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NON-EXISTENCE OF GLOBAL WEAK SOLUTIONS TO SEMI-LINEAR WAVE EQUATIONS INVOLVING TIME-DEPENDENT STRUCTURAL DAMPING TERMS

M. KIRANE[1,*], A.Z. FINO[2], S. KERBAL[3], A. LAADHARI[1]

Abstract. We consider a semilinear wave equation involving a time-dependent structural damping term of the form $\frac{1}{(1+t)^{\beta}}(-\Delta)^{\sigma/2}u_t$. Our results show the influence of the parameters β, σ on the nonexistence of global weak solutions under assumptions on the given system data. Our approach is based on the nonlinear capacity method combined with a pointwise estimate of the fractional Laplacian of some test functions, which was derived by Fujiwara (2018) (see also Fino and Dao (2022)). We extend the blowu-p results developed recently by Fino and Hamza (2022).

Keywords: Blow-Up, Damped Wave Equations, Variable Coefficients, Fractional Laplacian.

AMS Subject Classification: 35B44, 35A01, 35L20, 35L71.

1. Introduction

The aim of the paper is to establish a non-existence of global weak solutions to the Cauchy problem for the following semi-linear structurally damped wave equation

$$\begin{cases} u_{tt} - \Delta u + \dfrac{b_0}{(1+t)^{\beta}}(-\Delta)^{\sigma/2}u_t = |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $0 < \sigma < 2$, $n \geq 1$, $p > 1$, b_0 is a positive constant and $\beta \in \mathbb{R}$. Without loss of generality, we assume that $b_0 = 1$.

For the natural occurrence of fractional derivatives in physics, we may refer to [10].

Prior to our main blow-up results, we would like to emphasize that the special case of Eq. (1) for $b_0 = 0$ is governed by the classical semi-linear wave equation where the Strauss conjecture for this case states that: if $p \leq p_S$, then there is no global solution for Eq. (1) under suitable sign assumptions on the initial data, and for $p > p_S$, a global solution exists for small initial data; see e.g. [1,11,17,18,20] among many other references, where p_S is the positive solution of the quadratic equation such that $(n-1)p^2 - (n+1)p - 2 = 0$, and is given by

$$p_S = p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}.$$

[1]Department of Mathematics, College of Computing and Mathematical Sciences, Khalifa University of Science and Technology, Abu Dhabi, United Arab Emirates, e-mail: aymen.laadhari@ku.ac.ae
[2]College of Engineering and Technology, American University of the Middle East, Egaila 54200, Kuwait e-mail: ahmad.fino@aum.edu.kw
[3]Department of Mathematics, Sultan Qaboos University, FracDiff Research Group (DR/RG/03), Al-Khoud 123, Muscat, P.O. Box 36, Oman, e-mail: skerbal@squ.edu.om
*Corresponding author e-mail: mokhtar.kirane@ku.ac.ae
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In order to have more general initial data, but still with compact support, in [13] the author obtained with a slightly less sharp blow-up result an exponent of the form $(n+1)/(n-1)$ which is less than the Strauss exponent p_S, for $n \geq 2$.

We would like to emphasize here that the test function method, introduced by [19] and used by [7, 8, 14, 15], plays a similar role as of Kato's method in the proof of blow-up results. In fact, the test function is effective in the case of parabolic equations as it provide exactly the critical exponent p_c, but in the case of hyperbolic equations (cf. [14]) we get the so-called Kato's exponent p^* which is less than p_c. This is one of the weakness of the test function method but in general it can be applied to a more general equation and system with no positive condition on solutions.

When $\beta = 0$, $\sigma \to 2$, and $b_0 = 1$, the problem that is given by Eq. (1) is reduced to

$$u_{tt} - \Delta u - \Delta u_t = |u|^p, \qquad x \in \mathbb{R}^n, t > 0, \tag{2}$$

which is called the visco-elastic damping case. The authors [3, Theorem 4.2] proved that the solution of (2) blows-up in finite time when $1 < p \leq (n+1)/(n-1)_+$, where $(\cdot)_+ := \max\{0, \cdot\}$, by applying the test function method. Similar result has been obtained recently in the case of an exterior domain [8]. On the other hand, the authors of the paper [2] proved that there exists a global solution for the Eq.(2) when $p > 1 + \frac{3}{n-1}$ $(n \geq 2)$ for sufficiently small initial data. Therefore, the exact value of the critical exponent is still an open question.

Recently, the problem that is given by Eq. (1) with $\sigma \to 2$,

$$u_{tt} - \Delta u - \frac{b_0}{(1+t)^\beta}\Delta u_t = |u|^p, \qquad x \in \mathbb{R}^n, t > 0, \tag{3}$$

has been studied by the authors of the paper [6]. They proved that if $\int\limits_{\mathbb{R}^n} u_1(x)dx > 0$,

$$\begin{cases} p \in (1, \infty) & \text{for } n = 1, 2, \\ p \in (1, \frac{n}{n-2}] & \text{for } n \geq 3, \end{cases}$$

and

$$\begin{cases} 1 < p \leq \frac{n+1}{(n-1)_+}, & \text{if } \beta \geq -1, \\ 1 < p \leq \frac{n(1-\beta)+2}{(n(1-\beta)-2)_+}, & \text{if } \beta \leq -1, \end{cases}$$

then the mild solution of the Eq.(3) blows-up in finite time.

The goal of this paper is to prove the non-existence of global weak solutions of the Eq.(1) which is a generalization of the results of [6] to the case of fractional Laplacian damping term by using a recent version of the test function method that relies on Lemma 1 below. The novelty of this paper is the choice of this test function which is consistent with the suppression of positivity usually assumed by many authors.

In the case of $\beta \neq 0$ and $b_0 = 1$, we give an intuitive observation for understanding the influence of the damping term such as $(\frac{1}{(1+t)^\beta}(-\Delta)^{\sigma/2}u_t)$ by scaling argument. Let $u(t,x)$ be a solution of the linear strong damped wave equation given by:

$$u_{tt}(t,x) - \Delta u(t,x) + \frac{1}{(1+t)^\beta}(-\Delta)^{\sigma/2}u_t(t,x) = 0. \tag{4}$$

When $\sigma \geq 1 - \beta$, we put

$$u(t,x) = v(\lambda(1+t), \lambda x), \qquad \lambda(1+t) = s,\ \lambda x = y, \tag{5}$$

with a parameter $\lambda > 0$, we have

$$v_{ss}(s,y) - \Delta v(s,y) + \frac{\lambda^{\sigma+\beta-1}}{s^\beta}(-\Delta)^{\sigma/2}v_s(s,y) = 0.$$

Thus, when $\sigma = 1-\beta$ we notice that Eq. (4) is invariant, while $\sigma > 1-\beta$, letting $\lambda \to 0$, we obtain the wave equation without damping as:

$$v_{ss}(s,y) - \Delta v(s,y) = 0.$$

We note that $\lambda \to 0$ is corresponding to $t \to +\infty$.

On the other hand, when $\sigma < 1-\beta$, we put

$$u(t,x) = v(\lambda^{\frac{\sigma}{1-\beta}}(1+t), \lambda x), \qquad \lambda^{\frac{2}{1-\beta}}(1+t) = s,\ \lambda x = y,$$

with a parameter $\lambda > 0$, we have

$$v_{ss}(s,y) + \frac{1}{s^\beta}(-\Delta)^{\sigma/2} v_s(s,y) - \lambda^{\frac{2(1-\beta-\sigma)}{1-\beta}} \Delta v(s,y) = 0.$$

In this case, letting $\lambda \to 0$, we obtain the pseudo-parabolic equation as following:

$$v_{ss}(s,y) + \frac{1}{s^\beta}(-\Delta)^{\sigma/2} v_s(s,y) = 0.$$

This means that the asymptotic behaviour of solutions depends on the behaviour of the coefficient of the damping term.

This paper is organized as follows. We will start move to after our work. By stating the main theorem of our work and give the proof in the Section 4. Some preliminaries will be given in the Section 3.

2. Main results

This section is aimed to state the main results.

Theorem 1 (**Blow-up**). *We assume that*

$$(u_0, u_1) \in \left(L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)\right) \times \left(L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right)$$

satisfying the following condition:

$$\int_{\mathbb{R}^n} u_1(x)dx > 0. \tag{6}$$

If

$$\begin{cases} 1 < p < \min\left\{\dfrac{n+1}{(n-1)_+}; \dfrac{n}{(n-\sigma)_+}\right\}, & \text{if } \ \sigma \geq 1-\beta, \\ 1 < p < \min\left\{\dfrac{n(1-\beta)+\sigma}{(n(1-\beta)-\sigma)_+}; \dfrac{n}{(n-\sigma)_+}\right\}, & \text{if } \ \sigma \leq 1-\beta, \end{cases} \tag{7}$$

or

$$\begin{cases} p = \dfrac{n+1}{n-1}, & \text{if } \ \sigma \geq 1-\beta,\ \sigma > \dfrac{2n}{n+1},\ n \geq 2, \\ p = \dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}, & \text{if } \ n(1+\beta) < \sigma \leq 1-\beta,\ n \geq 1, \end{cases}$$

where $(\cdot)_+ := \max\{0, \cdot\}$*, then the problem that is given by Eq.(1) has no global weak solutions.*

Remark 1. We note that, by taking the limit case $\sigma \to 2$, we recover the same results as in [6].

Remark 2. We stress that the exponent $\dfrac{n+1}{(n-1)_+}$ appearing in (7) was introduced firstly in [13] to prove the non-existence of the global solutions to the semi-linear wave equation with the non-linearity $|u|^p$, subject to small initial data with compact support.

3. Preliminaries

Definition 1. [16] Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$ and $s \in (0,1)$. The fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ is a non-local operator defined on $\mathcal{S}$ by

$$
\begin{aligned}
(-\Delta)^s v(x) &:= C_{n,s}\ p.v. \int\limits_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2s}}\,dy \\
&= \begin{cases} C_{n,s} \displaystyle\int\limits_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2s}}\,dy, & \text{if } 0<s<1/2, \\[2ex] C_{n,s} \displaystyle\int\limits_{\mathbb{R}^n} \frac{v(x)-v(y)-\nabla v(x)\cdot(x-y)\mathcal{X}_{|x-y|<\delta}(y)}{|x-y|^{n+2s}}\,dy,\ \forall\,\delta>0, & \text{if } 1/2\le s<1, \end{cases}
\end{aligned}
$$

where $p.v.$ stands for Cauchy's principal value, and $C_{n,s} := \dfrac{s\,4^s\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(1-s)}$.

In fact, we are rarely going to use the fractional Laplacian operator in the Schwartz space; it can be extended to less regular functions as follows: for $s \in (0,1)$, $\varepsilon > 0$, it can be written as

$$
\mathcal{L}_{s,\varepsilon}(\Omega) := \begin{cases} L_s(\mathbb{R}^n) \cap C^{0,2s+\varepsilon}(\Omega), & \text{if } 0<s<1/2, \\[2ex] L_s(\mathbb{R}^n) \cap C^{1,2s+\varepsilon-1}(\Omega), & \text{if } 1/2\le s<1, \end{cases}
$$

where Ω is an open subset of $\mathbb{R}^n$, $C^{0,2s+\varepsilon}(\Omega)$ is the space of $2s+\varepsilon$- Hölder continuous functions on Ω, $C^{1,2s+\varepsilon-1}(\Omega)$ is the space of functions of $C^1(\Omega)$ whose first partial derivatives are Hölder continuous with exponent $2s+\varepsilon-1$, and

$$
L_s(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} \quad \text{such that} \quad \int\limits_{\mathbb{R}^n} \frac{u(x)}{1+|x|^{n+2s}}\,dx < \infty \right\}.
$$

Proposition 1. *[16, Proposition 2.4] Let Ω be an open subset of $\mathbb{R}^n$, $s \in (0,1)$, and $f \in \mathcal{L}_{s,\varepsilon}(\Omega)$ for some $\varepsilon > 0$. Then, $(-\Delta)^s f$ is a continuous function in Ω and $(-\Delta)^s f(x)$ is given by the pointwise formulas of Definition 1 for every $x \in \Omega$.*

Remark 3. A simple sufficient condition for the function f to satisfy the conditions in Proposition 1 is that $f \in L^1_{loc}(\mathbb{R}^n) \cap C^2(\Omega)$.

Using [4, Lemma 2.11] and its proof, we have the following.

Lemma 1. *Let $\langle x\rangle := (1+(|x|-1)^4)^{1/4}$ for all $x \in \mathbb{R}^n$, $n \ge 1$. Let $s \in (0,1]$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a function defined by*

$$
\phi(x) = \begin{cases} 1, & \text{if } |x| \le 1, \\[2ex] \langle x\rangle^{-n-2s}, & \text{if } |x| \ge 1. \end{cases} \tag{8}
$$

Then

$$
\phi \in \mathcal{C}^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap H^2(\mathbb{R}^n), \qquad \partial_x^2 \phi \in L^\infty(\mathbb{R}^n),
$$

and the following estimate holds:

$$
\max\left\{ |\Delta\phi(x)|\,,\ |(-\Delta)^s\phi(x)| \right\} \le C\,\phi(x) \quad \text{for all } x \in \mathbb{R}^n.
$$

Lemma 2. *Let h be a smooth function satisfying $\partial_x^2 h \in L^\infty(\mathbb{R}^n)$. For any $R > 0$, let h_R be a function defined by*

$$
h_R(x) := h(R^{-1}x) \quad \text{for all } x \in \mathbb{R}^n.
$$

Then, $(-\Delta)^s(h_R)$, $s \in (0,1]$, satisfies the following scaling property

$$
(-\Delta)^s h_R(x) = R^{-2s}(-\Delta)^s h(R^{-1}x), \quad \text{for all } x \in \mathbb{R}^n.
$$

Lemma 3. *[4, Lemma 2.14] Let $s, \widetilde{s} \in (0,1]$, $R > 0$ and $p > 1$. Then, the following estimate holds*

$$\int_{\mathbb{R}^n} (\phi_R(x))^{-\frac{1}{p-1}} \left|(-\Delta)^{\widetilde{s}} \phi_R(x)\right|^{\frac{p}{p-1}} dx \leq C\, R^{-\frac{2\widetilde{s}p}{p-1}+n},$$

where $\phi_R(x) := \phi(x/R)$ and ϕ as defined in (8).

4. Proof of Theorem 1

The proof of Theorem 1 relies mainly on the concept of weak solution of the Cauchy problem (1) and the use of the test function method. Let

$$X_{\delta,T} = \{\varphi \in C([0,\infty), H^2(\mathbb{R}^n)) \cap C^1([0,\infty), H^\delta(\mathbb{R}^n)) \cap C^2([0,\infty), L^2(\mathbb{R}^n)), \text{such that supp}\varphi \subset Q_T\},$$

where $Q_T := [0,T] \times \mathbb{R}^n$, and the homogeneous fractional Sobolev space $H^\delta(\mathbb{R}^n)$, $\delta \in (0,2)$, is defined by

$$H^\delta(\mathbb{R}^n) = \begin{cases} \{u \in L^2(\mathbb{R}^n); (-\Delta)^{\delta/2} u \in L^2(\mathbb{R}^n)\}, & \text{if } \delta \in (0,1), \\ H^1(\mathbb{R}^n), & \text{if } \delta = 1, \\ \{u \in H^1(\mathbb{R}^n); (-\Delta)^{\delta/2} u \in L^2(\mathbb{R}^n)\}, & \text{if } \delta \in (1,2), \end{cases}$$

endowed with the norm

$$\|u\|_{H^\delta(\mathbb{R}^n)} = \begin{cases} \|u\|_{L^2(\mathbb{R}^n)} + \left\|(-\Delta)^{\delta/2} u\right\|_{L^2(\mathbb{R}^n)}, & \text{if } \delta \in (0,1), \\ \|u\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)}, & \text{if } \delta = 1, \\ \|u\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)} + \left\|(-\Delta)^{\delta/2} u\right\|_{L^2(\mathbb{R}^n)}, & \text{if } \delta \in (1,2). \end{cases}$$

The weak formulation associated with the Eq.(1) reads as follows:

Definition 2. (Weak solution) Let $T > 0$, and $u_0, u_1 \in L^2(\mathbb{R}^n)$. A function

$$u \in L^1((0,T), L^2(\mathbb{R}^n)) \cap L^p((0,T); L^{2p}(\mathbb{R}^n)),$$

is said to be a weak solution of the Eq.(1) on $[0,T) \times \mathbb{R}^n$ if

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi \, dt\, dx + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\, dx \\ &+ \int_{\mathbb{R}^n} u_0(x)(-\Delta)^{\sigma/2}\varphi(0,x)\, dx - \int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)\, dx \\ &= \int_{Q_T} u\varphi_{tt}\, dt\, dx - \int_0^T \frac{1}{(1+t)^\beta} \int_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi_t\, dx\, dt \\ &- \int_{Q_T} u\Delta\varphi\, dt\, dx + \int_0^T \frac{\beta}{(1+t)^{\beta+1}} \int_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi\, dx\, dt, \end{aligned}$$

holds for all $\varphi \in X_{\sigma,T}$. We denote the lifespan for the weak solution by $T_w(u_0,u_1) := \sup\{T \in (0,\infty]$ for which there exists a unique weak solution u to the Eq.(1). Moreover, if $T > 0$ can be arbitrary chosen, i.e. $T_w(u_0,u_1) = \infty$, then u is called a global weak solution of the Eq.(1).

Proof of Theorem 1. Let u be a global weak solution of the Eq.(1), that is

$$\begin{aligned}
&\int\limits_{Q_T} |u|^p \varphi \, dt \, dx + \int\limits_{\mathbb{R}^n} u_1(x)\varphi(0,x)\, dx \\
&+ \int\limits_{\mathbb{R}^n} u_0(x)(-\Delta)^{\sigma/2}\varphi(0,x)\, dx - \int\limits_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)\, dx \\
&= \int\limits_{Q_T} u\varphi_{tt}\, dt\, dx - \int\limits_0^T \frac{1}{(1+t)^\beta} \int\limits_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi_t \, dx\, dt \qquad (9)\\
&- \int\limits_{Q_T} u\Delta\varphi \, dt\, dx + \int\limits_0^T \frac{\beta}{(1+t)^{\beta+1}} \int\limits_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi \, dx\, dt,
\end{aligned}$$

for all $T > 0$, and all $\varphi \in X_{\sigma,T}$.

Let $T > 0$. Now, we introduce the following test function

$$\varphi(t,x) = \psi^\eta(t)\phi_{T^d}(x) = \psi^\eta(t)\phi\left(\frac{x}{T^d}\right), \qquad (10)$$

where ϕ is defined in the Eq.(8), $\psi(t) = \Psi(\frac{t}{T})$, $\eta \gg 1$, $d > 0$ are constants that will be determined later, and $\Psi \in C^\infty(\mathbb{R}_+)$ is a cut-off non-increasing function such that

$$\Psi(r) = \begin{cases} 1, & \text{if} \quad 0 \le r \le 1/2, \\ \searrow, & \text{if} \quad 1/2 \le r \le 1, \\ 0, & \text{if} \quad r \ge 1. \end{cases}$$

From the formulation of the Eq.(9), we get the following inequality

$$\begin{aligned}
&\int\limits_{Q_T} |u|^p \varphi \, dt \, dx + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\, dx \\
&\le \int\limits_{\frac{T}{2}}^T \int_{\mathbb{R}^n} |u|\,|\varphi_{tt}|\, dx\, dt + \int\limits_{Q_T} |u|\,|\Delta\varphi|\, dt\, dx + \int\limits_{\frac{T}{2}}^T \frac{1}{(1+t)^\beta} \int\limits_{\mathbb{R}^n} |u||(-\Delta)^{\sigma/2}\varphi_t|\, dx\, dt \\
&+ \int\limits_0^T \frac{\beta}{(1+t)^{\beta+1}} \int\limits_{\mathbb{R}^n} |u|\ |(-\Delta)^{\sigma/2}\varphi|\, dx\, dt \qquad (11)\\
&+ \int\limits_{\mathbb{R}^n} |u_0| \left(|(-\Delta)^{\sigma/2}\varphi(0,x)| + |\varphi_t(0,x)| \right) dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Let $\varepsilon > 0$. By applying ε-Young's inequality

$$AB \le \varepsilon A^p + C(\varepsilon,p)B^{p'},\ A \ge 0,\ B \ge 0,\ p + p' = pp',\ C(\varepsilon,p) = (p-1)(\varepsilon\, p^p)^{-1/(p-1)},$$

we obtain the estimation for the first integral as following:

$$\begin{aligned} I_1 &\leq \int\limits_{Q_T} |u|\varphi^{1/p}\varphi^{-1/p}\phi_{T^d}|(\psi^\eta)_{tt}|\,dt\,dx \\ &\leq \varepsilon \int\limits_{Q_T} |u|^p\varphi\,dt\,dx + C\int\limits_{Q_T} \phi_{T^d}\psi^{-\eta/(p-1)}|(\psi^\eta)_{tt}|^{p'}\,dt\,dx. \end{aligned} \tag{12}$$

As $(\psi^\eta)_{tt} = \eta\psi^{\eta-1}\psi_{tt} + \eta(\eta-1)\psi^{\eta-2}|\psi_t|^2$, the inequality (12) becomes

$$I_1 \leq \varepsilon \int\limits_{Q_T} |u|^p\psi\,dt\,dx + C\int\limits_{Q_T} \phi_{T^d}\psi^{\eta-p'}|\psi_{tt}|^{p'}\,dt\,dx + C\int\limits_{Q_T} \phi_{T^d}\psi^{\eta-2p'}|\psi_t|^{2p'}\,dt\,dx. \tag{13}$$

Proceeding similarly as for (13), we get

$$I_2 \leq \varepsilon \int\limits_{Q_T} |u|^p\varphi\,dt\,dx + C\int\limits_{Q_T} \psi^\eta\phi_{T^d}^{-1/(p-1)}|\Delta\phi_{T^d}|^{p'}\,dt\,dx. \tag{14}$$

In the same way, we write

$$I_3 = \int\limits_{\frac{T}{2}}^{T} \frac{1}{(1+t)^\beta} \int\limits_{\mathbb{R}^n} |u|\varphi^{1/p}\varphi^{-1/p}\,|(-\Delta)^{\sigma/2}\phi_{T^d}||(\psi^\eta)_t|\,dx\,dt$$

$$\leq \varepsilon \int\limits_{Q_T} |u|^p\varphi\,dt\,dx + C\int\limits_{\frac{T}{2}}^{T} \frac{1}{(1+t)^{\beta p'}} \int\limits_{\mathbb{R}^n} \phi_{T^d}^{-1/(p-1)}|(-\Delta)^{\sigma/2}\phi_{T^d}|^{p'}\,\psi^{\eta-p'}\,|\psi_t|^{p'}\,dx\,dt. \tag{15}$$

Clearly,

$$\frac{1}{(1+t)^{\beta p'}} \leq C\,T^{-\beta p'}, \qquad \forall\, t \in \left(\frac{T}{2}, T\right),$$

therefore,

$$I_3 \leq \varepsilon \int\limits_{Q_T} |u|^p\varphi\,dt\,dx + C\,T^{-\beta p'} \int\limits_{Q_T} \phi_{T^d}^{-1/(p-1)}|(-\Delta)^{\sigma/2}\phi_{T^d}|^{p'}\,\psi^{\eta-p'}\,|\psi_t|^{p'}\,dt\,dx. \tag{16}$$

In the same manner,

$$\begin{aligned} I_4 &\leq C\int\limits_0^T \frac{1}{(1+t)^{\beta+1}} \int\limits_{\mathbb{R}^n} |u|\varphi^{1/p}\varphi^{-1/p}\,\psi^\eta\,|(-\Delta)^{\sigma/2}\phi_{T^d}|\,dx\,dt \\ &\leq \varepsilon \int\limits_{Q_T} |u|^p\varphi\,dt\,dx + C\int\limits_0^T \frac{1}{(1+t)^{(\beta+1)p'}} \int\limits_{\mathbb{R}^n} \phi_{T^d}^{-1/(p-1)}|(-\Delta)^{\sigma/2}\phi_{T^d}|^{p'}\,\psi^\eta\,dx\,dt. \end{aligned} \tag{17}$$

Finally, it remains only to control the term I_5. By exploiting the identity $\varphi_t(0,x) = \eta\psi_t(0)\phi_{T^d}(x)$, we infer

$$I_5 \leq C\int\limits_{\mathbb{R}^n} |u_0|\left(|(-\Delta)^{\sigma/2}\phi_{T^d}| + |\psi_t(0)|\phi_{T^d}\right)\,dx. \tag{18}$$

Plugging (11) together with (13)-(18) and choosing ε small enough, we deduce that

$$\begin{aligned}
&\int\limits_{Q_T} |u|^p \varphi \, dx \, dt + \int\limits_{\mathbb{R}^n} u_1(x) \phi_{T^d}(x) \, dx \\
&\leq C \int\limits_{Q_T} \phi_{T^d} \psi^{\eta - p'} |\psi_{tt}|^{p'} \, dt \, dx + C \int\limits_{Q_T} \phi_{T^d} \psi^{\eta - 2p'} |\psi_t|^{2p'} \, dt \, dx \\
&\quad + C \int\limits_{Q_T} \psi^{\eta} \phi_{T^d}^{-1/(p-1)} |\Delta \phi_{T^d}|^{p'} \, dt \, dx \\
&\quad + C \, T^{-\beta p'} \int\limits_{Q_T} \phi_{T^d}^{-1/(p-1)} |(-\Delta)^{\sigma/2} \phi_{T^d}|^{p'} \, \psi^{\eta - p'} \, |\psi_t|^{p'} \, dt \, dx \\
&\quad + C \int\limits_0^T \frac{1}{(1+t)^{(\beta+1)p'}} \int\limits_{\mathbb{R}^n} \phi_{T^d}^{-1/(p-1)} |(-\Delta)^{\sigma/2} \phi_{T^d}|^{p'} \, \psi^{\eta} \, dx \, dt \\
&\quad + C \int\limits_{\mathbb{R}^n} |u_0| \left(|(-\Delta)^{\sigma/2} \phi_{T^d}| + |\psi_t(0)| \phi_{T^d} \right) \, dx.
\end{aligned}$$

Taking account of the expression of φ given by (10) and Lemmas 1-3, we infer that

$$\begin{aligned}
&\int\limits_{Q_T} |u|^p \varphi \, dx \, dt + \int\limits_{\mathbb{R}^n} u_1(x) \phi_{T^d}(x) \, dx \\
&\leq C \, T^{-2p'+1+nd} + C \, T^{-2dp'+1+nd} + C \, T^{-\beta p' - p' - \sigma dp' + 1 + nd} \\
&\quad + C \, T^{-\sigma dp' + nd} \int\limits_0^T (1+t)^{-\frac{(\beta+1)\, p}{p-1}} \, dt + C \left(T^{-\sigma d} + T^{-1} \right) \int\limits_{\mathbb{R}^n} |u_0(x)| \, dx.
\end{aligned} \tag{19}$$

Since $0 < \sigma < 2$, we notice that the cases of $\sigma \geq 1 - \beta$ and $\sigma \leq 1 - \beta$ are equivalent to

$$\sigma \geq 1 - \beta, \quad \text{for all } \beta > -1, \qquad \text{and} \qquad \sigma < 1 - \beta, \quad \text{for all } \beta < 1.$$

Therefore, we distinguish two cases:

I. Case: $\sigma \geq 1 - \beta$, for all $\beta > -1$.

In this case, we choose $d = 1$.

Subcritical case $p < \min\{\dfrac{n+1}{(n-1)_+}; \dfrac{n}{(n-\sigma)_+}\}$. Note that,

$$\int\limits_0^T (1+t)^{-\frac{(\beta+1)\, p}{p-1}} \, dt \leq C \begin{cases} T^{1 - \frac{(\beta+1)\, p}{p-1}}, & \text{if } \beta\, p < -1, \\ \ln T, & \text{if } \beta\, p = -1, \\ 1, & \text{if } \beta\, p > -1. \end{cases}$$

We have two cases to distinguish.

a) If $\beta \geq 0$. In this case, we have $\beta\, p > -1$ and so

$$\int\limits_0^T (1+t)^{-\frac{(\beta+1)\, p}{p-1}} \, dt \leq C.$$

Therefore, the Eq.(19) implies

$$\int\limits_{Q_T} |u|^p \varphi \, dx \, dt + \int\limits_{\mathbb{R}^n} u_1(x)\phi_T(x) \, dx$$
$$\leq C \, T^{-2p'+1+n} + C \, T^{-(\beta+1)p'-\sigma p'+1+n} + C \, T^{-\sigma p'+n} + C \left(T^{-\sigma} + T^{-1}\right) \int\limits_{\mathbb{R}^n} |u_0(x)| \, dx.$$

Using the fact $\beta \, p > -1 \Longrightarrow -(\beta+1)p' - \sigma p' + 1 + n < -\sigma p' + n$, we conclude that

$$\int\limits_{Q_T} |u|^p \varphi \, dx \, dt + \int\limits_{\mathbb{R}^n} u_1(x)\phi_T(x) \, dx$$
$$\leq C \, T^{-2p'+1+n} + C \, T^{-\sigma p'+n} + C \left(T^{-\sigma} + T^{-1}\right) \int\limits_{\mathbb{R}^n} |u_0(x)| \, dx. \tag{20}$$

Note that, we can easily see that

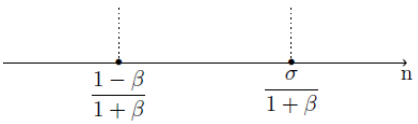
$$-2p' + 1 + n < 0 \Longleftrightarrow p < \frac{n+1}{(n-1)_+} \quad \text{and} \quad -\sigma p' + n < 0 \Longleftrightarrow p < \frac{n}{(n-\sigma)_+}.$$

Letting $T \to \infty$, and using the Lebesgue dominated convergence theorem together with $u_1 \in L^1(\mathbb{R}^n)$, we conclude that

$$\int\limits_{\mathbb{R}^n} u_1(x) \, dx \leq 0.$$

This contradicts with our assumption (6).

b) If $-1 < \beta < 0$. We have three cases for n.



Note that when $n = 1$, we have $n = 1 < \frac{1-\beta}{1+\beta}$.

i) If $n \geq \dfrac{\sigma}{1+\beta}$, then $n \geq \dfrac{1-\beta}{1+\beta}$ and therefore

$$p < \min\left\{\frac{n+1}{n-1}; \frac{n}{n-\sigma}\right\} \leq -\frac{1}{\beta}, \quad \text{for all } n \geq 2,$$

i.e. $\beta \, p > -1$, which implies a contradiction by following the same calculations as in part a).

ii) If $\dfrac{1-\beta}{1+\beta} < n < \dfrac{\sigma}{1+\beta}$ when $\sigma > 1 - \beta$, then we have

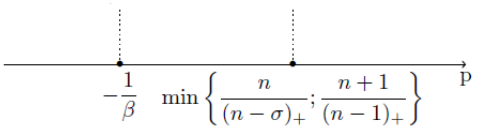
$$n < \frac{\sigma}{1+\beta} < \frac{\sigma}{2-\sigma},$$

and therefore

$$p < \min\left\{\frac{n+1}{n-1}; \frac{n}{n-\sigma}\right\} = \frac{n+1}{n-1} < -\frac{1}{\beta}, \quad \text{for all } n \geq 2,$$

i.e. $\beta \, p > -1$, which implies a contradiction by following the same calculations as in part a).

iii) If $n \leq \dfrac{1-\beta}{1+\beta}$, then $n \leq \dfrac{\sigma}{1+\beta}$, i.e.



• If $p \le -1/\beta$, i.e. $p\beta \ge -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \le C\,\ln T, \qquad \text{for all } T \gg 1.$$

Then, the Eq.(19) implies

$$\int_{Q_T} |u|^p \varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx \le C\,T^{-2p'+1+n}$$

$$+C\,T^{-(\beta+1)p'-\sigma p'+1+n} + C\,T^{-\sigma p'+n}\,\ln T + C\left(T^{-\sigma}+T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\,dx,$$

for all $T \gg 1$. As $p\beta \ge -1 \Longrightarrow (\beta+1)p' \ge 1$, we conclude that

$$\int_{Q_T} |u|^p \varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx$$

$$\le C\,T^{-2p'+1+n} + C\,T^{-\sigma p'+n}\,\ln T + C\left(T^{-\sigma}+T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\,dx, \quad \forall\, T \gg 1. \tag{21}$$

Note that, we can easily see that

$$-2p'+1+n<0 \Longleftrightarrow p<\frac{n+1}{(n-1)_+} \quad \text{and} \quad -\sigma p'+n<0 \Longleftrightarrow p<\frac{n}{(n-\sigma)_+}.$$

Letting $T \to \infty$, using the fact that $\ln T \le T^{\frac{\sigma p'-n}{2}}$ (because $p < \frac{n}{(n-\sigma)_+}$) and the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\,dx \le 0.$$

This contradicts with our assumption (6).

• If $p > -1/\beta$, i.e. $p\beta < -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \le C\,T^{1-\frac{(\beta+1)\,p}{p-1}}, \qquad \text{for all } T > 0.$$

Then, the Eq.(19) implies

$$\int_{Q_T} |u|^p \varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx$$

$$\le C\,T^{-2p'+1+n} + C\,T^{-(\beta+1)p'-\sigma p'+1+n} + C\left(T^{-\sigma}+T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\,dx,$$

for all $T \gg 1$. As $\sigma \ge 1-\beta \Longrightarrow n \le \frac{1-\beta}{1+\beta} \le \frac{\sigma}{2-\sigma}$, we conclude that

$$p < \min\left\{\frac{n+1}{(n-1)_+};\frac{n}{(n-\sigma)_+}\right\} =$$

$$\frac{n+1}{(n-1)_+} \leq \frac{n+1}{(n-\beta-\sigma)_+} \Longrightarrow -(\beta+1)p' - \sigma p' + 1 + n < 0,$$

and

$$p < \min\left\{\frac{n+1}{(n-1)_+}; \frac{n}{(n-\sigma)_+}\right\} = \frac{n+1}{(n-1)_+} \Longleftrightarrow -2p' + 1 + n < 0.$$

Note that $\sigma \geq 1-\beta > 1$, and $2 > \sigma > \sigma + \beta$, therefore

$$\begin{cases} (n-\beta-\sigma)_+ = (n-1)_+ = (n-\sigma)_+ = 0, & \text{when } n = 1, \\ (n-\beta-\sigma)_+ = n-\beta-\sigma,\ (n-1)_+ = n-1,\ (n-\sigma)_+ = n-\sigma, & \text{when } n \geq 2, \end{cases}$$

Letting $T \to \infty$, using the Lebesgue dominated convergence theorem, we conclude that

$$\int\limits_{\mathbb{R}^n} u_1(x)\,dx \leq 0.$$

This contradicts with our assumption (6).

Critical case: $p < \infty$ and $\sigma \geq 1$ when $n = 1$, or $p = \dfrac{n+1}{n-1}$ and $\sigma > \dfrac{2n}{n+1}$ when $n \geq 2$.

Note that when $2 \leq n < \dfrac{\sigma}{2-\sigma}$, we have

$$p = \frac{n+1}{n-1} < \frac{n}{n-\sigma} \quad \text{i.e.} \quad -\sigma p' + n < 0,$$

and when $\sigma \geq 1$ and $n = 1$, we also have $-\sigma p' + n < 0$.
We have two cases to distinguish.

a) If $\beta \geq 0$. In this case, we have $\beta\, p > -1$ and so

$$\int\limits_0^T (1+t)^{-\dfrac{(\beta+1)\,p}{p-1}}\,dt \leq C.$$

From the subcritical case, we can see that we have

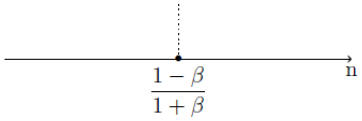
$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{22}$$

On the other hand, by applying Hölder's inequality instead of Young's inequality, we get

$$\begin{aligned} \int\limits_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx \quad \leq \quad & C\left(\int\limits_{\frac{T}{2}}^{T}\int\limits_{\mathbb{R}^n} |u|^p\varphi\,dx\,dt\right)^{1/p} + C\left(\int\limits_0^T \int\limits_{|x|\geq T} |u|^p\varphi\,dx\,dt\right)^{1/p} \\ & + C\,T^{-\sigma p'+n} + C\left(T^{-\sigma} + T^{-1}\right)\int\limits_{\mathbb{R}^n} |u_0(x)|\,dx. \end{aligned}$$

Letting $T \longrightarrow \infty$ and taking into consideration (22), we get a contradiction.

b) If $-1 < \beta < 0$. We have three cases for n.



Note that when $n = 1$, we have $n = 1 < \dfrac{1-\beta}{1+\beta}$.

i) If $n > \dfrac{1-\beta}{1+\beta}$, then

$$p = \frac{n+1}{n-1} < -\frac{1}{\beta},$$

i.e. $\beta\, p > -1$, which implies a contradiction by following the same calculations as in part a).

ii) If $n = \frac{1-\beta}{1+\beta}$, then

$$p = \frac{n+1}{n-1} = -\frac{1}{\beta},$$

i.e. $\beta\, p = -1$, and therefore

$$\int_0^T (1+t)^{-\dfrac{(\beta+1)\,p}{p-1}}\, dt \le C\, \ln T, \qquad \text{for all } T > 0.$$

From the subcritical case (20), we can see easily have

$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{23}$$

On the other hand, by applying Hölder's inequality instead of Young's inequality and using $p = \dfrac{n+1}{n-1}$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\phi_T(x)\, dx \;\le\; & C\left(\int_{\frac{T}{2}}^{T}\int_{\mathbb{R}^n} |u|^p\varphi\, dx\, dt\right)^{1/p} + C\left(\int_0^T\int_{|x|\ge T} |u|^p\varphi\, dx\, dt\right)^{1/p} \\ & + C\, T^{-\sigma p'+n}\ln T + C\left(T^{-\sigma}+T^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\, dx.\end{aligned}$$

Letting $T \longrightarrow \infty$ and taking into consideration (23) and the fact that $\ln T \le T^{\dfrac{\sigma p'-n}{2}}$ (because $p < \dfrac{n}{n-\sigma}$), we get

$$\int_{\mathbb{R}^n} u_1(x)\, dx \le 0.$$

This contradicts with our assumption (6).

iii) If $n < \dfrac{1-\beta}{1+\beta}$, then

$$p = \frac{n+1}{(n-1)_+} > -\frac{1}{\beta},$$

i.e. $p\beta < -1$. In this case, we change the test function ψ by $\psi(t) = \Psi\left(\dfrac{t}{K^{-1}T}\right)$ where $K \ge 1$ is independent of T. Then

$$\int_0^{K^{-1}T} (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\, dt \le C\, K^{-1+(\beta+1)p'}\, T^{1-(\beta+1)p'}, \qquad \text{for all } T > 0.$$

From the subcritical case (20), we can see easily have

$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{24}$$

On the other hand, by applying Hölder's inequality in the Eq.(19) instead of Young's inequality, we get

$$\int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx \le C\,K^{2p'-1}\left(\int_{\frac{K^{-1}T}{2}}^{K^{-1}T}\int_{\mathbb{R}^n}|u|^p\varphi\,dx\,dt\right)^{1/p}$$

$$+C\,K^{-1}\left(\int_0^{K^{-1}T}\int_{|x|\ge T}|u|^p\varphi\,dx\,dt\right)^{1/p} + C\;K^{-1+(\beta+1)p'}\,T^{-(\beta+1)p'-\sigma p'+1+n}$$

$$+C\left(K^{\sigma}T^{-\sigma}+KT^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx.$$

As $\sigma \ge 1-\beta$ and $2>\sigma>\sigma+\beta$, we conclude that

$$p=\frac{n+1}{n-1}\le\frac{n+1}{n-\beta-\sigma}\Longrightarrow -(\beta+1)p'-\sigma p'+1+n\le 0,\quad \text{when } n\ge 2,$$

and

$$-(\beta+1)p'-\sigma p'+1+n<0,\quad \text{when } n=1,$$

so

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx &\le C\,K^{2p'-1}\left(\int_{\frac{K^{-1}T}{2}}^{K^{-1}T}\int_{\mathbb{R}^n}|u|^p\varphi\,dx\,dt\right)^{1/p}\\ &\quad +C\,K^{-1}\left(\int_0^{K^{-1}T}\int_{|x|\ge T}|u|^p\varphi\,dx\,dt\right)^{1/p}\\ &\quad +C\;K^{-1+(\beta+1)p'}+C\left(K^{\sigma}T^{-\sigma}+KT^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx.\end{aligned}$$

Letting $T\longrightarrow\infty$ and taking into consideration (24), we get

$$\int_{\mathbb{R}^n} u_1(x)\,dx \le C\;K^{-1+(\beta+1)p'}.$$

Letting $K\longrightarrow\infty$ and using $p\beta<-1\Longrightarrow -1+(\beta+1)p'<0$, we infer that

$$\int_{\mathbb{R}^n} u_1(x)\,dx\le 0.$$

This contradicts with our assumption (6).

II. Case: $\sigma<1-\beta$, for all $\beta<1$.
In this case, we take $d=\dfrac{1-\beta}{\sigma}>1$.

Subcritical case $p < \min\left\{\dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}; \dfrac{n}{n-\sigma}\right\}$.

Note that,

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \le C \begin{cases} T^{1-\frac{(\beta+1)\,p}{p-1}}, & \text{if } \beta\,p < -1,\\ \ln T, & \text{if } \beta\,p = -1,\\ 1, & \text{if } \beta\,p > -1. \end{cases}$$

We have two cases to distinguish.

a) If $0 \le \beta < 1$. In this case, we have $\beta\,p > -1$ and so

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \le C.$$

Therefore, the Eq.(19) implies

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx\\ &\le C\,T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\,T^{-\frac{2(1-\beta)}{\sigma}p'+1+\frac{n(1-\beta)}{\sigma}}\\ &\quad + C\,T^{-(1-\beta)p'+\frac{n(1-\beta)}{\sigma}} + C\left(T^{-(1-\beta)}+T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\,dx. \end{aligned}$$

As

$$\sigma < 1-\beta \Longrightarrow -\frac{2(1-\beta)}{\sigma}p' + 1 + \frac{n(1-\beta)}{\sigma} < -2p' + 1 + \frac{n(1-\beta)}{\sigma},$$

we conclude that

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx\\ &\le C\,T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\,T^{(1-\beta)\left[-p'+\frac{n}{\sigma}\right]} + C\left(T^{-(1-\beta)}+T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\,dx, \end{aligned}$$

Note that, we can easily see

$$(1-\beta)\left[-p'+\frac{n}{\sigma}\right] < 0 \Longleftrightarrow p < \frac{n}{(n-\sigma)_+}$$

and

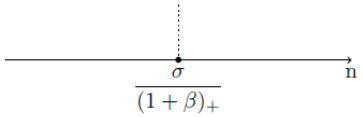
$$-2p'+1+\frac{n(1-\beta)}{\sigma} < 0 \Longleftrightarrow p < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}.$$

Note that $n(1-\beta)-\sigma \ge 1-\beta-\sigma > 0$, for all $n \ge 1$. Letting $T \to \infty$, and using the Lebesgue dominated convergence theorem together with $u_1 \in L^1(\mathbb{R}^n)$, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\,dx \le 0.$$

This contradicts with our assumption (6).

b) If $\beta < 0$. We have two cases for n.



Note that $n(1-\beta)-\sigma \geq 1-\beta-\sigma > 0$ and $\sigma < \beta+1 < 1 \leq n \Longrightarrow n-\sigma > 0$, for all $n \geq 1$.

i) If $n \geq \dfrac{\sigma}{(1+\beta)_+}$, we have

$$p < \min\left\{\frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}; \frac{n}{n-\sigma}\right\} = \frac{n}{n-\sigma} \leq -\frac{1}{\beta},$$

i.e. $\beta\, p > -1$, which implies a contradiction by following the same calculations as in part a).

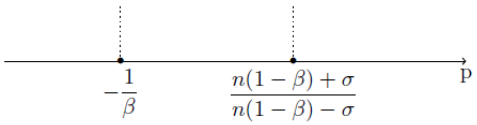
ii) If $n < \dfrac{\sigma}{(1+\beta)_+}$, we get

$$p < \min\left\{\frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}; \frac{n}{n-\sigma}\right\} = \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma},$$

and

$$-\frac{1}{\beta} < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma},$$

i.e.



• If $p \leq -1/\beta$, i.e. $p\beta \geq -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\, dt \leq C\, \ln T, \qquad \text{for all } T \gg 1.$$

Then, the Eq. (19) implies

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\, dx \\
&\leq C\, T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\, T^{-\frac{2(1-\beta)}{\sigma}p'+1+\frac{n(1-\beta)}{\sigma}} \\
&\quad + C\, T^{-(1-\beta)p'+\frac{n(1-\beta)}{\sigma}}\, \ln T + C\left(T^{-(1-\beta)} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx,
\end{aligned}$$

for all $T \gg 1$. As

$$\sigma < 1-\beta \Longrightarrow -\frac{2(1-\beta)}{\sigma}p' + 1 + \frac{n(1-\beta)}{\sigma} < -2p' + 1 + \frac{n(1-\beta)}{\sigma},$$

we conclude that

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\, dx \\
&\leq C\, T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\, T^{(1-\beta)\left[-p'+\frac{n}{\sigma}\right]}\, \ln T + C\left(T^{-(1-\beta)} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx,
\end{aligned}$$

Note that, we can easily see that

$$(1-\beta)\left[-p' + \frac{n}{\sigma}\right] < 0 \Longleftrightarrow p < \frac{n}{n-\sigma}$$

and

$$-2p' + 1 + \frac{n(1-\beta)}{\sigma} < 0 \Longleftrightarrow p < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}.$$

Letting $T \to \infty$, using the fact that $\ln T \leq T^{\frac{(1-\beta)(p'-\frac{n}{\sigma})}{2}}$ (because $p < \frac{n}{n-\sigma}$) and the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\,dx \leq 0.$$

This contradicts with our assumption (6).

• If $p > -1/\beta$, i.e. $p\beta < -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \leq C\,T^{1-\frac{(\beta+1)\,p}{p-1}}, \qquad \text{for all } T > 0.$$

Then, the Eq.(19) implies

$$\int_{Q_T} |u|^p \varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx$$
$$\leq C\,T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\,T^{-\frac{2(1-\beta)}{\sigma}p'+1+\frac{n(1-\beta)}{\sigma}} + C\left(T^{-(1-\beta)} + T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\,dx.$$

As

$$\sigma < 1-\beta \Longrightarrow -\frac{2(1-\beta)}{\sigma}p' + 1 + \frac{n(1-\beta)}{\sigma} < -2p' + 1 + \frac{n(1-\beta)}{\sigma},$$

we conclude that

$$\int_{Q_T} |u|^p \varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx$$
$$\leq C\,T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\left(T^{-(1-\beta)} + T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\,dx,$$

Note that, we can easily see

$$-2p' + 1 + \frac{n(1-\beta)}{\sigma} < 0 \Longleftrightarrow p < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}.$$

Letting $T \to \infty$, using the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\,dx \leq 0.$$

This contradicts with our assumption (6).

Critical case: $p = \dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}$ and $\sigma > n(1+\beta)$.

Note that when $n(1+\beta) < \sigma$, we have

$$p = \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma} < \frac{n}{n-\sigma} \quad \text{i.e.} \quad -\sigma p' + n < 0.$$

We have two cases to distinguish.

a) If $0 \leq \beta < 1$. In this case, we have $\beta\,p > -1$ and so

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \leq C.$$

From the subcritical case (20), we can see easily that we have

$$u \in L^p((0,\infty);L^p(\mathbb{R}^n)). \tag{25}$$

On the other hand, by applying Hölder's inequality instead of Young's inequality, the Eq. (19) implies

$$\begin{aligned}\int\limits_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx \;\leq\; & C\left(\int\limits_{\frac{T}{2}}^{T}\int\limits_{\mathbb{R}^n}|u|^p\varphi\,dx\,dt\right)^{1/p} + C\left(\int\limits_{0}^{T}\int\limits_{|x|\geq T^d}|u|^p\varphi\,dx\,dt\right)^{1/p}\\ & + C\,T^{(1-\beta)\left[-p'+\frac{n}{\sigma}\right]} + C\left(T^{-(1-\beta)}+T^{-1}\right)\int\limits_{\mathbb{R}^n}|u_0(x)|\,dx.\end{aligned}$$

Letting $T\longrightarrow\infty$ and taking into consideration (25) we get

$$\int\limits_{\mathbb{R}^n} u_1(x)\,dx \leq 0.$$

This contradicts with our assumption (6).

b) If $\beta<0$. As $n(1+\beta)<\sigma \Longrightarrow n<\dfrac{\sigma}{(1+\beta)_+}$, so

$$p=\frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}>-\frac{1}{\beta},$$

i.e. $p\beta<-1$. In this case, we change the test function ψ by $\psi(t)=\Psi\left(\dfrac{t}{K^{-1}T}\right)$ where $K\geq 1$ is independent of T. Then

$$\int\limits_{0}^{K^{-1}T}(1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \leq C\,K^{-1+(\beta+1)p'}\,T^{1-(\beta+1)p'}, \qquad \text{for all } T>0.$$

From the subcritical case (20), we can see easily that we have

$$u \in L^p((0,\infty);L^p(\mathbb{R}^n)). \tag{26}$$

On the other hand, by applying Hölder's inequality in the Eq. (19) instead of Young's inequality and using $p=\dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}$, the Eq. (19) implies

$$\begin{aligned}&\int\limits_{Q_T}|u|^p\varphi\,dx\,dt+\int\limits_{\mathbb{R}^n}u_1(x)\phi_{T^d}(x)\,dx\\ &\leq C\,K^{2p'-1}\left(\int\limits_{\frac{K^{-1}T}{2}}^{K^{-1}T}\int\limits_{\mathbb{R}^n}|u|^p\varphi\,dx\,dt\right)^{1/p} + C\,K^{-1}\left(\int\limits_{0}^{K^{-1}T}\int\limits_{|x|\geq T^d}|u|^p\varphi\,dx\,dt\right)^{1/p}\\ &+C\,K^{-1+(\beta+1)p'}+C\left(K^{1-\beta}T^{-(1-\beta)}+KT^{-1}\right)\int\limits_{\mathbb{R}^n}|u_0(x)|\,dx.\end{aligned}$$

Letting $T\longrightarrow\infty$ and taking into consideration (26), we get

$$\int\limits_{\mathbb{R}^n} u_1(x)\,dx \;\leq\; C\,K^{-1+(\beta+1)p'}.$$

Letting $K \longrightarrow \infty$ and using $p\beta < -1 \Longrightarrow -1 + (\beta + 1)p' < 0$, we infer that

$$\int_{\mathbb{R}^n} u_1(x)\, dx \leq 0.$$

This contradicts with our assumption (6).

□

5. Conclusion

We have established non-existence of the global weak solutions to the Cauchy problem for a semi-linear structurally damped wave equation. Moreover, we have shown the influence of the parameters of the equations on the threshold of the parameter leading to blowing-up solutions.

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References

[1] Chirila, A., Marin, M. Well-posedness for micro-stretch elasticity in the case of mass and thermal diffusion, *TWMS J. Pure Appl. Math.,* V.13, N.2, 2022, pp.191-210.

[2] D'Abbicco, M., Reissig, M. Semilinear structural damped waves, *Math. Meth. Appl. Sci.*, V.37, 2014, pp.1570-1592.

[3] D'Ambrosio, L., Lucente, S. Nonlinear Liouville theorems for Grushin and Tricomi operators, *Journal Differential Equations*, V.193, N.1, 2003, pp.511-541.

[4] Dao, T.A., Fino, A.Z. Blow up results for semi-linear structural damped wave model with nonlinear memory, *Mathematische Nachrichten*, V.295, N.2, 2022, pp.309-322.

[5] Fino, A.Z. Finite time blow up for wave equations with strong damping in an exterior domain, *Mediterranean Journal of Mathematics*, V.17, N.6, 2020, p.21

[6] Fino, A.Z., Hamza, M.A. Blow-up of solutions to semilinear wave equations with a time-dependent strong damping, *J. Evolution Equations & Control Theory*, V.11, N.6, 2022, pp.1955-1966.

[7] Fino, A., Karch, G. Decay of mass for nonlinear equation with fractional Laplacian, *J. Monatsh. Math.*, V.160, 2010, pp.375-384.

[8] Fino A.Z., Kirane, M. Qualitative properties of solutions to a time-space fractional evolution equation, *J. Quarterly of Applied Mathematics*, V.70, N.1, 2012, pp.133-157.

[9] Gecmen, M.Z., Celik, E., Dokuyucu, M.A. Numerical Solution of Volterra integral equations using Hosoya polynomial, *Turkish Journal of Science,* V.6, N.3, 2021, pp.110-117.

[10] Holm, S. Natural occurrence of fractional derivatives in physics, *In 2023 International Conference on Fractional Differentiation and its Applications, IEEE* March, 2023, pp.1-5.

[11] John, F. Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.*, V.28, N.1-3, 1979, pp.235-268.

[12] Kalantarova, V., Aliyeva, G.N. Decay of solutions of damped Kirchhoff and Beam equations, *TWMS J. Pure Appl. Math.,* V.13, N.1, 2022, pp.119-128.

[13] Kato, T. Blow-up of solutions of some nonlinear hyperbolic equations, *Comp. Pure Appl. Math.*, V.33, 1980, pp.501-505.

[14] Mitidieri, E., Pohozaev, S.I. A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, *Proc. Steklov. Inst. Math.*, V.234, 2001, pp.3-383.

[15] Oqielat, M.N., Eriqat, T., Al-Zhour, Z.,El-Ajou, A., Momani, S. Numerical solutions of time-fractional nonlinear water wave partial differential equation via Caputo fractional derivative: an effective analytical method and some applications, *Appl. Comput. Math.,* V.21, N.2, 2022, pp.207-222.

[16] Silvestre, L. Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.*, V.60, N.1, 2007, pp.67-112.

[17] Strauss, W.A. Nonlinear scattering theory at low energy, *J. Functional Analysis*, V.41, N.1, 1981, pp.110-133.

[18] Yordanov, B., Zhang, Q.S. Finite time blow up for critical wave equations in high dimensions, *J. Functional Analysis*, V.231, N.2, 2006, pp.361-374.

[19] Zhang, Q.S. A blow up result for a nonlinear wave equation with damping: the critical case, *C. R. Acad. Sci. Paris*, V.333, N.2, 2001, pp.109-114.
[20] Zhou, Y. Blow up of solutions to semilinear wave equations with critical exponent in high dimensions, *Chin. Ann. Math. Ser. B*, V.28, N.2, 2007, pp.205-212.



Mokhtar Kirane - is a Professor at the Department of Mathematical Sciences in Khalifa University of Sciences and Technology, Abu Dhabi, United Arab Emirates. He received his M.Sc. degree in Mathematics from the University of Nice in France in 1981 and his Ph.D. degree in Mathematics from University of Pierre et Marie Curie, in 1983. His research interests are differential equations, fractional calculus, inverse problems, image processing and biomathematics.



Ahmad Ziad Fino - is an Associate Professor at the Department of Mathematics in the American University of the Middle East, Egaila, Kuwait. He received his M.Sc. degree in Applied Mathematics from the Lebanese University in Lebanon in 2007 and his Ph.D. degree in Applied Mathematics from La Rochelle University, France in 2010. His research interests are qualitative and quantitative properties in partial differential equations involving fractional Laplacian, memory term and variable coefficients.



Sebti Kerbal - is an Associate Professor at the Department of Mathematics in Sultan Qaboos University, Muscat, Oman. He received his M.Sc. degree in Mathematics from the University of Ottawa in Canada in 1993 and his Ph.D. degree in Applied Mathematics from Ottawa University in 1998. His research interests are differential equations and fractional calculus.



Aymen Laadhari - is an Assistant Professor at the Department of Mathematics in Khalifa University of Science and Technology, United Arab Emirates. He earned his M.Sc. and Ph.D. degrees in Applied Mathematics from Grenoble University, France, in 2007 and 2011, respectively. His research interests are encompass mathematical modeling, numerical analysis, scientific computing, computational fluid dynamics and differential equations.