_5c_{10}) \\ \times(c_1c_{15} + c_2c_{14} + c_3c_{13} + c_5c_{12} + c_6c_{11} + c_7c_{10}), \\ d_4 = 2(c_1c_{13} + c_5c_{10})(c_1c_{16} + c_2c_{15} + c_3c_{14} + c_6c_{12} + c_7c_{11}) \\ +2(c_1c_{14} + c_2c_{13} + c_5c_{11} + c_6c_{10})(c_1c_{15} + c_2c_{14} + c_3c_{13} \\ +c_5c_{12} + c_6c_{11} + c_7c_{10}), \\ d_5 = (c_1c_{15} + c_2c_{14} + c_3c_{13} + c_5c_{12} + c_6c_{11} + c_7c_{10})^2 \\ +2(c_1c_{13} + c_5c_{10})(c_2c_{16} + c_3c_{15} + c_4c_{14} + c_7c_{12}) \\ +2(c_1c_{14} + c_2c_{13} + c_5c_{11} + c_6c_{10})(c_1c_{16} + c_2c_{15} \\ +c_3c_{14} + c_4c_{13} + c_6c_{12} + c_7c_{11}) - (c_1c_{17} - c_8c_{10})^2 - (c_8c_{13})^2, \\ d_6 = 2(c_1c_{13} + c_5c_{10})(c_3c_{16} + c_4c_{15}) + 2(c_1c_{14} + c_2c_{13} \\ +c_5c_{11} + c_6c_{10})(c_2c_{16} + c_3c_{15} + c_4c_{14} + c_7c_{12}) \\ +2(c_1c_{15} + c_2c_{14} + c_3c_{13} + c_5c_{12} + c_6c_{11} + c_7c_{10}) \\ \times(c_1c_{16} + c_2c_{15} + c_3c_{14} + c_4c_{13} + c_6c_{12} + c_7c_{11}) \\ -2(c_1c_{17} - c_8c_{10})(c_2c_{17} - c_9c_{10} - c_8c_{11}) \\ -2c_8c_{13}(c_9c_{13} + c_8c_{14} + c_6c_{17}), \\ d_7 = (c_1c_{16} + c_2c_{15} + c_3c_{14} + c_4c_{13} + c_6c_{12} + c_7c_{11})^2 \\ +2c_4c_{16}(c_1c_{13} + c_5c_{10}) + 2(c_3c_{16} + c_4c_{15}) \\ \times(c_1c_{14} + c_2c_{13} + c_5c_{11} + c_6c_{10}) + 2(c_1c_{15} + c_2c_{14} + c_3c_{13} \\ +c_5c_{12} + c_6c_{11} + c_7c_{10})(c_2c_{16} + c_3c_{15} + c_4c_{14} + c_7c_{12}) \\ -(c_2c_{17} - c_9c_{10} - c_8c_{11})^2 - (c_9c_{13} + c_8c_{14} + c_6c_{17})^2 \\ -2c_8c_{13}(c_9c_{14} + c_8c_{15} + c_7c_{17}), \\ d_8 = 2c_4c_6(c_1c_{14} + c_2c_{13} + c_5c_{11} + c_6c_{10}) + 2(c_1c_{15} + c_2c_{14} \\ +c_3c_{13} + c_5c_{12} + c_6c_{11} + c_7c_{10})(c_3c_{16} + c_4c_{15}) \\ +2(c_1c_{16} + c_2c_{15} + c_3c_{14} + c_4c_{13} + c_6c_{12} + c_7c_{11}) \\ \times(c_2c_{16} + c_3c_{15} + c_4c_{14} + c_7c_{12}) - 2(c_1c_{17} \\ -c_8c_{10})(c_4c_{17} - c_9c_{12}) - 2(c_2c_{17} - c_9c_{10} - c_8c_{11}) \\ \times(c_3c_{17} - c_9c_{11} - c_8c_{12}) - 2c_8c_{13}(c_9c_{15} + c_8c_{16}) \\ -2(c_9c_{13} + c_8c_{14} + c_6c_{17})(c_9c_{14} + c_8c_{15} + c_7c_{17}), \end{cases}$$

and

$$\begin{cases} d_9 = (c_2c_{16} + c_3c_{15} + c_4c_{14} + c_7c_{12})^2 + 2(c_3c_{16} + c_4c_{15}) \\ \times(c_1c_{16} + c_2c_{15} + c_3c_{14} + c_4c_{13} + c_6c_{12} + c_7c_{11}) \\ +2c_4c_{16}(c_1c_{15} + c_2c_{14} + c_3c_{13} + c_5c_{12} + c_6c_{11} + c_7c_{10}) \\ -(c_3c_{17} - c_9c_{11} - c_8c_{12})^2 - 2(c_2c_{17} - c_9c_{10} - c_8c_{11}) \\ \times(c_4c_{17} - c_9c_{12}) - (c_9c_{14} + c_8c_{15} + c_7c_{17})^2 \\ -2(c_9c_{15} + c_8c_{16})(c_9c_{13} + c_8c_{14} + c_6c_{17}), \\ d_{10} = 2c_4c_{16}(c_1c_{16} + c_2c_{15} + c_3c_{14} + c_4c_{13} + c_6c_{12} + c_7c_{11}) \\ +2(c_3c_{16} + c_4c_{15})(c_2c_{16} + c_3c_{15} + c_4c_{14} + c_7c_{12}) \\ -2(c_3c_{17} - c_9c_{11} - c_8c_{12})(c_4c_{17} - c_9c_{12}) \\ -2c_9c_{16}(c_9c_{13} + c_8c_{14} + c_6c_{17}) - 2(c_9c_{15} + c_8c_{16}) \\ \times(c_9c_{14} + c_8c_{15} + c_7c_{17}), \\ d_{11} = (c_3c_{16} + c_4c_{15})^2 + 2c_4c_{16}(c_2c_{16} + c_3c_{15} + c_4c_{14} + c_7c_{12}) \\ -(c_4c_{17} - c_9c_{12})^2 \\ -(c_9c_{15} + c_8c_{16})^2 - 2c_9c_{16}(c_9c_{14} + c_8c_{15} + c_7c_{17}), \\ d_{12} = 2c_4c_{16}(c_3c_{16} + c_4c_{15}) - 2c_9c_{16}(c_9c_{15} + c_8c_{16}), \\ d_{13} = (c_4c_{16})^2 - (c_9c_{16})^2. \end{cases}$$

Set

$$M_1(\varepsilon) = d_1\varepsilon^{12p} + d_2\varepsilon^{11p} + d_3\varepsilon^{10p} + d_4\varepsilon^{9p} + d_5\varepsilon^{8p} + d_6\varepsilon^{7p} + d_7\varepsilon^{6p} + d_8\varepsilon^{5p} + d_9\varepsilon^{4p} + d_{10}\varepsilon^{3p} + d_{11}\varepsilon^{2p} + d_{12}\varepsilon^{p} + d_{13}. \tag{11}$$

and

$$M_2(\varepsilon) = d_1\varepsilon^{12} + d_2\varepsilon^{11p} + d_3\varepsilon^{10} + d_4\varepsilon^9 + d_5\varepsilon^8 + d_6\varepsilon^7 + d_7\varepsilon^6 + d_8\varepsilon^5 + d_9\varepsilon^4 + d_{10}\varepsilon^3 + d_{11}\varepsilon^2 + d_{12}\varepsilon + d_{13}.$$

Lemma 2. *(1) Assume that $b_3 + b_5 + b_7 \neq 0$ and $d_k > 0 (k = 1, 2, 3, \cdots, 13)$, then Eq.(4) owns no root involving zero real part. (2) Assume that $d_{13} > 0$ and $\exists\ \varepsilon_0 > 0$ satisfying $M_2(\varepsilon_0) < 0$, then Eq.(4) owns at least two couples of purely imaginary roots.*

Proof. (1) It follows from (11) that

$$\begin{aligned}\frac{dM_1(\varepsilon)}{d\varepsilon} &= 12pd_1\varepsilon^{12p-1} + 11pd_2\varepsilon^{11p-1} + 10pd_3\varepsilon^{10p-1} + 9pd_4\varepsilon^{9p-1}\\ &+8pd_5\varepsilon^{8p-1} + 7pd_6\varepsilon^{7p-1} + 6pd_7\varepsilon^{6p-1} + 5pd_8\varepsilon^{5p-1}\\ &+4pd_9\varepsilon^{4p-1} + 3pd_{10}\varepsilon^{3p-1} + 2pd_{11}\varepsilon^{2p-1} + pd_{12}\varepsilon^{p-1}.\end{aligned}$$

Since $d_l > 0 (l =, 2, \cdots, 12)$, one gains $\frac{dM_1(\varepsilon)}{d\varepsilon} > 0, \forall\ \varepsilon > 0$. In addition, $M_1(0) = d_{13} > 0$, one knows that Eq.(10) admits no positive real root. According to $b_3 + b_5 + b_7 \neq 0$, we can know that $s = 0$ is not the root of (4), which completes the proof of (1).

(2) Clearly, $M_2(0) = d_{13} > 0, M_2(\varepsilon_0) < 0 (\varepsilon_0 > 0)$ and $\lim_{\varepsilon\to+\infty} \frac{M_2(\varepsilon)}{d\varepsilon} = +\infty$, then there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $\varepsilon_2 \in (\varepsilon_0, +\infty)$ satisfying $M_2(\varepsilon_1) = M_2(\varepsilon_2) = 0$, then Eq.(10) owns at least two positive real roots. Then, (4) owns at least two couples of purely imaginary roots, which completes the proof of (2). □

Suppose that Eq.(10) owns twelve positive real roots $\varepsilon_i (i = 1, 2, \cdots, 12)$. By (9), we get

$$\vartheta_j^k = \frac{1}{\varepsilon_j}\left[\arccos\left(-\frac{\Phi_3(\varepsilon_j)\Phi_5(\varepsilon_j) + \Phi_2(\varepsilon_j)\Phi_6(\varepsilon_j)}{\Phi_1(\varepsilon_j)\Phi_5(\varepsilon_j) + \Phi_2(\varepsilon_j)\Phi_4(\varepsilon_j)}\right) + 2l\pi\right],$$

where $l = 0, 1, 2, \cdots, j = 1, 2, \cdots, 12$. Let

$$\vartheta_0 = \min_{j=1,2,\cdots,12}\{\vartheta_j^0\}, \varepsilon_0 = \varepsilon|_{\vartheta=\vartheta_0}.$$

Now we give the following condition:

$$(Q_3)\ T_{1R}T_{2R} + T_{1I}T_{2I} > 0,$$

where

$$\begin{cases}
\begin{aligned}
T_{1R} &= 3p\varepsilon_0^{3p-1}\cos\frac{(3p-1)\pi}{2} + 2pb_1\varepsilon_0^{2p-1}\cos\frac{(2p-1)\pi}{2}\\
&+pb_2\varepsilon_0^{p-1}\cos\frac{(p-1)\pi}{2} + pb_4\varepsilon_0^{p-1}\cos\frac{(p-1)\pi}{2}\cos\varepsilon_0\vartheta_0\\
&+pb_4\varepsilon_0^{p-1}\sin\frac{(p-1)\pi}{2}\sin\varepsilon_0\vartheta_0 + pb_6\varepsilon_0^{p-1}\cos\frac{(p-1)\pi}{2}\cos 2\varepsilon_0\vartheta_0\\
&+pb_6\varepsilon_0^{p-1}\sin\frac{(p-1)\pi}{2}\sin 2\varepsilon_0\vartheta_0,\\
T_{1I} &= 3p\varepsilon_0^{3p-1}\sin\frac{(3p-1)\pi}{2} + 2pb_1\varepsilon_0^{2p-1}\sin\frac{(2p-1)\pi}{2}\\
&+pb_2\varepsilon_0^{p-1}\sin\frac{(p-1)\pi}{2} - pb_4\varepsilon_0^{p-1}\cos\frac{(p-1)\pi}{2}\sin\varepsilon_0\vartheta_0\\
&+pb_4\varepsilon_0^{p-1}\sin\frac{(p-1)\pi}{2}\cos\varepsilon_0\vartheta_0 - pb_6\varepsilon_0^{p-1}\cos\frac{(p-1)\pi}{2}\sin 2\varepsilon_0\vartheta_0\\
&+pb_6\varepsilon_0^{p-1}\sin\frac{(p-1)\pi}{2}\cos 2\varepsilon_0\vartheta_0,\\
T_{2R} &= \left(b_4\varepsilon_0^p\cos\frac{p\pi}{2} + b_5\right)\varepsilon_0\sin\varepsilon_0\vartheta_0 - \left(b_4\varepsilon_0^p\sin\frac{p\pi}{2}\right)\varepsilon_0\cos\varepsilon_0\vartheta_0\\
&+\left(b_6\varepsilon_0^p\cos\frac{p\pi}{2} + b_7\right)\varepsilon_0\sin 2\varepsilon_0\vartheta_0 - \left(b_6\varepsilon_0^p\sin\frac{p\pi}{2}\right)\varepsilon_0\cos 2\varepsilon_0\vartheta_0,\\
T_{2I} &= \left(b_4\varepsilon_0^p\cos\frac{p\pi}{2} + b_5\right)\varepsilon_0\cos\varepsilon_0\vartheta_0 + \left(b_4\varepsilon_0^p\sin\frac{p\pi}{2}\right)\varepsilon_0\sin\varepsilon_0\vartheta_0\\
&+\left(b_6\varepsilon_0^p\cos\frac{p\pi}{2} + b_7\right)\varepsilon_0\cos 2\varepsilon_0\vartheta_0 + \left(b_6\varepsilon_0^p\sin\frac{p\pi}{2}\right)\varepsilon_0\sin 2\varepsilon_0\vartheta_0.
\end{aligned}
\end{cases}$$

Lemma 3. *Let $s(\vartheta) = \varsigma_1(\vartheta) + i\varsigma_2(\vartheta)$ be the root of Eq.(4) near $\vartheta = \vartheta_0$ satisfying $\varsigma_1(\vartheta_0) = 0, \varsigma_2(\vartheta_0) = \varepsilon_0$, then $Re\left(\frac{ds}{d\vartheta}\right)\Big|_{\vartheta=\vartheta_0,\varepsilon=\varepsilon_0} > 0$.*

Proof. It follows from Eq.(4) that

$$\begin{aligned}
&\left(3ps^{3p-1} + 2pb_1s^{2p-1} + pb_2s^{p-1}\right)\frac{ds}{d\vartheta} + pb_4s^{p-1}e^{-s\vartheta}\frac{ds}{d\vartheta}\\
&-e^{-s\vartheta}\left(\frac{ds}{d\vartheta}\vartheta + s\right)(b_4s^p + b_5) + pb_6s^{p-1}e^{-2s\vartheta}\frac{ds}{d\vartheta}\\
&-2e^{-2s\vartheta}(b_6s^p + b_7)\left(\frac{ds}{d\vartheta}\theta + s\right) = 0,
\end{aligned}$$

which results in

$$\left(\frac{ds}{d\vartheta}\right)^{-1} = \frac{T_1(s)}{T_2(s)} - \frac{\vartheta}{s},$$

where

$$\begin{cases}
T_1(s) = 3ps^{3p-1} + 2pb_1s^{2p-1} + pb_2s^{p-1}\\
\quad +pb_4s^{p-1}e^{-s\vartheta} + pb_6s^{p-1}e^{-2s\vartheta},\\
T_2(s) = se^{-s\vartheta}(b_4s^p + b_5) + 2se^{-2s\vartheta}(b_6s^p + b_7).
\end{cases}$$

Then

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_0,\varepsilon=\varepsilon_0} = \mathrm{Re}\left[\frac{T_1(s)}{T_2(s)}\right]_{\vartheta=\vartheta_0,\varepsilon=\varepsilon_0} = \frac{T_{1R}T_{2R} + T_{1I}T_{2I}}{T_{2R}^2 + T_{2I}^2}.$$

In view of (Q_3), we gain

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_0,\varepsilon=\varepsilon_0} > 0.$$

The proof is completed. □

Applying Lemma 1, the following assertion can be lightly is acquired.

Theorem 1. *If (Q_1)-(Q_3) are fulfilled, then $W(w_{1*}, w_{2*}, w_{3*})$ of system (2) keeps locally asymptotically stability when $\vartheta \in [0, \vartheta_0)$ and a cluster of Hopf bifurcations of system (2) happen near $W(w_{1*}, w_{2*}, w_{3*})$ when $\vartheta = \vartheta_0$.*

Remark 1. *In this paper, we only deal with the effect of the delay on the stability and Hopf bifurcation of model (2). As for the effect of the order "p" on the stability and Hopf bifurcation of model (2), we leave it for future works.*

4. Bifurcation control via extended hybrid controller

In this section, we are to use a novel extended hybrid controller which includes a common hybrid controller that consists of state feedback and parameter perturbation and a PD controller to control the stability and Hopf bifurcation for model (2). Following the ideas of [15,50,51], we get the following fractional-order controlled delayed turbidostat model:

$$\begin{cases}
\dfrac{d^pw_1(t)}{dt^p} = [a + \kappa_1w_2(t-\vartheta) + \kappa_2w_3(t-\vartheta)](w_0 - w_1) - \dfrac{w_2}{\gamma_1}g_1(w_1) - \dfrac{w_3}{\gamma_2}g_2(w_1),\\
\dfrac{d^pw_2(t)}{dt^p} = \rho_1\{w_2[g_1(w_1) - (a + \kappa_1w_2(t-\vartheta) + \kappa_2w_3(t-\vartheta))]\} + \rho_2[w_2(t-\vartheta) - w_2],\\
\dfrac{d^pw_3(t)}{dt^p} = w_3[g_2(w_1) - (a + \kappa_1w_2(t-\vartheta) + \kappa_2w_3(t-\vartheta))]\\
\quad +\tau_p[w_3 - w_{3*}] + \tau_d\dfrac{d[^pw_3 - w_{3*}]}{dt^p},
\end{cases} \tag{12}$$

where ρ_1, ρ_2 stands for feedback gain parameters and $\tau_p, \tau_d \neq 1$ stand for the proportional control parameter and the derivative parameter, respectively. Clearly, model (12) and model (2) admit the same positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*})$. The linear system of system (2) at $W(w_{1*}, w_{2*}, w_{3*})$ owns the following form:

$$\mathcal{D}^p w(t) = C_1 w(t) + C_2 w(t-\theta), \tag{13}$$

where

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, C_1 = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & 0 \\ \alpha_6 & 0 & \alpha_7 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & \beta_1 & \beta_2 \\ 0 & \beta_3 & \beta_4 \\ 0 & \beta_5 & \beta_6 \end{bmatrix},$$

and

$$\begin{cases} \alpha_1 = -\left[a + \kappa_1 w_{2*} + \kappa_{3*} + \dfrac{g_1^{'}(w_{1*})}{\gamma_1} + \dfrac{g_2^{'}(w_{1*})}{\gamma_2}\right], \\ \alpha_2 = -\dfrac{g_1(w_{1*}) + g_1^{'}(w_{1*})}{\gamma_1}, \alpha_3 = -\dfrac{g_2(w_{1*}) + g_2^{'}(w_{1*})}{\gamma_2}, \\ \alpha_4 = \rho_1 w_{2*} g_1^{'}(w_{1*}), \alpha_5 = -\rho_2, \\ \alpha_6 = \dfrac{w_{3*} g_2^{'}(w_{1*})}{1-\tau_d}, \alpha_7 = \dfrac{\tau_p}{1-\tau_d}, \\ \beta_1 = \kappa_1(w_0 - w_{1*}), \beta_2 = \kappa_2(w_0 - w_{1*}), \\ \beta_3 = -\kappa_1 w_{2*} + \rho_2, \beta_4 = -\kappa_2 w_{2*}, \\ \beta_5 = -\dfrac{\kappa_1 w_{3*}}{1-\tau_d}, \beta_6 = -\dfrac{\kappa_2 w_{3*}}{1-\tau_d}. \end{cases}$$

The characteristic equation of system (13) reads as

$$\det \begin{bmatrix} s^p - \alpha_1 & -\alpha_2 - \beta_1 e^{-s\vartheta} & -\alpha_3 - \beta_2 e^{-s\vartheta} \\ -\alpha_4 & s^p - \alpha_5 - \beta_3 e^{-s\vartheta} & -\beta_4 e^{-s\vartheta} \\ -\alpha_5 & -\beta_5 e^{-s\vartheta} & s^p - \alpha_7 - \beta_6 e^{-s\vartheta} \end{bmatrix} = 0,$$

which results in

$$s^{3p} + e_1 s^{2p} + e_2 s^p + e_3 + (e_4 s^p + e_5) e^{-s\vartheta} + (e_6 s^p + e_7) e^{-2s\vartheta} = 0, \tag{14}$$

where

$$\begin{cases} e_1 = \alpha_7\beta_3 - \beta_3 - 2\alpha_1 - \alpha_5, \\ e_2 = \alpha_1\alpha_7 + \alpha_5\alpha_7 + \alpha_1\alpha_5 - \alpha_3\alpha_5 - \alpha_2\alpha_4 + \alpha_7\beta_3 + \alpha_1\beta_3 + \beta_6(\alpha_1 + \alpha_5), \\ e_3 = \alpha_3\alpha_5^2 - \alpha_1\alpha_5\alpha_7 + \alpha_4\alpha_7\beta_1 + \alpha_2\alpha_4\alpha_7, \\ e_4 = -(\alpha_5\beta_2 + \alpha_4\beta_1), \\ e_5 = -(\alpha_1\alpha_5\beta_6 + \alpha_1\alpha_7\beta_3 + \alpha_2\alpha_5\beta_4 + \alpha_3\alpha_4\beta_5 - \alpha_3\alpha_5\beta_3 - \alpha_2\alpha_4\beta_6), \\ e_6 = \beta_3\beta_6 - \beta_4\beta_5, \\ e_7 = \alpha_1\beta_4\beta_5 - \alpha_1\beta_3\beta_6 - \alpha_4\beta_2\beta_5 + \alpha_5\beta_2\beta_3 + \alpha_4\beta_1\beta_6. \end{cases}$$

When $\vartheta = 0$, then Eq.(14) has the following form:

$$\lambda^3 + e_1\lambda^2 + (e_2 + e_4 + e_6)\lambda + e_3 + e_5 + e_7 = 0. \tag{15}$$

If

$$(Q_4) \begin{cases} \Lambda_1 = e_1 > 0, \\ \Lambda_2 = \det \begin{bmatrix} e_1 & 1 \\ e_3 + e_5 + e_7 & e_2 + e_4 + e_6 \end{bmatrix}, \\ \Lambda_3 = (e_3 + e_5 + e_7)\Delta_2 > 0 \end{cases}$$

is fulfilled, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq.(15) obey $|\arg(\lambda_1)| > \frac{p\pi}{2}, |\arg(\lambda_2)| > \frac{p\pi}{2}, |\arg(\lambda_3)| > \frac{p\pi}{2}$. In view of Lemma 1, we can understand that the positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*})$ of system (12) with the delay $\vartheta = 0$ maintains locally asymptotically stability.

By (14), we have

$$(s^{3p} + e_1 s^{2p} + e_2 s^p + e_3) e^{s\vartheta} + (e_4 s^p + e_5) + (e_6 s^p + e_7) e^{-s\vartheta} = 0. \tag{16}$$

Suppose that $s = i\epsilon = \epsilon\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ is the root of Eq.(16). Then by Eq.(16), one gains

$$\begin{aligned}&\left[\epsilon^{3p}\left(\cos\frac{3p\pi}{2} + i\sin\frac{3p\pi}{2}\right) + e_1\epsilon^{2p}(\cos p\pi + i\sin p\pi)\right.\\ &\left.+e_2\epsilon^p\left(\cos\frac{p\pi}{2} + i\sin\frac{p\pi}{2}\right) + e_3\right](\cos\epsilon\vartheta + i\sin\epsilon\vartheta)\\ &+\left[e_4\epsilon^p\left(\cos\frac{p\pi}{2} + i\sin\frac{p\pi}{2}\right) + e_5\right] + \left[e_6\epsilon^p\left(\cos\frac{p\pi}{2} + i\sin\frac{p\pi}{2}\right) + e_7\right]\\ &\times(\cos\epsilon\vartheta - i\sin\epsilon\vartheta) = 0.\end{aligned} \tag{17}$$

Then, it follows from (17) that

$$\begin{cases}\Psi_1(\epsilon)\cos\epsilon\vartheta - \Psi_2(\epsilon)\sin\epsilon\vartheta = -\Psi_3(\epsilon),\\ \Psi_4(\epsilon)\cos\epsilon\vartheta + \Psi_5(\epsilon)\sin\epsilon\vartheta = -\Psi_6(\epsilon),\end{cases} \tag{18}$$

where

$$\begin{cases}\Psi_1(\epsilon) = f_1\epsilon^{3p} + f_2\epsilon^{2p} + f_3\epsilon^p + f_4,\\ \Psi_2(\epsilon) = f_5\epsilon^{3p} + f_6\epsilon^{2p} + f_7\epsilon^p,\\ \Psi_3(\epsilon) = f_8\epsilon^p + f_9,\\ \Psi_4(\epsilon) = f_{10}\epsilon^{3p} + f_{11}\epsilon^{2p} + f_{12}\epsilon^p,\\ \Psi_5(\epsilon) = f_{13}\epsilon^{3p} + f_{14}\epsilon^{2p} + f_{15}\epsilon^p + f_{16},\\ \Psi_6(\epsilon) = f_{17}\epsilon^p,\end{cases}$$

and

$$\begin{cases}f_1 = \cos\frac{3p\pi}{2},\\ f_2 = e_1\cos p\pi,\\ f_3 = (e_2 + e_6)\cos\frac{p\pi}{2},\\ f_4 = e_3 + e_7,\\ f_5 = \sin\frac{3p\pi}{2},\\ f_6 = e_1\sin p\pi,\\ f_7 = (e_2 - e_6)\sin\frac{p\pi}{2},\\ f_8 = e_4\cos\frac{p\pi}{2},\\ f_9 = e_5,\\ f_{10} = \sin\frac{3p\pi}{2},\\ f_{11} = e_1\sin p\pi,\\ f_{12} = (e_2 + e_6)\sin\frac{p\pi}{2},\\ f_{13} = \cos\frac{3p\pi}{2},\\ f_{14} = e_1\cos p\pi,\\ f_{15} = (e_2 - e_6)\cos\frac{p\pi}{2},\\ f_{16} = e_3 - e_7,\\ f_{17} = e_4\sin\frac{p\pi}{2}.\end{cases}$$

By (18), we have

$$\begin{cases}\cos\epsilon\vartheta = -\dfrac{\Psi_3(\epsilon)\Psi_5(\epsilon) + \Psi_2(\epsilon)\Psi_6(\epsilon)}{\Psi_1(\epsilon)\Psi_5(\epsilon) + \Psi_2(\epsilon)\Psi_4(\epsilon)},\\ \sin\epsilon\vartheta = \dfrac{\Psi_1(\epsilon)\Psi_6(\epsilon) - \Psi_3(\epsilon)\Psi_4(\epsilon)}{\Psi_1(\epsilon)\Psi_5(\epsilon) + \Psi_2(\epsilon)\Psi_4(\epsilon)}.\end{cases} \tag{19}$$

In view of $\cos^2 \epsilon\vartheta + \sin^2 \epsilon\vartheta = 1$, it follows from (19) that

$$\begin{aligned}&[\Psi_3(\epsilon)\Psi_5(\epsilon) + \Psi_2(\epsilon)\Psi_6(\epsilon)]^2 + [\Psi_1(\epsilon)\Psi_6(\epsilon) - \Psi_3(\epsilon)\Psi_4(\epsilon)]^2\\ &= [\Psi_1(\epsilon)\Psi_5(\epsilon) + \Psi_2(\epsilon)\Psi_4(\epsilon)]^2,\end{aligned}$$

which results in

$$\begin{aligned}&g_1\epsilon^{12p} + g_2\epsilon^{11p} + g_3\epsilon^{10p} + g_4\epsilon^{9p} + g_5\epsilon^{8p} + g_6\epsilon^{7p} + g_7\epsilon^{6p}\\ &+g_8\epsilon^{5p} + g_9\epsilon^{4p} + g_{10}\epsilon^{3p} + g_{11}\epsilon^{2p} + g_{12}\epsilon^{p} + g_{13} = 0,\end{aligned} \tag{20}$$

where

$$\left\{\begin{aligned}
&g_1 = (f_1f_{13} + f_5f_{10})^2,\\
&g_2 = 2(f_1f_{13} + f_5f_{10})(f_1f_{14} + f_2f_{13} + f_5f_{11} + f_6f_{10})\\
&g_3 = (f_1f_{14} + f_2f_{13} + f_5f_{11} + f_6f_{10})^2 - 2(f_1f_{13} + f_5f_{10})\\
&\times(f_1f_{15} + f_2f_{14} + f_3f_{13} + f_5f_{12} + f_6f_{11} + f_7f_{10}),\\
&g_4 = 2(f_1f_{13} + f_5f_{10})(f_1f_{16} + f_2f_{15} + f_3f_{14} + f_6f_{12} + f_7f_{11})\\
&+2(f_1f_{14} + f_2f_{13} + f_5f_{11} + f_6f_{10})(f_1f_{15} + f_2f_{14} + f_3f_{13}\\
&+f_5f_{12} + f_6f_{11} + f_7f_{10}),\\
&g_5 = (f_1f_{15} + f_2f_{14} + f_3f_{13} + f_5f_{12} + f_6f_{11} + f_7f_{10})^2\\
&+2(f_1f_{13} + f_5f_{10})(f_2f_{16} + f_3f_{15} + f_4f_{14} + f_7f_{12})\\
&+2(f_1f_{14} + f_2f_{13} + f_5f_{11} + f_6f_{10})(f_1f_{16} + f_2f_{15}\\
&+f_3f_{14} + f_4f_{13} + f_6f_{12} + f_7f_{11}) - (f_1f_{17} - f_8f_{10})^2 - (f_8f_{13})^2,\\
&g_6 = 2(f_1f_{13} + f_5f_{10})(f_3f_{16} + f_4f_{15}) + 2(f_1f_{14} + f_2f_{13}\\
&+f_5f_{11} + f_6f_{10})(f_2f_{16} + f_3f_{15} + f_4f_{14} + f_7f_{12})\\
&+2(f_1f_{15} + f_2f_{14} + f_3f_{13} + f_5f_{12} + f_6f_{11} + f_7f_{10})\\
&\times(f_1f_{16} + f_2f_{15} + f_3f_{14} + f_4f_{13} + f_6f_{12} + f_7f_{11})\\
&-2(f_1f_{17} - f_8f_{10})(f_2f_{17} - f_9f_{10} - f_8f_{11})\\
&-2f_8f_{13}(f_9f_{13} + f_8f_{14} + f_6f_{17}),\\
&g_7 = (f_1f_{16} + f_2f_{15} + f_3f_{14} + f_4f_{13} + f_6f_{12} + f_7f_{11})^2\\
&+2f_4f_{16}(f_1f_{13} + f_5f_{10}) + 2(f_3f_{16} + f_4f_{15})\\
&\times(f_1f_{14} + f_2f_{13} + f_5f_{11} + f_6f_{10}) + 2(f_1f_{15} + f_2f_{14} + f_3f_{13}\\
&+f_5f_{12} + f_6f_{11} + f_7f_{10})(f_2f_{16} + f_3f_{15} + f_4f_{14} + f_7f_{12})\\
&-(f_2f_{17} - f_9f_{10} - f_8f_{11})^2 - (f_9f_{13} + f_8f_{14} + f_6f_{17})^2\\
&-2f_8f_{13}(f_9f_{14} + f_8f_{15} + f_7f_{17}),\\
&g_8 = 2f_4f_6(f_1f_{14} + f_2f_{13} + f_5f_{11} + f_6f_{10}) + 2(f_1f_{15} + f_2f_{14}\\
&+f_3f_{13} + f_5f_{12} + f_6f_{11} + f_7f_{10})(f_3f_{16} + f_4f_{15})\\
&+2(f_1f_{16} + f_2f_{15} + f_3f_{14} + f_4f_{13} + f_6f_{12} + f_7f_{11})\\
&\times(f_2f_{16} + f_3f_{15} + f_4f_{14} + f_7f_{12}) - 2(f_1f_{17}\\
&-f_8f_{10})(f_4f_{17} - f_9f_{12}) - 2(f_2f_{17} - f_9f_{10} - f_8f_{11})\\
&\times(f_3f_{17} - f_9f_{11} - f_8f_{12}) - 2f_8f_{13}(f_9f_{15} + f_8f_{16})\\
&-2(f_9f_{13} + f_8f_{14} + f_6f_{17})(f_9f_{14} + f_8f_{15} + f_7f_{17}),\\
&g_9 = (f_2f_{16} + f_3f_{15} + f_4f_{14} + f_7f_{12})^2 + 2(f_3f_{16} + f_4f_{15})\\
&\times(f_1f_{16} + f_2f_{15} + f_3f_{14} + f_4f_{13} + f_6f_{12} + f_7f_{11})\\
&+2f_4f_{16}(f_1f_{15} + f_2f_{14} + f_3f_{13} + f_5f_{12} + f_6f_{11} + f_7f_{10})\\
&-(f_3f_{17} - f_9f_{11} - f_8f_{12})^2 - 2(f_2f_{17} - f_9f_{10} - f_8f_{11})\\
&\times(f_4f_{17} - f_9f_{12}) - (f_9f_{14} + f_8f_{15} + f_7f_{17})^2\\
&-2(f_9f_{15} + f_8f_{16})(f_9f_{13} + f_8f_{14} + f_6f_{17}),\\
&g_{10} = 2f_4f_{16}(f_1f_{16} + f_2f_{15} + f_3f_{14} + f_4f_{13} + f_6f_{12} + f_7f_{11})\\
&+2(f_3f_{16} + f_4f_{15})(f_2f_{16} + f_3f_{15} + f_4f_{14} + f_7f_{12})\\
&-2(f_3f_{17} - f_9f_{11} - f_8f_{12})(f_4f_{17} - f_9f_{12})\\
&-2f_9f_{16}(f_9f_{13} + f_8f_{14} + f_6f_{17}) - 2(f_9f_{15} + f_8f_{16})\\
&\times(f_9f_{14} + f_8f_{15} + f_7f_{17}),
\end{aligned}\right.$$

and

$$\begin{cases} g_{11} = (f_3f_{16} + f_4f_{15})^2 + 2f_4f_{16}(f_2f_{16} + f_3f_{15} + f_4f_{14} + f_7f_{12}) \\ \quad -(f_4f_{17} - f_9f_{12})^2 \\ \quad -(f_9f_{15} + f_8f_{16})^2 - 2f_9f_{16}(f_9f_{14} + f_8f_{15} + f_7f_{17}), \\ g_{12} = 2f_4f_{16}(f_3f_{16} + f_4f_{15}) - 2f_9f_{16}(f_9f_{15} + f_8f_{16}), \\ g_{13} = (f_4f_{16})^2 - (f_9f_{16})^2. \end{cases}$$

Set

$$\begin{aligned} &N_1(\epsilon) = g_1\epsilon^{12p} + g_2\epsilon^{11p} + g_3\epsilon^{10p} + g_4\epsilon^{9p} + d_5\epsilon^{8p} + g_6\epsilon^{7p} \\ &+g_7\epsilon^{6p} + g_8\epsilon^{5p} + g_9\epsilon^{4p} + g_{10}\epsilon^{3p} + g_{11}\epsilon^{2p} + g_{12}\epsilon^{p} + g_{13}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} &N_2(\epsilon) = g_1\epsilon^{12} + g_2\epsilon^{11} + g_3\epsilon^{10} + g_4\epsilon^{9} + d_5\epsilon^{8} + g_6\epsilon^{7} \\ &+g_7\epsilon^{6} + g_8\epsilon^{5} + g_9\epsilon^{4} + g_{10}\epsilon^{3} + g_{11}\epsilon^{2} + g_{12}\epsilon + g_{13}, \end{aligned}$$

Lemma 4. *(1) Assume that $e_3 + e_5 + e_7 \neq 0$ and $g_k > 0 (k = 1, 2, 3, \cdots, 13)$, then Eq.(14) owns no root involving zero real part. (2) Assume that $g_{13} > 0$ and $\exists\ \epsilon_0 > 0$ satisfying $N_2(\epsilon_0) < 0$, then Eq.(14) owns at least two couples of purely imaginary roots.*

Proof. (1) It follows from (21) that

$$\begin{aligned} \frac{dN_1(\epsilon)}{d\epsilon} &= 12pg_1\epsilon^{12p-1} + 11pg_2\epsilon^{11p-1} + 10pg_3\epsilon^{10p-1} + 9pg_4\epsilon^{9p-1} \\ &+8pg_5\epsilon^{8p-1} + 7pg_6\epsilon^{7p-1} + 6pg_7\epsilon^{6p-1} + 5pg_8\epsilon^{5p-1} \\ &+4pg_9\epsilon^{4p-1} + 3pg_{10}\epsilon^{3p-1} + 2pg_{11}\epsilon^{2p-1} + pg_{12}\epsilon^{p-1}. \end{aligned}$$

Since $g_l > 0 (l =, 2, \cdots, 12)$, one gains $\frac{dN_1(\epsilon)}{d\epsilon} > 0, \forall\ \epsilon > 0$. In addition, $N_1(0) = g_{13} > 0$, one knows that Eq.(20) admits no positive real root. According to $e_3 + e_5 + e_7 \neq 0$, we can know that $s = 0$ is not the root of (14), which completes the proof of (1).

(2) Clearly, $N_2(0) = g_{13} > 0, N_2(\epsilon_0) < 0 (\epsilon_0 > 0)$ and $\lim_{\epsilon\to+\infty} \frac{N_2(\epsilon)}{d\epsilon} = +\infty$, then there exist $\epsilon_1 \in (0, \epsilon_0)$ and $\epsilon_2 \in (\epsilon_0, +\infty)$ satisfying $N_2(\epsilon_1) = N_2(\epsilon_2) = 0$, then Eq.(20) owns at least two positive real roots. Then, (14) owns at least two couples of purely imaginary roots, which completes the proof of (2). □

Suppose that Eq.(20) owns twelve positive real roots $\epsilon_i (i = 1, 2, \cdots, 12)$. By (19), we get

$$\vartheta_i^k = \frac{1}{\epsilon_j}\left[\arccos\left(-\frac{\Psi_3(\epsilon)\Psi_5(\epsilon) + \Psi_2(\epsilon)\Psi_6(\epsilon)}{\Psi_1(\epsilon)\Psi_5(\epsilon) + \Psi_2(\epsilon)\Psi_4(\epsilon)}\right) + 2k\pi\right],$$

where $k = 0, 1, 2, \cdots, i = 1, 2, \cdots, 12$. Let

$$\vartheta_* = \min_{i=1,2,\cdots,12}\{\vartheta_i^0\}, \epsilon_0 = \epsilon|_{\vartheta=\vartheta_*}.$$

Now, we give the following condition:

$$(Q_5)\ S_{1R}S_{2R} + S_{1I}S_{2I} > 0,$$

where

$$\begin{cases} S_{1R} = 3p\epsilon_0^{3p-1}\cos\dfrac{(3p-1)\pi}{2} + 2pe_1\epsilon_0^{2p-1}\cos\dfrac{(2p-1)\pi}{2} \\ +pe_2\epsilon_0^{p-1}\cos\dfrac{(p-1)\pi}{2} + pe_4\epsilon_0^{p-1}\cos\dfrac{(p-1)\pi}{2}\cos\epsilon_0\vartheta_* \\ +pe_4\epsilon_0^{p-1}\sin\dfrac{(p-1)\pi}{2}\sin\epsilon_0\vartheta_* + pe_6\epsilon_0^{p-1}\cos\dfrac{(p-1)\pi}{2}\cos 2\epsilon_0\vartheta_* \\ +pe_6\epsilon_0^{p-1}\sin\dfrac{(p-1)\pi}{2}\sin 2\epsilon_0\vartheta_*, \\ S_{1I} = 3p\epsilon_0^{3p-1}\sin\dfrac{(3p-1)\pi}{2} + 2pe_1\epsilon_0^{2p-1}\sin\dfrac{(2p-1)\pi}{2} \\ +pe_2\epsilon_0^{p-1}\sin\dfrac{(p-1)\pi}{2} - pe_4\epsilon_0^{p-1}\cos\dfrac{(p-1)\pi}{2}\sin\epsilon_0\vartheta_* \\ +pe_4\epsilon_0^{p-1}\sin\dfrac{(p-1)\pi}{2}\cos\epsilon_0\vartheta_* - pe_6\epsilon_0^{p-1}\cos\dfrac{(p-1)\pi}{2}\sin 2\epsilon_0\vartheta_* \\ +pe_6\epsilon_0^{p-1}\sin\dfrac{(p-1)\pi}{2}\cos 2\epsilon_0\vartheta_*, \\ S_{2R} = \left(e_4\epsilon_0^p\cos\dfrac{p\pi}{2} + e_5\right)\epsilon_0\sin\epsilon_0\vartheta_* - \left(e_4\epsilon_0^p\sin\dfrac{p\pi}{2}\right)\epsilon_0\cos\epsilon_0\vartheta_* \\ +\left(e_6\epsilon_0^p\cos\dfrac{p\pi}{2} + e_7\right)\epsilon_0\sin 2\epsilon_0\vartheta_* - \left(e_6\epsilon_0^p\sin\dfrac{p\pi}{2}\right)\epsilon_0\cos 2\epsilon_0\vartheta_*, \\ S_{2I} = \left(e_4\epsilon_0^p\cos\dfrac{p\pi}{2} + e_5\right)\epsilon_0\cos\epsilon_0\vartheta_* + \left(e_4\epsilon_0^p\sin\dfrac{p\pi}{2}\right)\epsilon_0\sin\epsilon_0\vartheta_* \\ +\left(e_6\epsilon_0^p\cos\dfrac{p\pi}{2} + e_7\right)\epsilon_0\cos 2\epsilon_0\vartheta_* + \left(e_6\epsilon_0^p\sin\dfrac{p\pi}{2}\right)\epsilon_0\sin 2\epsilon_0\vartheta_*. \end{cases}$$

Lemma 5. *Let $s(\vartheta) = \eta_1(\vartheta) + i\eta_2(\vartheta)$ be the root of Eq.(14) near $\vartheta = \vartheta_*$ satisfying $\eta_1(\vartheta_*) = 0, \eta_2(\vartheta_*) = \epsilon_0$, then $Re\left(\frac{ds}{d\vartheta}\right)\Big|_{\vartheta=\vartheta_*,\epsilon=\epsilon_0} > 0$.*

Proof. It follows from Eq.(14) that

$$\begin{aligned} &\left(3ps^{3p-1} + 2pe_1s^{2p-1} + pe_2s^{p-1}\right)\frac{ds}{d\vartheta} + pe_4s^{p-1}e^{-s\vartheta}\frac{ds}{d\vartheta} \\ &-e^{-s\vartheta}\left(\frac{ds}{d\vartheta}\vartheta + s\right)(e_4s^p + e_5) + pe_6s^{p-1}e^{-2s\vartheta}\frac{ds}{d\vartheta} \\ &-2e^{-2s\vartheta}(e_6s^p + e_7)\left(\frac{ds}{d\vartheta}\theta + s\right) = 0, \end{aligned}$$

which results in

$$\left(\frac{ds}{d\vartheta}\right)^{-1} = \frac{S_1(s)}{S_2(s)} - \frac{\vartheta}{s},$$

where

$$\begin{cases} S_1(s) = 3ps^{3p-1} + 2pe_1s^{2p-1} + pe_2s^{p-1} \\ \quad +pe_4s^{p-1}e^{-s\vartheta} + pe_6s^{p-1}e^{-2s\vartheta}, \\ S_2(s) = se^{-s\vartheta}(e_4s^p + e_5) + 2se^{-2s\vartheta}(e_6s^p + e_7). \end{cases}$$

Then

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_*,\epsilon=\epsilon_0} = \mathrm{Re}\left[\frac{S_1(s)}{S_2(s)}\right]_{\vartheta=\vartheta_*,\epsilon=\epsilon_0} = \frac{S_{1R}S_{2R} + S_{1I}S_{2I}}{S_{2R}^2 + S_{2I}^2}.$$

In view of (Q_5), we gain

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_*,\epsilon=\epsilon_0} > 0.$$

The proof is completed. □

Applying Lemma 1, the following assertion can be lightly is acquired.

Theorem 2. *If $(Q_1),(Q_4),(Q_5)$ are fulfilled, then $W(w_{1*},w_{2*},w_{3*})$ of system (12) keeps locally asymptotically stability when $\vartheta\in[0,\vartheta_*)$ and a cluster of Hopf bifurcations of system (12) happen near $W(w_{1*},w_{2*},w_{3*})$ when $\vartheta=\vartheta_*$.*

5. Software experiments

In this section, we give software experiments via Matlab software [1].

Example 1. Give the fractional-order delayed turbidostat model as follows:

$$\begin{cases} \dfrac{d^p w_1(t)}{dt^p} = [a+\kappa_1 w_2(t-\vartheta)+\kappa_2 w_3(t-\vartheta)](w_0-w_1)-\dfrac{w_2}{\gamma_1}g_1(w_1)-\dfrac{w_3}{\gamma_2}g_2(w_1),\\ \dfrac{d^p w_2(t)}{dt^p} = w_2[g_1(w_1)-(a+\kappa_1 w_2(t-\vartheta)+\kappa_2 w_3(t-\vartheta))],\\ \dfrac{d^p w_3(t)}{dt^p} = w_3[g_2(w_1)-(a+\kappa_1 w_2(t-\vartheta)+\kappa_2 w_3(t-\vartheta))], \end{cases} \tag{22}$$

where $p=0.93, a=1.4, \kappa_1=2.2, \kappa_2=1.05, w_0=1, \gamma_1=1, \gamma_2=1, g_1(w_1)=\frac{3w_1}{0.25+w_2}, g_2(w_1)=\frac{5w_1}{0.8+w_2}$. We can easily obtain that system (22) owns the unique positive equilibrium point $W(w_{1*},w_{2*},w_{3*})=W(0.5750,0.2127,0.2123)$. The three assumptions $(Q_1),(Q_2),(Q_3)$ of Theorem 1 are fulfilled. With the aid of Matlab software, one can get $\varepsilon_0=3.4456, \vartheta_0=1.91$. To check the correctness of the main conclusions of Theorem 1, we take two delays as examples. Firstly, select $\vartheta=1.82$ which means $\vartheta<\vartheta_0=1.91$, that is to say, ϑ falls into the range $[0,\vartheta_0)$. To this case, the Matlab simulation figures are displayed in Fig.1. Obviously, Fig.1 manifests that the limiting nutrient concentration w_1 will be close to 0.5750, the concentration of the first competitor w_2 will be close to 0.2127 and the concentration of the second competitor w_3 will be close to 0.2123 as $t\to+\infty$. Secondly, choose $\vartheta=2.13$ which means $\vartheta>\vartheta_0=1.91$, that is to say, ϑ exceeds the threshold delay number ϑ_0. To this case, the Matlab simulation figures are displayed in Fig.2. Obviously, Fig.2 manifests that the limiting nutrient concentration w_1 is about to remain a periodic oscillation state around 0.5750, the concentration of the first competitor w_2 is about to remain a periodic oscillation state around 0.2127 and the concentration of the second competitor w_3 is about to remain a periodic oscillation state around 0.2123. That is to say, system (22) are to produce a limit cycle (namely, Hopf bifurcation) near $W(w_{1*},w_{2*},w_{3*})=W(0.5750,0.2127,0.2123)$. Furthermore, the bifurcation figures of system (22) are provided to embody the bifurcation point $\vartheta_0=1.91$ (see Figs.3-5).

Example 2. Give the fractional-order controlled delayed turbidostat model as follows:

$$\begin{cases} \dfrac{d^p w_1(t)}{dt^p} = [a+\kappa_1 w_2(t-\vartheta)+\kappa_2 w_3(t-\vartheta)](w_0-w_1)-\dfrac{w_2}{\gamma_1}g_1(w_1)-\dfrac{w_3}{\gamma_2}g_2(w_1),\\ \dfrac{d^p w_2(t)}{dt^p} = \rho_1\{w_2[g_1(w_1)-(a+\kappa_1 w_2(t-\vartheta)+\kappa_2 w_3(t-\vartheta))]\}+\rho_2[w_2(t-\vartheta)-w_2],\\ \dfrac{d^p w_3(t)}{dt^p} = w_3[g_2(w_1)-(a+\kappa_1 w_2(t-\vartheta)+\kappa_2 w_3(t-\vartheta))]\\ \quad +\tau_p[w_3-w_{3*}]+\tau_d\dfrac{d[^p w_3-w_{3*}]}{dt^p}, \end{cases} \tag{23}$$

where $p=0.93, a=1.4, \kappa_1=2.2, \kappa_2=1.05, w_0=1, \gamma_1=1, \gamma_2=1, g_1(w_1)=\frac{3w_1}{0.25+w_2}, g_2(w_1)=\frac{5w_1}{0.8+w_2}$. Let $\rho_1=0.2, \rho_2=0.5, \tau_p=0.5, \tau_d=0.4$. We can easily obtain that system (23) owns the unique positive equilibrium point $W(w_{1*},w_{2*},w_{3*})=W(0.5750,0.2127,0.2123)$. The three assumptions $(Q_1),(Q_4),(Q_5)$ of Theorem 2 are fulfilled. With the aid of Matlab software, one can get $\epsilon_0=2.8963, \vartheta_*=2.32$. To check the correctness of the main conclusions of Theorem 2, we take two delays as examples. Firstly, select $\vartheta=2.15$ which means $\vartheta<\vartheta_*=2.32$, that is to say, ϑ falls into the range $[0,\vartheta_*)$. To this case, the Matlab simulation figures are displayed in Fig.6.

Obviously, Fig.6 manifests that the limiting nutrient concentration w_1 will be close to 0.5750, the concentration of the first competitor w_2 will be close to 0.2127 and the concentration of the second competitor w_3 will be close to 0.2123 as $t \to +\infty$. Secondly, choose $\vartheta = 2.53$ which means $\vartheta > \vartheta_* = 1.91$, that is to say, ϑ exceeds the threshold delay number ϑ_*. To this case, the Matlab simulation figures are displayed in Fig.7. Obviously, Fig.7 manifests that the limiting nutrient concentration w_1 is about to remain a periodic oscillation state around 0.5750, the concentration of the first competitor w_2 is about to remain a periodic oscillation state around 0.2127 and the concentration of the second competitor w_3 is about to remain a periodic oscillation state around 0.2123. That is to say, system (23) are to produce a limit cycle (namely, Hopf bifurcation) near $W(w_{1*}, w_{2*}, w_{3*}) = W(0.5750, 0.2127, 0.2123)$. Furthermore, the bifurcation figures of system (23) are provided to embody the bifurcation point $\vartheta_* = 2.32$ (see Figs.8-10).

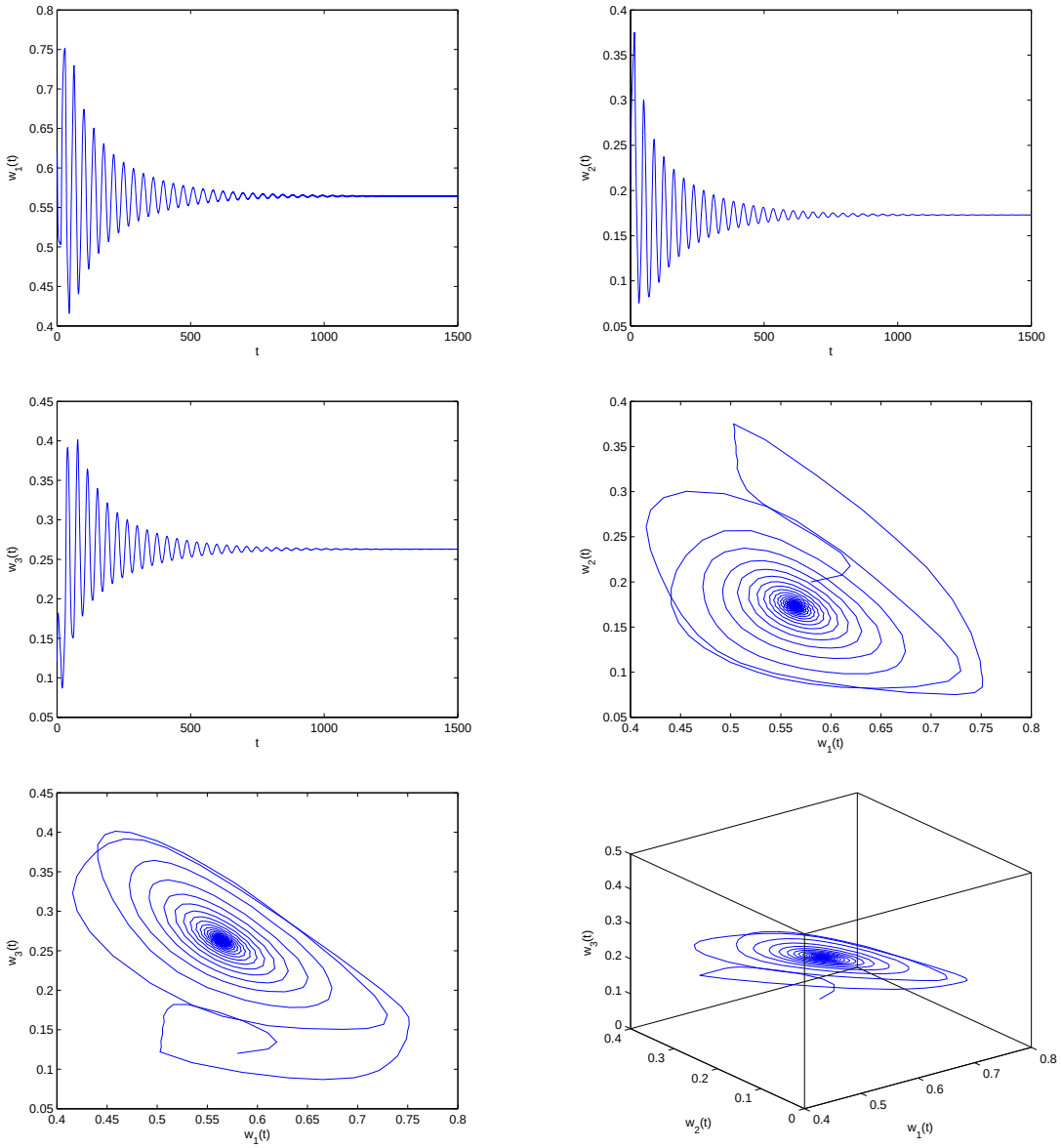


Figure 1. Computer experiment figures of system (22) with the time delay $\vartheta = 1.82 < \vartheta_0 = 1.91$. The positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*}) = W(0.5750, 0.2127, 0.2123)$ maintains locally asymptotically stability.

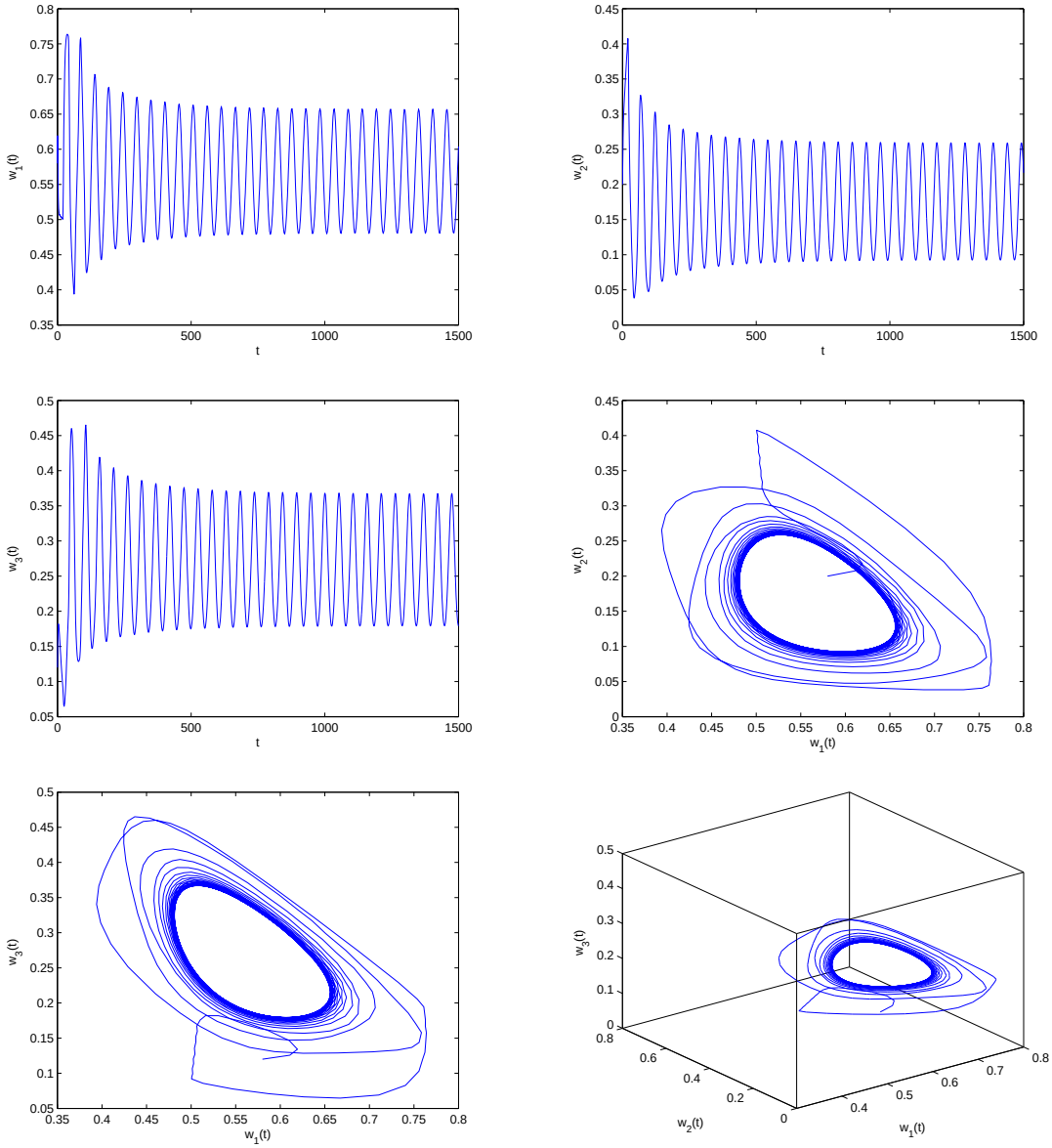


Figure 2. Computer experiment figures of system (22) with the time delay $\vartheta = 2.13 > \vartheta_0 = 1.91$. A limit cycle (Hopf bifurcation) happens near the positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*}) = W(0.5750, 0.2127, 0.2123)$.

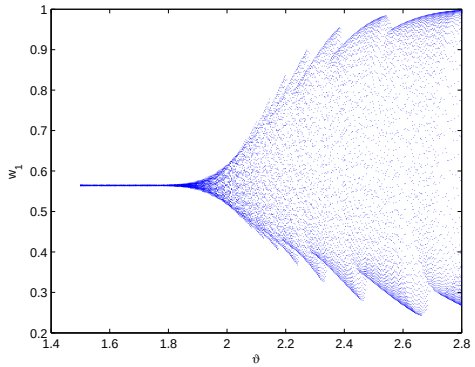


Figure 3. Bifurcation plot of system (22): The evolutionary relation of the time t and the variable w_1. The bifurcation point of system (22) is 1.91.

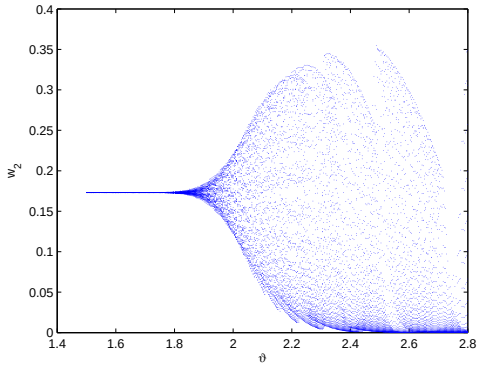


Figure 4. Bifurcation plot of system (22): The evolutionary relation of the time t and the variable w_2. The bifurcation point of system (22) is 1.91.

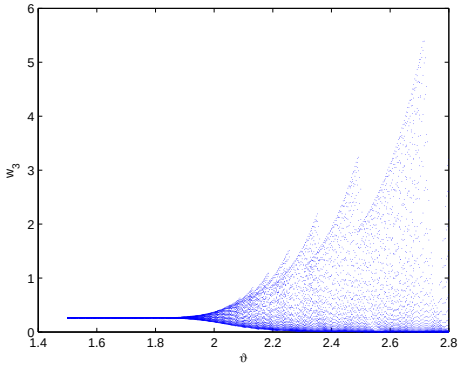
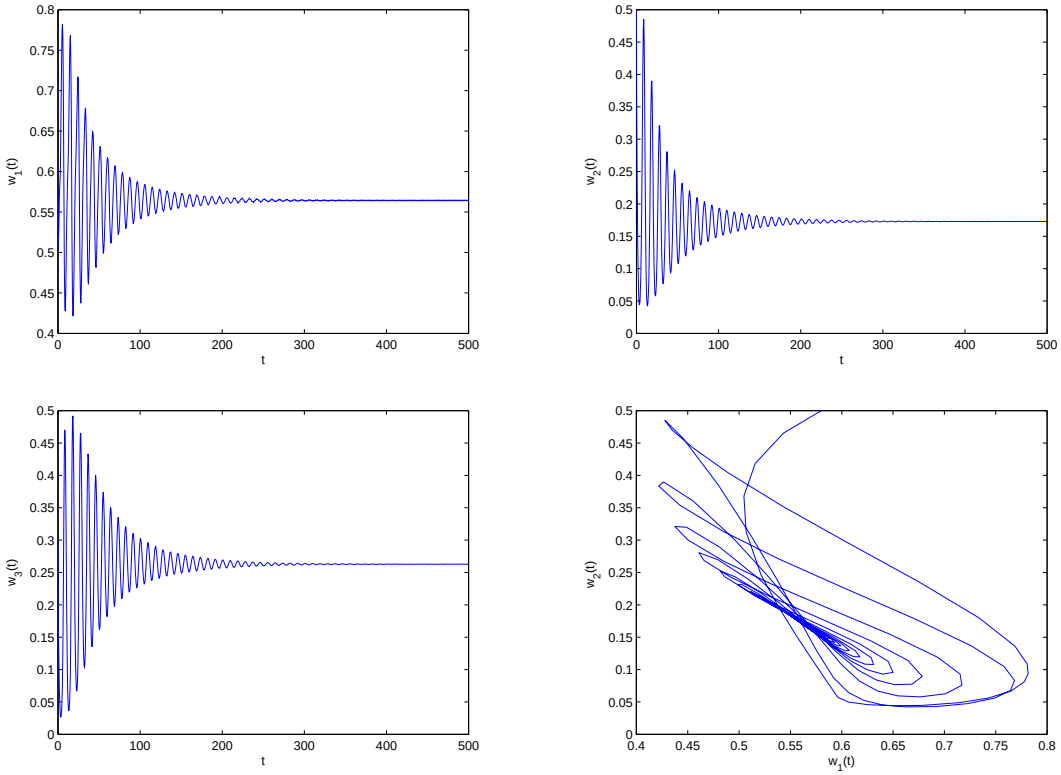


Figure 5. Bifurcation plot of system (22): The evolutionary relation of the time t and the variable w_3. The bifurcation point of system (22) is 1.91.



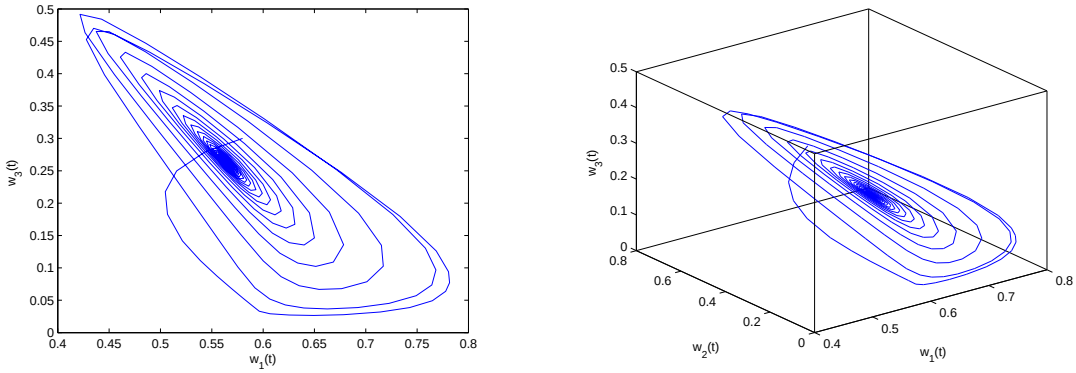


Figure 6. Computer experiment figures of system (23) with the time delay $\vartheta = 2.15 < \vartheta_* = 2.32$. The positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*}) = W(0.5750, 0.2127, 0.2123)$ maintains locally asymptotically stability.

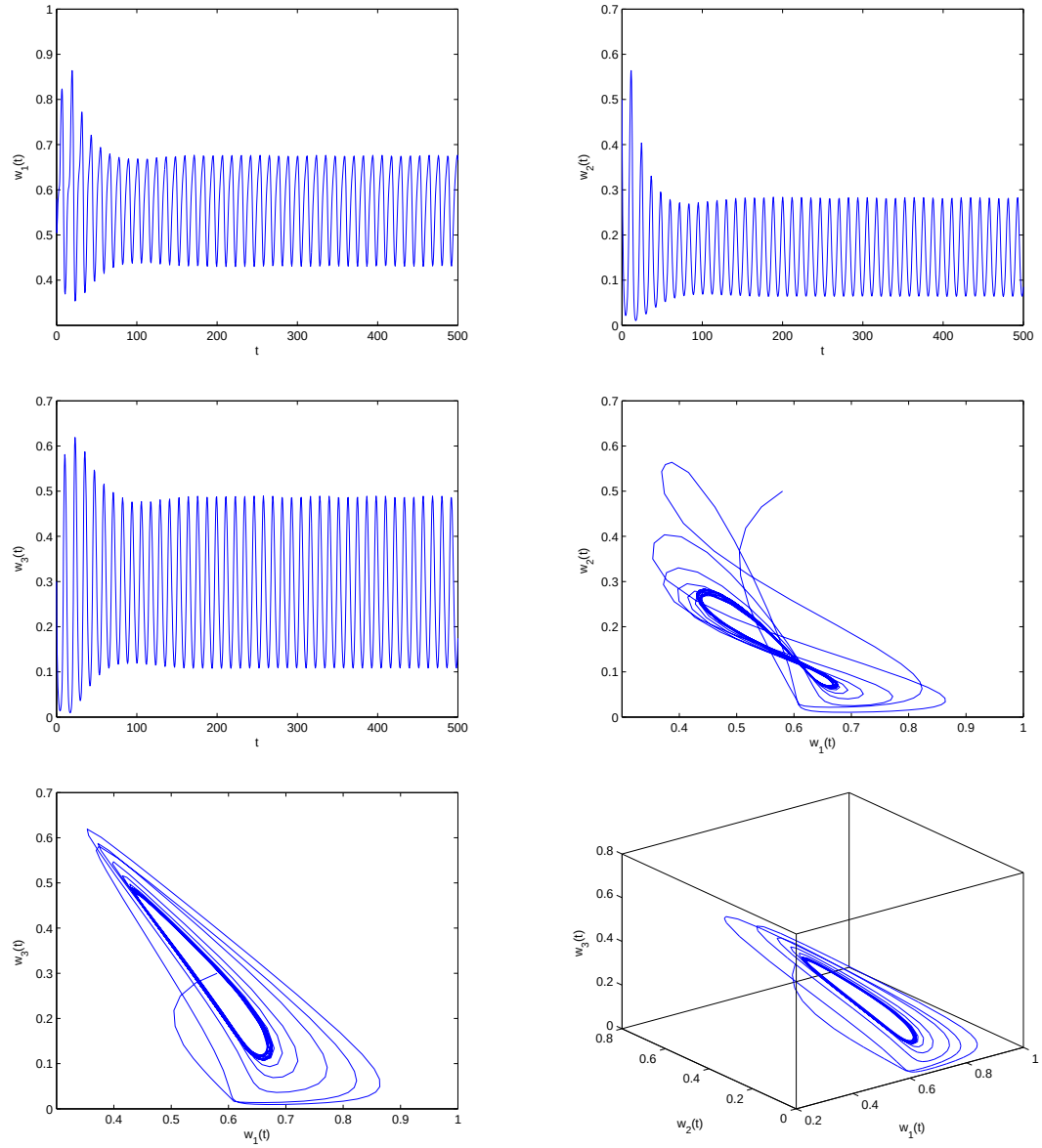


Figure 7. Computer experiment figures of system (23) with the time delay $\vartheta = 2.15 > \vartheta_* = 2.32$. A limit cycle (Hopf bifurcation) happens near the positive equilibrium point $W(w_{1*}, w_{2*}, w_{3*}) = W(0.5750, 0.2127, 0.2123)$.

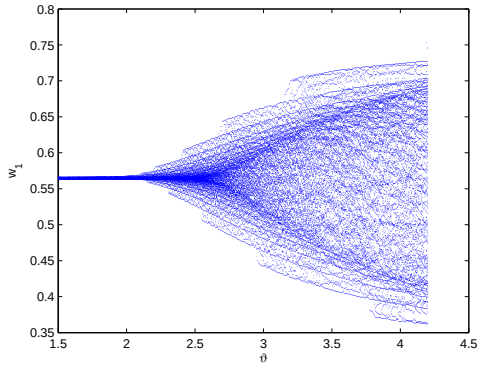


Figure 8. Bifurcation plot of system (23): The evolutionary relation of the time t and the variable w_1. The bifurcation point of system (23) is 2.32.

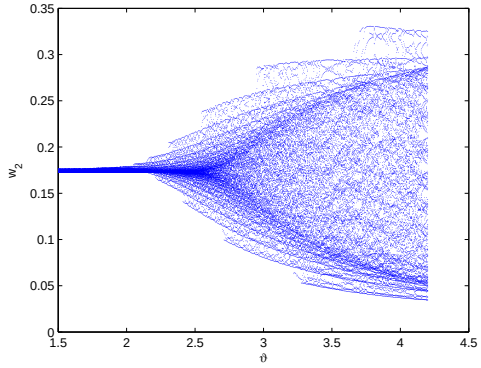


Figure 9. Bifurcation plot of system (23): The evolutionary relation of the time t and the variable w_2. The bifurcation point of system (23) is 2.32.

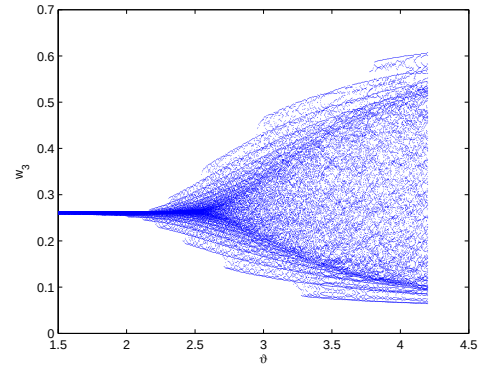


Figure 10. Bifurcation plot of system (23): The evolutionary relation of the time t and the variable w_3. The bifurcation point of system (23) is 2.32.

Remark 2. *From software simulation figures of Example 1 and Example 2, we can see that the bifurcation point of system (23) is* 1.91 *and the bifurcation point of system (23) is* 2.32, *which implies that the stability region of system (22) is enlarged and the time of onset of Hopf bifurcation of system (22) is postponed via the designed extended hybrid controller. Based on this fact, we think that designing some suitable controllers to achieve control objectives is an interesting topic.*

Remark 3. *When calculating the bifurcation diagrams in Example 1 (Figs.3-5) and Example 2 (Figs.8-10), based on the bifurcation diagrams, we can not determine the both minimum and maximum concentrations of the variables* w_1, w_2, w_3.

Remark 4. *We calculate the Hopf bifurcation points directly using the computation of* ε_0 *and* ϵ_0 *not using graphs.*

Remark 5. *The initial conditions in Example 1 and Example 2 are* $(0.58, 0.2, 0.12)$ *and* $(0.58, 0.5, 0.3)$*, respectively.*

6. Conclusion

In recent year, in order to reveal the intrinsic evolution law of various chemical substances, many scholars have paid great attention to the exploration of differential chemical reaction models. Depending on the previous works, we set up a novel fractional-order delayed turbidostat model. Making use of Laplace transform, stability and bifurcation knowledge of fractional-order dynamical system, we have investigated the stability and bifurcation behavior of the formulated fractional-order delayed turbidostat model. A new sufficient criterion about bifurcation and stability of this model is gained. By virtue of extended hybrid controller, we have succeed in dominating the stability domain and the time of generation of Hopf bifurcation of the formulated fractional-order delayed turbidostat model. In order to verify the validity of the acquired results, we give some Matlab simulations. The acquired outcomes have significant value in balancing and adjusting the concentrations of various chemical substances. The research methods are also exploited to handle many control problems in lots of differential dynamical systems.

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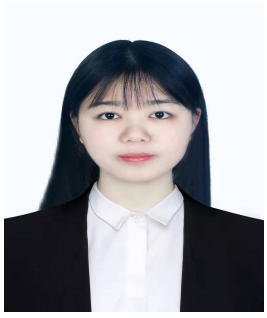
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