Appl. Comput. Math., V.23, N.4, 2024, pp.462-473
DOI: 10.30546/1683-6154.23.4.2024.462

THE IMPACT OF SELECTIVE CONVEXITY ON THE STRUCTURES OF PRODUCTION FUNCTIONS IN DATA ENVELOPMENT ANALYSIS

A.P. AFANASYEV[1,2], V.E. KRIVONOZHKO[3,4,*], A.V. LYCHEV[3]

Abstract. The Data Envelopment Analysis (DEA) technology was proposed at the end of the last century. This technology enables managers to find important characteristics of production units behavior. It also permits to analyze various situations with production units, to compute important characteristics of production units: Marginal rates, efficiency scores, and so on. The Free Disposal Hull (FDH) technology appeared almost at the same time. The production possibility set (PPS) of the DEA models is convex, and the PPS of the FDH models is non-convex. For this reason, optimization methods are widely used for applications and development of the DEA models. Recently, a lot of models have appeared in the literature that expand capabilities of the DEA technologies. Moreover, the notion of selective convexity was introduced in the DEA models. This technology allows managers to consider problems in which there are variables such as averages, ratios, percentages, etc. In this paper, constructions of production functions are considered in DEA models with selective convexity. Computational experiments with real-life datasets confirmed that the proposed algorithms work efficiently.

Keywords: Data Envelopment Analysis, Free Disposal Hull, Production Function, Selective Convexity.

AMS Subject Classification: 90B30, 90C11, 90C90.

1. Introduction

Data envelopment analysis (DEA) is a recognized non-parametric approach used to evaluate the efficiency of production units [7,32]. This technology was initially proposed in papers [4,6] at the end of the last century. DEA not only empowers managers to identify critical characteristics of production unit behaviors, but it also facilitates the analysis of various scenarios that may arise within the business context: Merging some units, split-up production units, and other potential situations. This technology permits also to compute some important characteristics of production units development: Scale elasticity, marginal rates of transformation, marginal rates of substitution, efficiency scores, and so on [27, 28, 31]. To analyze the behavior of production units, DEA utilizes optimization models. The Free Disposal Hull (FDH) technology appeared almost at the same time in paper [8]. However, constraints of the DEA models are convex; production possibility set (PPS) of the FDH model is non-convex. For this reason, optimization methods and software are widely applied for applications of the DEA models. In [10], the author considered DEA technology from a geometrical concept. Such approach permits now

[1]Centre for Distributed Computing, Institute for Information Transmission Problems, Russian Academy of Sciences, 36-1 Nakhimovsky Prospekt, Moscow 117218, Russia, e-mail: apa@iitp.ru
[2]Peoples Friendship University (RUDN University), 6 Miklukho-Maklaya st., 117198 Moscow, Russia
[3]National University of Science and Technology "MISIS", 4 Leninsky ave., Bldg. 1, Moscow 119049, Russia, e-mail: lychev@misis.ru
[4]Federal Research Center "Computer Science and Control", Russian Academy of Sciences, 44-2 Vavilov st., Moscow 119333, Russia
*Corresponding author e-mail: krivonozhkove@mail.ru
Manuscript received 06 October 2023.

applications for nonparametric frontiers. In the paper [9], the author generated a two-phase algorithm for the DEA models to find the frame of any one of four production possibility sets.

The DEA and FDH technologies are two extreme cases for investigation and application of production units. However, a lot of other models have appeared recently for investigation of production units [18, 19, 23]. Such models expand capabilities of the DEA technology by considering the notion of selective convexity introduced in [29]. This technology allows managers to consider problems in which there are variables such as averages, ratios, percentages, integers, etc.

The earlier discussion on the suitability of including ratio data in DEA models was mainly focused on the selection the most suitable DEA models from the existing options [11,13,15]. The subsequent research has indicated that the use of ratio measures is generally inconsistent with the production assumption that permits the possibility of convex combinations of production units [12,24]. In particular, the authors of the paper [12] suggested to utilize the volume variables (numerator and denominator) that were employed to calculate the ratio. However, this approach may not always be feasible, primarily due to the absence of initial data. The other interesting approaches [25, 26] on dealing with ratio data and their limitations are discussed in [20].

In [20], the authors explicitly formulate new production assumptions that allow managers to introduce ratio measures. The proposed ratio CRS (R-CRS) and ratio VRS (R-VRS) models give managers theoretical foundation to use DEA approach in different practical applications where variables are both volume and ratio measures. In their next paper [21], the authors considered how the DEA models with ratio measures can be solved. It is worth noting that R-VRS model extends the approach that was given in [30], which is used in practical applications for handling of environmental variables [5, 14, 16].

Visualization methods are widely used in many areas of human activity. Visualization of the PPS helps to discover some specific features of production units behavior which cannot be revealed by other methods. At the same time, there exist very few studies devoted to the visualization in the DEA area. In [1], multidimensional visualization methods were described for the DEA models. In [2, 3], some methods were proposed for construction of input and output isoquants in DEA models with selective convexity. In [22], the scale characteristics of the variable returns to scale production frontiers with ratio inputs and outputs are considered.

In this paper, constructions of production functions are considered for the DEA models with selective convexity. Two cases are taken:

(1) production function is built on the basis of convex variables (inputs and outputs);
(2) production function is built on the basis of non-convex variables (inputs and outputs).

Computational experiments confirmed that the proposed algorithms work efficiently with real-life datasets.

2. Background

Let a set of production units be (X_j, Y_j), $j = 1, \ldots, n$, where the vector of outputs $Y_j = (y_{1j}, \ldots, y_{rj}) \geq 0$ is generated from the vector of inputs $X_j = (x_{1j}, \ldots, x_{mj}) \geq 0$. All data are assumed to be nonnegative, but at least one component of every input and output vector is positive.

The notion of selective convexity can be found in [29]. Assume that the input and output sets I and O have the following partition

$$I = I^C \cup I^{NC}, \quad O = O^C \cup O^{NC},$$

where the subsets I^C and I^{NC}, and O^C and O^{NC}, are mutually disjoint.

The subsets I^C and O^C are called the subsets of volume inputs and outputs (volume measures). The complementary subsets $I^{NC} = I \setminus I^C$ and $O^{NC} = O \setminus O^C$ are denoted, respectively, as ratio inputs and outputs (ratio measures).

Assume that the set I^C contains the input components from 1 to m', at the same time the set I^{NC} contains the inputs from $(m'+1)$ to m. Then, it is evident that any vector of inputs can be represented in the form $X = (X^C, X^{NC})$, where X^C is the vector of the first m' components of X, and X^{NC} is the vector of the last components of X.

Next, let us assume that the set O^C contains the output components from 1 to r', and the set O^{NC} contains the outputs from $(r'+1)$ to r. Therefore, any vector of outputs can be written in the form $Y = (Y^C, Y^{NC})$.

The production possibility set T of the technology with selective convexity can be determined by the following postulates [29]:

(P1) Feasibility of observed data. Unit $(X_j, Y_j) \in T$ for any $j = 1, \ldots, n$.
(P2) Free disposability. $(X, Y) \in T$, and $Y \geq Y' \geq 0$ and $X' \leq X$ implies $(X', Y') \in T$.
(P3) Selective convexity. Let $(X', Y') \in T$ and $(X'', Y'') \in T$. Assume that $(X')_i = (X'')_i$ for all $i \in I^{NC}$, and $(Y')_r = (Y'')_r$ for all $r \in O^{NC}$. Then, for any $\lambda \in [0, 1]$, the unit $\lambda(X', Y') + (1-\lambda)(X'', Y'') \in T$.

The production possibility set T, which satisfies (P1)–(P3) can be written in the following form:

$$T = \left\{ (X^C, X^{NC}, Y^C, Y^{NC}) \geq 0 \;\middle|\; \sum_{j=1}^{n} X_j^C \lambda_j \leq X^C,\ \sum_{j=1}^{n} Y_j^C \lambda_j \geq Y^C,\ \text{if } \lambda_j \geq 0, \right.$$
$$\left. \text{then } X_j^{NC} \leq X^{NC} \text{ and } Y_j^{NC} \geq Y^{NC},\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0,\ j = 1, \ldots, n \right\}. \quad (1)$$

Now, define the vector $\hat{Y}$ of dimension $(r - r')$; components of this vector are as follows:

$$\hat{Y}_k = \max\big\{ (Y_j)_k \;\big|\; j = 1, \ldots, n \big\}, \quad \forall k \in O^{NC}.$$

Each component $\hat{Y}_k$ is equal to the maximum level of output across all observed production units. This vector is used in the description of Algorithm 2 below.

We also need constant $\underline{\lambda}$ in our description of algorithms. This is a small parameter, its value is less than any other parameter used in Algorithm 2.

Next, define the two-dimensional plane in the input-output space E^{m+r} as follows:

$$\mathrm{Pl}(X_o, Y_o, d_1, d_2) = (X_o, Y_o) + \alpha d_1 + \beta d_2,$$

where $Z_o = (X_o, Y_o) \in T$, α and β are any real numbers, directions d_1, $d_2 \in E^{m+r}$ and d_1 is not parallel to d_2. The plane goes through point Z_o in E^{m+r} and is spanned by vectors d_1 and d_2.

Now, define the intersection of the frontier with the two-dimensional plane as in [17]

$$\mathrm{Sec}(X_o, Y_o, d_1, d_2) = \big\{ (X, Y) \;\big|\; (X, Y) \in \mathrm{Pl}(X_o, Y_o, d_1, d_2) \cap \mathrm{Bound}\, T \big\}, \quad (2)$$

where $\mathrm{Bound}\, T$ is a set of boundary points of T.

A production function for unit (X_o, Y_o) is a section (2) with the following directions: $d_1 = (X_o, 0) \in E^{m+r}$, $d_2 = (0, Y_o) \in E^{m+r}$.

Now, we are ready to describe the algorithm for constructing a production function.

3. Algorithm for construction of the convex production function

Algorithm 1. *Construction of the convex production function.*

Step 1. Find a leftmost vertex of the production function for unit (X_o, Y_o).

Step 1a. Solve the following optimization problem.

$$\begin{aligned}
&\min \theta_1 \\
&\sum_{j=1}^{n} X_j^C \lambda_j \le \theta_1 X_o, \\
&\sum_{j=1}^{n} Y_j^C \lambda_j \ge \tau_1 Y_o \\
&(X_j^{NC} - X_o^{NC})\lambda_j \le 0,\ j = 1, \dots, n, \\
&(Y_j^{NC} - Y_o^{NC})\lambda_j \ge 0,\ j = 1, \dots, n, \\
&\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \ge 0,\ j = 1, \dots, n,
\end{aligned} \tag{3}$$

where τ_1 and θ_1 are free variables.

Step 1b. Let θ_1^* be optimal objective of (3). Solve the following problem.

$$\begin{aligned}
&\max \tau_1 \\
&\sum_{j=1}^{n} X_j^C \lambda_j \le \theta_1^* X_o, \\
&\sum_{j=1}^{n} Y_j^C \lambda_j \ge \tau_1 Y_o \\
&(X_j^{NC} - X_o^{NC})\lambda_j \le 0,\ j = 1, \dots, n, \\
&(Y_j^{NC} - Y_o^{NC})\lambda_j \ge 0,\ j = 1, \dots, n, \\
&\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \ge 0,\ j = 1, \dots, n,
\end{aligned} \tag{4}$$

where τ_1 is a free variable.

Let $Z_1^1 = (\theta_1^* X_o^C, X_o^{NC}, \tau_1^* Y_o^C, Y_o^{NC})$, where θ_1^* and τ_1^* are optimal objectives of problems (3) and (4), respectively.

Step 2. Find the rightmost vertex of the production function for unit (X_o, Y_o).

Step 2a. Solve the following optimization problem.

$$\begin{aligned}
&\max \tau_2 \\
&\sum_{j=1}^{n} X_j^C \lambda_j \le \theta_2 X_o, \\
&\sum_{j=1}^{n} Y_j^C \lambda_j \ge \tau_2 Y_o \\
&(X_j^{NC} - X_o^{NC})\lambda_j \le 0,\ j = 1, \dots, n, \\
&(Y_j^{NC} - Y_o^{NC})\lambda_j \ge 0,\ j = 1, \dots, n, \\
&\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \ge 0,\ j = 1, \dots, n,
\end{aligned} \tag{5}$$

where τ_2 and θ_2 are free variables.

Step 2b. Let τ_2^* be optimal objective of (5). Solve the following problem.

$$\begin{aligned}
&\min \theta_2 \\
&\sum_{j=1}^{n} X_j^C \lambda_j \le \theta_2 X_o, \\
&\sum_{j=1}^{n} Y_j^C \lambda_j \ge \tau_2^* Y_o \\
&(X_j^{NC} - X_o^{NC})\lambda_j \le 0,\ j = 1, \dots, n, \\
&(Y_j^{NC} - Y_o^{NC})\lambda_j \ge 0,\ j = 1, \dots, n, \\
&\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \ge 0,\ j = 1, \dots, n,
\end{aligned} \tag{6}$$

where τ_2 is a free variable.

Let $Z_2^1 = (\theta_2^* X_o^C, X_o^{NC}, \tau_2^* Y_o^C, Y_o^{NC})$, where θ_2^* and τ_2^* are optimal objectives of problems (5) and (6), respectively.

Step 3. Set $l := 1$, $k := 1$, $i_1 := 1$, $i_2 := 2$. Create flow F_k^l with points $Z_{i_1}^l = Z_1^1$, $Z_{i_2}^l = Z_2^1$ of production possibility set T. Define set $M = \{Z_1^1, Z_2^1\}$.

Step 4. Perform the following operations. Take any unprocessed flow F_k^l, solve optimization problem of the following type:

$$\begin{aligned}
&\max \beta_1 \\
&(Z_{i_1}^l + Z_{i_2}^l)/2 + \beta_1 d_1 + \tau d_2 \in T,
\end{aligned} \tag{7}$$

where β_1 and τ are scalar variables, vector d_1 is perpendicular to the vector d_2, it lies in the plane of the section, and is directed to the upper left corner of a production function, vector $d_2 = Z_{i_1}^l - Z_{i_2}^l$.

If the optimal objective value of problem (7) is $\beta_1^* > 0$, then start new flows $F_{k_1}^{l+1}$ and $F_{k_2}^{l+1}$ and solve optimization sub-problems.

Flow $F_{k_1}^{l+1}$ contains points

$$Z_{i_1}^{l+1} = Z_{i_1}^l, \quad Z_{i_2}^{l+1} = (Z_{i_1}^l + Z_{i_2}^l)/2 + \beta_1^* d_1 + \tau^* d_2,$$

where β_1^* and τ^* are optimal values of variables in (7).

Flow $F_{k_2}^{l+1}$ contains points

$$Z_{i_3}^{l+1} = Z_{i_2}^{l+1}, \quad Z_{i_4}^{l+1} = Z_{i_2}^l, \quad d_2 = Z_{i_4}^{l+1} - Z_{i_3}^{l+1},$$

where d_1 is perpendicular to the vector d_2 .

If the optimal solution value in (7) is $\beta_1^* \le 0$, then points $Z_{i_1}^l$ and $Z_{i_2}^l$ are angular points of the segment of production function. Include these points to the set of corner points M. Flow F_k^l is deleted from the list of flow tasks.

Step 5. Set $l := l + 1$. If exist unprocessed flows F_k^l, then go to the Step 4, else go to the Step 6.

Step 6. Points of set M are angular points of production function. Connect adjacent pair points by line segments. At last, add a vertical line after the first point, and a horizontal line starting from the last point. This completes construction of the production function.

Fig.1 illustrates the construction of production function with the help of the algorithm. At first, the angular points Z_1^1 and Z_2^1 are found by solving models (3)–(6). Then, the flow F_1^1 containing these two points is started. At the next steps, point Z_2^2 will be found with the help of solving problem (7); and the flow F_1^1 will be split into two flows F_1^2 and F_2^2, that will have vertices Z_1^1 and Z_2^2 for flow F_1^2 and vertices Z_2^2 and Z_2^1 for flow F_2^2.

After this, the computations are repeated until all segments of the production function are found.

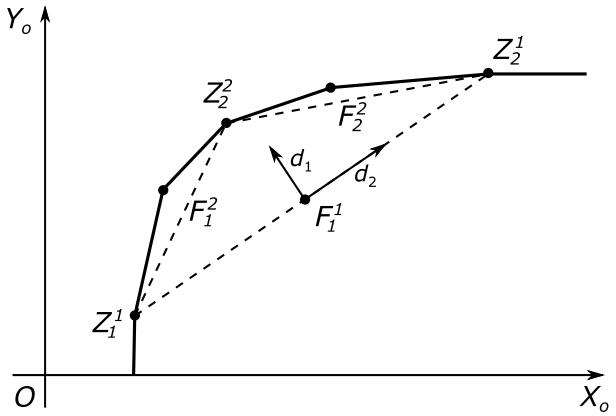


FIGURE 1. Construction of the convex production function.

Taking into account everything said above, we can formulate the following result.

Assertion 1. *Algorithm 1 constructs a convex production function for production possibility set (1) in a finite number of steps.*

4. Algorithm for construction of the non-convex production function

Algorithm 2.

Step 1. Find a leftmost vertex of the production function for unit (X_o, Y_o). Solve the following problem:

$$\begin{aligned}
& \min \theta \\
\text{subject to } & \sum_{j=1}^{n} X_j^C \lambda_j \le X_o^C, \\
& \sum_{j=1}^{n} Y_j^C \lambda_j \ge Y_o^C, \\
& X_j^{NC} \delta_j \le \theta X_o^{NC},\ j = 1, \dots, n, \\
& \tau Y_o^{NC} - Y_j^{NC} \le \hat{Y}(1 - \delta_j),\ j = 1, \dots, n, \\
& \underline{\lambda} \delta_j \le \lambda_j \le \delta_j,\ j = 1, \dots, n, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0,\ \delta_j \in \{0, 1\},\ j = 1, \dots, n, \\
& \tau, \theta \text{ are free variables.}
\end{aligned} \tag{8}$$

Set $\alpha_1 := \theta^*$, $l := 1$.

Step 2. Find two adjacent angular points on the production function. This requires solving two optimization problems in the following steps.

a) Solve the problem:

$$\begin{aligned} & \max \tau \\ \text{subject to } & \sum_{j=1}^{n} X_j^C \lambda_j \le X_o^C, \\ & \sum_{j=1}^{n} Y_j^C \lambda_j \ge Y_o^C, \\ & X_j^{NC} \delta_j \le \alpha_l X_o^{NC},\ j = 1, \dots, n, \\ & \tau Y_o^{NC} - Y_j^{NC} \le \hat{Y}(1 - \delta_j),\ j = 1, \dots, n, \\ & \underline{\lambda} \delta_j \le \lambda_j \le \delta_j,\ j = 1, \dots, n, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \ge 0,\ \delta_j \in \{0, 1\},\ j = 1, \dots, n, \\ & \tau \text{ is a free variable.} \end{aligned} \tag{9}$$

Set $\beta_l := \tau^*$.

b) Solve the problem:

$$\begin{aligned} & \min \theta \\ \text{subject to } & \sum_{j=1}^{n} X_j^C \lambda_j \le X_o^C, \\ & \sum_{j=1}^{n} Y_j^C \lambda_j \ge Y_o^C, \\ & X_j^{NC} \delta_j \le \theta X_o^{NC},\ j = 1, \dots, n, \qquad (10a) \\ & \tau Y_o^{NC} - Y_j^{NC} \le \hat{Y}(1 - \delta_j),\ j = 1, \dots, n, \\ & \underline{\lambda} \delta_j \le \lambda_j \le \delta_j,\ j = 1, \dots, n, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \tau \ge \beta_l (1 + \varepsilon), \qquad (10b) \\ & \lambda_j \ge 0,\ \delta_j \in \{0, 1\},\ j = 1, \dots, n, \\ & \tau, \theta \text{ are free variables.} \end{aligned}$$

Here, ε is a small parameter.

If the solution of the problem (10) is infeasible, then go to Step 3.

Else set $l := l + 1$, $\alpha_l := \theta^*$, $\beta_l := \beta_{l-1}$. Set $l := l + 1$, $\alpha_l := \alpha_{l-1}$, and go to the beginning of Step 2.

Step 3. Stop.

Points (α_l, β_l), $l = 1, \dots, L$, are angular points of the stepwise production function of the model with selective convexity for unit (X_o, Y_o), where L is a number of angular points of production function.

Fig.2 illustrates production function with selective convexity on the two-dimensional plane determined by X_o^{NC} and Y_o^{NC}. Point A in the figure is an angular point of the production function and point A' does not belong to the set T. According to the step 2b of Algorithm 2, the trajectory point is moving along the segment $A'B'$ until it reaches the point B' that belongs

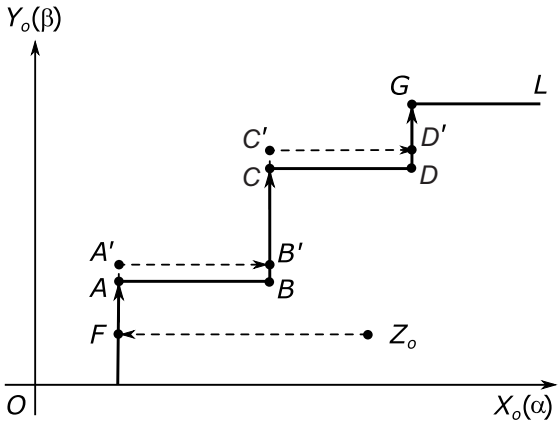


FIGURE 2. Construction of the non-convex production function.

to the set T. The segment AB is parallel to the segment $A'B'$, and this segment belongs to the set T according to the postulate P2.

The model (10) is a mixed-integer linear programming problem, where variables δ_j take values 0 or 1. So, when trajectory point moves along segment AB (or $A'B'$), variable τ takes constant value, and only when trajectory point reaches point B (or B'), and δ_j changes its value from 0 to 1 (or vice versa) the solution point changes its trajectory and goes vertically up, until the solution point reaches the point C (or C'). Next, the solution point moves along all segments until it reaches the ray GL. So, the production function with selective convexity has stepwise form and the following assertion is valid.

Assertion 2. *Algorithm 2 constructs a non-convex production function for production possibility set (1) in a finite number of steps.*

Proof. Every optimal objective value θ^* of the problem (10) gives us a length of the step, see Fig.2. The number of optimal objective values of problems (10) is finite under any value of parameter β_l, that determines a lower bound of variable τ according to (10b). Variable τ determines the feasible values of variables δ_j. Optimal objective value θ^* is found on the basis of optimal values δ_j^* according to (10a). Since δ_j takes values 0 or 1, then the number of optimal objective values of (10) is finite, what does not depends on τ^*.

Since the problems (8), (9), and (10) are mixed-integer linear programming problems, the number of iterations for their solution with the help of finite methods is also finite. Thus, the Algorithm 2 constructs a production function in a finite number of steps.

5. Computational experiments

We conducted computational experiments to test our algorithms for construction of production functions in the DEA models with selective convexity. For this purpose, we took dataset on national research universities in Russia, which included 26 units. The dataset was for 2011 year. Min, max and mean statistics are presented in Table 1.

As inputs, we took the following variables:
x_1^C — research and academic staff;
x_2^{NC} — average income (per research and academic staff);
x_3^{NC} — ratio of research income to total income;
As outputs, the following variables were taken:
y_1^C — number of students studying under priority directions of development (PDD);
y_2^{NC} — share of graduates employed in a specialty area;
y_3^C — number of students in additional professional education programs of the PDD;
y_4^{NC} — average number of papers in Web of Science, Scopus, and RSCI (per student).

TABLE 1. Descriptive statistics.

Variable	Min	Max	Mean	St. dev.
x_1^C	536	2701.165	1355.882	555.7691
x_2^{NC}	1.18	3.271833	2.138693	0.663743
x_3^{NC}	0.036424	0.468207	0.245307	0.102233
y_1^C	3445	14334.52	8156.616	2747.184
y_2^{NC}	0.613406	1.526	0.865491	0.190053
y_3^C	9.731	2709	459.5555	548.6029
y_4^{NC}	0.282	2.549	0.776564	0.488595

Fig.3 presents production functions, where as inputs we took x_1, and as outputs y_1 and y_3, all these variables are convex. Curve 1 in the figure corresponds to the DEA model with selective convexity. Curve 2 in the figure was constructed for the same variables (x_1, y_1, and y_3), however it was the BCC model.

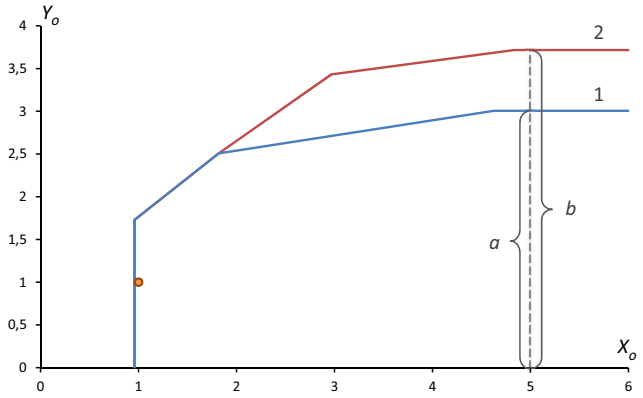


FIGURE 3. Production functions for convex variables.

In Fig.4, production functions for the BCC model and for the model with selective convexity are presented. Curve 1 is constructed for the BCC model with selective convexity, where as inputs we took (x_2, x_3) and as outputs variables (y_2, y_4) were taken. Curve 2 is the DEA model with convex variables, which are (x_2, x_3) and (y_2, y_4).

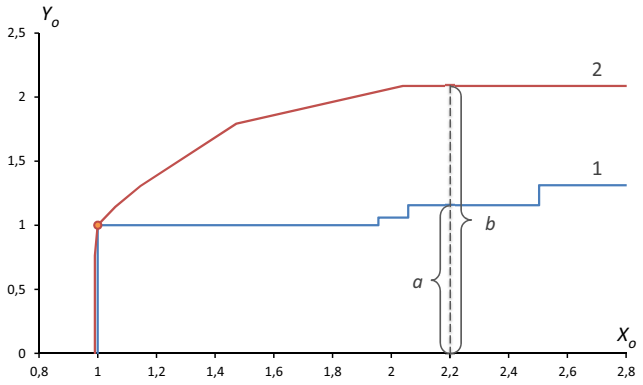


FIGURE 4. Production functions for the DEA model with selective convexity.

Even fluent sight on the figures shows rather evidently that functions differ from each other quite significantly.

We can measure this difference. For Fig.3, the relative difference is $(b-a)/a = (3.7165 - 3.006)/3.006 = 0.2364$, (23.64 %), for Fig.4 the relative difference is $(b-a)/a = (2.0876 - 1.1523)/1.1523 = 0.8061$, (80.61 %). So, we can see rather clear that models with relative convexity presented us more reliable results, 23.64 % in the first case and 80.61 % in the second case.

These simple results show us quite obviously the necessity to improve and to develop new models.

6. Conclusion

In [29], it was shown that some variables in the DEA models may have ratio measures in addition to the usual volume measures. Such ratio measures include averages, ratios, percentages, integers, etc. In [20, 21], new theoretical foundations were presented for the DEA models where variables may have ratio and volume measures. In this paper, we considered methods for constructions of production functions in the DEA models with selective convexity.

In the DEA models, the dependencies between variables are not explicitly specified. Important characteristics of unit's activities have to be determined by indirect methods by solving additional problems what makes it difficult for managers and analysts to use DEA models.

In our papers [1,2], we have managed to show that for analysis of the DEA models, it is possible to construct explicit dependencies between variables based on the construction of sections of multidimensional efficient frontier, or, in other words, to visualize the efficient frontier.

It is worth noting that visualization plays a huge role in the science and practice of the mankind. For example, the invention of the microscope in the 17th century allow scientists to study the secrets of the microcosm and discover new laws used now in biology, medicine, electronics, etc. However, very few methods are used now in the analysis of large-scale socio-economic and industrial systems.

In this paper, construction of production functions are considered for two cases: a) production function is built on the basis of volume variables (inputs and outputs); and b) production function is constructed on the basis of ratio variables (inputs and outputs).

So, it is quite natural to develop an algorithm for construction of production functions on the basis of mixed types of variables (volume and ratio).

Our computational experiments with real-life dataset confirmed that the proposed methods work efficiently and reliably.

Moreover, we can present with the help of our visualization methods that including ratio measures in the DEA models give us significantly more precise results.

7. Acknowledgments

This work was supported by the Russian Science Foundation (Project No.23-11-00197) https://rscf.ru/en/project/23-11-00197/.

References

[1] Afanasyev, A.P., Krivonozhko, V.E., Frsund, F.R., Lychev, A.V. Multidimensional visualization of data envelopment analysis models, *Data Envel.Anal. J.,* V.5, N.2, 2021, pp.339-361.

[2] Afanasyev, A.P., Krivonozhko, V.E., Lychev, A.V., Sukhoroslov, O.V. Constructions of input and output isoquants in DEA models with selective convexity, *Appl. Comput. Math.,* V.21, N.3, 2022, pp.317-328.

[3] Afanasiev, A.P., Krivonozhko, V.E., Lychev, A.V., Sukhoroslov, O.V. Convex isoquants in DEA models with selective convexity, *Automat. Rem. Contr.,* 2024, V.85, N.1, pp.79-89.

[4] Banker, R.D., Charnes, A., Cooper, W.W. Some models for estimating technical and scale efficiency in data envelopment analysis, *Manag. Sci.,* V.30, N.9, 1984, pp.1078-1092.
[5] Blackburn, V., Brennan, S., Ruggiero, J. Measuring efficiency in Australian schools: A preliminary analysis, *Socio-Econ. Plan. Sci.,* V.48, N.1, 2014, pp.4-9.
[6] Charnes, A., Cooper, W.W., Rhodes, E. Measuring the efficiency of decision making units, *Eur. J. Operat. Res.,* V.2, N.6, 1978, pp.429-444.
[7] Cooper, W.W., Seiford, L.M., Tone, K. Data envelopment analysis, A *Comprehensive Text with Models, Applications, References and DEA-Solver Software,* 2nd Ed. New York: Springer Science and Business Media, 2007.
[8] Deprins, D., Simar, L., Tulkens, H. Measuring labor efficiency in post offices, *The Performance of Public Enterprises: Concepts and Measurements,* Marchand M., Pestieau P., Tulkens H. (Eds.), Chapter 10, 1984, pp.243-268.
[9] Dulá, J.H. An Algorithm for data envelopment analysis, *INFORMS J. Comput.,* V.23, N.2, 2011, pp.284-296.
[10] Dulá, J.H. A geometrical approach for generalizing the production possibility set in DEA, *J. Oper. Res. Soc.,* V.60, N.11, 2009, pp.1546-1555.
[11] Dyson, R.G., Allen, R., Camanho, A.S., Podinovski, V.V., Sarrico, C., Shale E.A. Pitfalls and protocols in DEA, *Eur. J. Operat. Res.* V.132, N.2, 2001, pp.245-259.
[12] Emrouznejad, A., Amin, G.R. DEA models for ratio data: Convexity consideration, *Appl. Math. Model.,* V.33, N.1, 2009, pp.486-498.
[13] Golany, B., Thore, S. The economic and social performance of nations: Efficiency and returns to scale, *Socio-Econ. Plan. Sci.*, V.31, N.3, 1997, pp.191-204.
[14] Haelermans, C., Ruggiero, J. Estimating technical and allocative efficiency in the public sector: A nonparametric analysis of Dutch schools, *Eur. J. Oper. Res.*, V.227, N.1. 2013, pp.174-181.
[15] Hollingsworth, B., Smith, P. Use of ratios in data envelopment analysis, *Appl. Econ. Lett.*, V.10, N.11, 2003, pp.733-735.
[16] Johnson, A., Ruggiero, J. Nonparametric measurement of productivity and efficiency in education, *Ann. Oper. Res.,* V.221, N.1, 2014, pp.197-210.
[17] Krivonozhko, V.E., Utkin, O.B., Volodin, A.V., Sablin, I.A., Patrin M.V. Constructions of economic functions and calculations of marginal rates in DEA using parametric optimization methods, *J. Oper. Res. Soc.,* V.55, N.10, 2004, pp.1049-1058.
[18] Mehdiloo M., Papaioannou, G., Podinovski V.V. Efficient targets and reference sets in selectively convex technologies, *Omega,* V.129, 2024, pp.103155.
[19] Olesen, O.B., Papaioannou, G., Podinovski, V.V. Constant returns-to-scale production technologies with fixed ratio inputs and outputs, *J. Prod. Anal.,* 2024.
[20] Olesen, O.B., Petersen, N.C., Podinovski, V.V. Efficiency analysis with ratio measures, *Eur. J. Oper. Res.,* V.245, N.2, 2015, pp.446-462.
[21] Olesen, O.B., Petersen, N.C., Podinovski, V.V. Efficiency measures and computational approaches for data envelopment analysis models with ratio inputs and outputs, *Eur. J. Oper. Res.,* V.261, N.2, 2017, pp.640-655.
[22] Olesen, O.B., Petersen, N.C., Podinovski, V.V. Scale characteristics of variable returns-to-scale production technologies with ratio inputs and outputs, *Ann. Oper. Res.*, V.318, 2022, pp.383-423.
[23] Olesen, O.B., Petersen, N.C., Podinovski, V.V. The structure of production technologies with ratio inputs and outputs, *J. Prod. Anal.,* V.57, 2022, pp.255-267.
[24] Olesen, O.B., Petersen, N.C. Target and technical efficiency in DEA: Controlling for environmental characteristics, *J. Prod. Anal.,* V.32, N.1, 2009, pp.27-40.
[25] Paradi, J.C., Vela, S.A., Zhu, H. Adjusting for cultural differences, a new DEA model applied to a merged bank, *J. Prod. Anal.,* V.33, N.2, 2009, pp.109-123.
[26] Paradi, J.C., Vela, S., Yang, Z. Assessing bank and bank branch performance, *Handbook on Data Envelopment Analysis,* Cooper W.W., Seiford L.M., Zhu J. (Eds.), Boston: Kluwer Academic Publishers, 2004, pp.349-400.
[27] Podinovski, V.V., Chambers, R.G., Atici, K.B., Deineko, I.D. Marginal values and returns to scale for nonparametric production frontiers, *Oper. Res.,* V.64, N.1, 2016, pp.236-250.
[28] Podinovski, V.V. Returns to scale in convex production technologies, *Eur. J. Operat. Res.,* V.258, N.3, 2017, pp.970-982.
[29] Podinovski, V.V. Selective convexity in DEA models, *Eur. J. Operat. Res.,* V.161, N.2, 2005, pp.552-563.
[30] Ruggiero, J. On the measurement of technical efficiency in the public sector, *Eur. J. Oper. Res.,* V.90, N.3, 1996, pp.553-565.
[31] Sahoo, B.K., Tone, K. Scale elasticity in non-parametric DEA approach, *Data Envelopment Analysis: A Handbook of Models and Methods,* Zhu J. (Ed.) Springer, 2015, pp.269-290.
[32] Thanassoulis, E., Portela, M.C.S., Despić, O. Data envelopment analysis: The mathematical programming approach to efficiency analysis, *The Measurement of Productive Efficiency and Productivity Growth,* Fried H.O., Lovell C.A.K., Schmidt S.S. (Eds.). New York: Oxford University Press, 1984, pp.251-420.

Alexander Afanasyev - for his biography, see Appl. Comput. Math., V.21, N.3, 2022, p.327.

Vladimir Krivonozhko - for his biography, see Appl. Comput. Math., V.14, N.1, 2015, p.327.

Andrey Lychev - for his biography, see Appl. Comput. Math., V.14, N.1, 2015, p.328.