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PICTURE FUZZY INTERACTIONAL AGGREGATION OPERATORS VIA STRICT TRIANGULAR NORMS AND APPLICATIONS TO MULTI-CRITERIA DECISION MAKING

X.X. WU[1,2], Z. ZHU[3], T. WANG[4], X. ZHANG[5,*]

Abstract. This paper mainly studies the structure of the closed operational laws in the picture fuzzy numbers (PFNs) and proposes efficient picture fuzzy multi-criteria decision making (MCDM) methods. We first introduce an admissible order for the PFNs and prove that all PFNs form a complete lattice under this order. Then, we give some specific examples to show the non-closeness of some existing picture fuzzy aggregation operators. To ensure the closeness of the operational laws in the PFNs, we construct a new class of picture fuzzy operators based on strict triangular norms, which consider the interaction between the positive degrees (negative degrees) and the neutral degrees. Based on these new operators, we obtain the picture fuzzy interactional weighted average (PFIWA) operator, the picture fuzzy interactional weighted geometric (PFIWG) operator, and the picture fuzzy mean interactional weighted average and geometric (PFMIWAG) operator. They are proved to be monotonous, idempotent, and bounded. We also establish a novel MCDM method under the picture fuzzy environment applying the PFMIWAG operator. Furthermore, the practical applicability and the comparative analysis with other MCDM methods are furnished to show the advantages of the proposed MCDM method.

Keywords: Picture Fuzzy Set, Aggregation Operator, Linear Order, Triangular Norm, Multi-Criteria Decision Making.

AMS Subject Classification: 03E72.

1. Introduction

For the voting process, the voters present multiple selection opinions as follows: 'yes', 'no', 'abstain', and 'refusal'. This case cannot be exactly described by fuzzy sets [59], intuitionistic fuzzy sets (IFSs) [3], and hesitant fuzzy sets [45]. To deal with such issues, being a direct generalization of fuzzy sets and IFSs, the concept of picture fuzzy sets (PFSs) as a modern fundamental concept about computational intelligence problems was introduced in [8, 9], which are characterized by three membership functions: a positive, a neutral, and a negative, meeting the requirement that the sum of the positive, neutral, and negative membership degree of every point is between 0 and 1. In particular, every triple of positive, neutral, and negative membership degrees for the PFSs was called a picture fuzzy number (PFN) in [21]. Then, some basic operations for the PFSs, including conjunctions, disjunctions, negations, and implications, were defined in [8, 10]. The authors also described the picture fuzzy (PF) distance, the picture fuzzy

[1]School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, China
[2]Zhuhai College of Science and Technology, Zhuhai, Guangdong 519041, China
e-mail: wuxinxing5201314@mail.gufe.edu.cn
[3]Business School, Sichuan University, Chengdu, Sichuan 610000, China, e-mail: zhuzhiyi2019@163.com
[4]School of Mathematics, Chengdu Normal University, Chengdu, Sichuan 611130, China
e-mail: 061063@cdnu.edu.cn
[5]Department of Mathematics, Shandong University, Weihai, Shandong 264209, China
*Corresponding author e-mail: xu_zhang_sdu@mail.sdu.edu.cn
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relation, and Zadeh's extension principle in the PFSs. In analogy to the IFSs, considering the uncertainty of three membership degrees for the PFSs, in [8] the author introduced the concept of interval-valued PFSs (IvPFSs).

To solve some practical multi-criteria decision-making (MCDM) problems described by PFSs, the PF aggregation operator, which monotonously transforms a collection of PF inputs into a single PFN, is an effective tool. In addition, the PF similarity measure, the methods of PF TOPSIS, PF AHP, and PF TODIM-VIKOR were also applied for PF MCDM problems [1, 2, 13, 18, 28, 39, 41, 43]. In order to get logical and influential aggregation results, we must work out the following basic problems: (1) Construct an admissible order for ranking all PFNs (according to our knowledge, until now there is no general methodology that ranks any two PFNs); (2) Define a 'good' aggregation operator, meeting the monotonicity and closeness. The monotonicity ensures that the higher the score of each criterion, the better the aggregation result. The closeness ensures that the aggregation output is still a PFN; and so, our evaluation criteria are under the unified framework.

Up to the present, many scholars have devoted themselves to the investigation of aggregation operators under PF environment (see [2, 12, 14, 21–23, 27, 33, 37, 38, 44, 47, 48, 51, 52, 60]) or intuitionistic fuzzy environment (see [11, 19, 35, 42, 46, 50, 54, 57]) and their applications (see [15, 25, 29, 32, 36, 49]). For example, in [51, 52] the authors developed some PF aggregation operators, including the PF weighted average operator, the PF weighted geometric operator, and the PF Hamacher (Heronian mean) aggregation operator. They used these operators to clarify the PF MCDM. Through triangular norms (t-norms) and triangular conorms (t-conorms), the authors of [2, 14] developed the weighted average, the ordered weighted average, and the hybrid average aggregation operators to deal with the MCDM for the PFSs. Meanwhile, in [21] the authors provided various PF aggregation operators using Dombi t-norms and t-conorms with a view to concerning the PF MCDM. Moreover, in [23] the authors constructed the PF Einstein (ordered) weighted operators by means of the Einstein sum and product. Considering the relationship among the criterions in MCDM problems, some interactional aggregation operators were used to model these possible relationships [4, 26, 30, 34, 47, 48, 53]. In [53], the authors introduced the Heronian mean aggregation operators in the PF situations where each criterion always has a relationship with others. Furthermore, in [26] the authors proposed the interactional operational laws for the PFNs, which get hold of the interaction between the positive degrees and the neutral degrees (negative degrees) for any two PFNs. In [4], the authors extended the Bonferroni mean to the PFNs. They also introduced the Bonferroni mean, the normalized weighted Bonferroni mean, and the ordered weighted Bonferroni mean for the PFSs. However, all above aggregation operators have a common shortcoming (see Section 3): They do not need to satisfy the fact that the sum of the three degrees cannot exceed one. Thus, such operators are not closed in the PFNs. This trouble mainly arises from the three-dimensional degree structure of the PFNs. The aggregation outcomes can be larger than one whenever two degrees calculated by t-conorms exist. Since there is no linear order for ranking all PFNs, all ordered aggregation operators are of little or no value for decision-making.

Inspired by the above results on PF aggregation operators, this paper is devoted to constructing the closed operational laws in the PFNs and establishing effective PF MCDM methods. In Sections 2 and 3, by introducing a total order $\preccurlyeq_{\mathrm{W}}$ for the PFNs, we prove that it is an admissible order. We also demonstrate that all PFNs form a complete lattice under this order. In Section 4, we consider some examples to explain the non-closeness of operational laws in [2, 4, 7, 14, 21, 23, 26, 51, 52, 56]. Intending to construct the closed operational laws in the PFNs, we introduce in Section 5 a new kind of PF operators via strict t-norms. These operators consider the interaction between the positive degrees (negative degrees) and the neutral degrees for any two PFNs. Note that the positive degrees and the negative degrees are more significant than the neutral degrees for evaluation and decision-making. Hence, we only consider the interactional operations that occur in the position of the neutral degrees. Meanwhile, we prove

that our operators are closed in the PFNs and obtain their basic properties. In Section 6, by using new PF operators, we obtain the PF interactional weighted average operator $\mathrm{PFIWA}_{T,\mathbf{\Omega}}$, the PF interactional weighted geometric operator $\mathrm{PFIWG}_{T,\mathbf{\Omega}}$, and the PF mean interactional weighted average and geometric operator $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$ induced by a t-norm T. In addition, we show that they have the monotonicity, idempotency, boundedness, shift-invariance, and homogeneity. In Section 7, through the operator $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$, a novel MCDM method under the PF environment is developed. Finally, we present an illustrative example involving the investment options of a commercial company that shows the decision procedure of our proposed MCDM method. We also describe the comparative analysis among the operators induced by other closed operational laws presented in [22, 33, 44, 48] and some famous t-norms, such as the algebraic product $T_{\mathbf{P}}$, Schweizer-Sklar t-norm $T_\gamma^{\mathbf{SS}}$, Hamacher t-norm $T_\gamma^{\mathbf{H}}$, Frank t-norm $T_\gamma^{\mathbf{F}}$, Dombi t-norm $T_\gamma^{\mathbf{D}}$, and Aczél-Alsina t-norm $T_\gamma^{\mathbf{AA}}$.

2. Preliminaries

2.1. Order and lattice.

Definition 1. [20, Definition 1.1.3] A *partial order* on a set X is a binary relation $\preccurlyeq$ satisfying the properties:

(1) $a \preccurlyeq a$ (reflexivity).
(2) From $a \preccurlyeq b$ and $b \preccurlyeq a$, it follows $a = b$ (antisymmetry).
(3) From $a \preccurlyeq b$ and $b \preccurlyeq c$, it follows $a \preccurlyeq c$ (transitivity).

The set X with this partial order is denoted by $(X, \preccurlyeq)$, which is called a *poset*.

Let $(L, \preceq)$ be a poset. An *upper bound* of a subset X of L is an element $u \in L$ satisfying $x \preceq u$ for all $x \in X$. If an upper bound u of X is not larger than any other upper bound of X, then it is said to be the *smallest upper bound* or *supremum* of X, denoted by $\bigvee X$ or $\sup X$. Similarly, the *greatest lower bound* or *infimum* of a subset X of L can be defined, denoted by $\bigwedge X$ or $\inf X$. Given any pair of elements, denote by

$$x \vee y = \sup\{x, y\} \text{ and } x \wedge y = \inf\{x, y\}.$$

Definition 2. [5] A *lattice* is a poset satisfying that every pair of two elements have the infimum as well as the supremum. A lattice is *bounded* if it has the bottom and the top elements.

Definition 3. [16, Definition O-2.1] A lattice L is called *complete* if, for any subset A of L, the greatest lower bound and the smallest upper bound of A exist. A totally ordered complete lattice is called a *complete chain*.

2.2. Triangular norm.

Definition 4. [24, Definition 1.1], [40] A function $T : [0,1]^2 \longrightarrow [0,1]$ is called a *triangular norm* (*t-norm*) if, for any x_1, x_2, $x_3 \in [0,1]$, the following are satisfied:

(T1) $T(x_1, x_2) = T(x_2, x_1)$ (commutativity);
(T2) $T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3)$ (associativity);
(T3) $T(x_1, x_3) \leq T(x_2, x_3)$ for $x_1 \leq x_2$ (monotonicity);
(T4) $T(x_1, 1) = x_1$ (neutrality).

A function $S : [0,1]^2 \longrightarrow [0,1]$ is called a *t-conorm* if, for any x_1, x_2, $x_3 \in [0,1]$, it satisfies (Tl)–(T3) and (S4) described by:

(S4) $S(x_1, 0) = x_1$ (neutrality).

The duality expression of t-norms and t-conorms is the following result.

TABLE 1. Basic t-norms

Name	Formula
Łukasiewicz	$T_{\mathbf{LK}}(x,y) = \max\{x+y-1, 0\}$
minimum	$T_{\mathbf{M}}(x,y) = \min\{x,y\}$
algebraic product	$T_{\mathbf{P}}(x,y) = x \cdot y$
drastic product	$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & x,y \in [0,1) \\ \min\{x,y\}, & \text{otherwise} \end{cases}$
nilpotent minimum	$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & x+y \leq 1 \\ \min\{x,y\}, & \text{otherwise} \end{cases}$

Proposition 1. *[24, Proposition 1.15] A binary function $S : [0,1]^2 \longrightarrow [0,1]$ is a t-conorm if and only if there exists a t-norm T such that, for any $(x,y) \in [0,1]^2$,*

$$S(x,y) = 1 - T(1-x, 1-y). \tag{1}$$

The t-norm given by formula (1) is called the *dual t-conorm* of T, analogously, we can give the definition of the dual t-norm of a t-conorm S.

The associativity of t-norms $T : [0,1]^2 \longrightarrow [0,1]$ helps us to define their extensions to n-ary functions $T^{(n)} : [0,1]^n \longrightarrow [0,1]$ in the following way (see [24, Remark 1.10] and [17, Definition 3.23])

$$T^{(n)}(x_1, \ldots, x_{n-1}, x_n) \triangleq T(T^{(n-1)}(x_1, \ldots, x_{n-1}), x_n).$$

If, in particular, we have $x_1 = x_2 = \cdots = x_n = x$, we shall briefly write $x_T^{(n)} = T^{(n)}(x, x, \ldots, x)$. Finally we put, by convention, $x_T^{(0)} = 1$ and $x_T^{(1)} = x$ for any $x \in [0,1]$.

Definition 5. [24, Definition 3.2] Let $f : [a,b] \longrightarrow [c,d]$ be a monotone function, where $[a,b]$ and $[c,d]$ are two closed subintervals of $[-\infty, +\infty]$. The *pseudo-inverse* $f^{(-1)} : [c,d] \longrightarrow [a,b]$ of f is defined by:

$$f^{(-1)}(y) = \sup\{x \in [a,b] \mid (f(x) - y)(f(b) - f(a)) < 0\}.$$

Definition 6. [24, Definition 3.25] A strictly decreasing function $\tau : [0,1] \longrightarrow [0, +\infty]$ is called an *additive generator* of a t-norm T if it is right-continuous in 0, $\tau(1) = 0$, and, for any $(x,y) \in [0,1]^2$, we have $\tau(x) + \tau(y) \in \mathrm{Ran}(\tau) \cup [\tau(0), +\infty]$ and $T(x,y) = \tau^{(-1)}(\tau(x) + \tau(y))$.

Definition 7. [24, Definitions 2.9 and 2.13] A t-norm T is

(i) *strictly monotone*, if $T(x,y_1) < T(x,y_2)$ whenever $x > 0$ and $y_1 < y_2$.
(ii) *Archimedean*, if for any $(x,y) \in (0,1)^2$, there exists some $n \in \mathbb{N}$ such that $x_T^{(n)} < y$.
(iii) *strict*, if it is strictly monotone and continuous.

Remark 1. By [24, Proposition 2.15], it follows that every strict t-norm is Archimedean.

Lemma 1. *[24] For a function $T : [0,1]^2 \longrightarrow [0,1]$, the following are equivalent:*

(i) *T is a strict t-norm;*
(ii) *T has a continuous additive generator τ with $\tau(0) = +\infty$.*

2.3. **Picture fuzzy set.**

Definition 8. [8, Definition 3.1] Let X be the universe of discourse. A *picture fuzzy set* (PFS) P in X is defined as an object with the following form:

$$P = \{\langle x, \mu_P(x), \eta_P(x), \nu_P(x)\rangle \mid x \in X\},$$

where $\mu_P(x)$, $\eta_P(x)$, $\nu_P(x) \in [0,1]$, $\mu_P(x) + \eta_P(x) + \nu_P(x) \leq 1$, and $\pi_P(x) = 1 - (\mu_P(x) + \eta_P(x) + \nu_P(x))$ for any $x \in X$. $\mu_P(x)$, $\eta_P(x)$, and $\nu_P(x)$ denote the *degree of positive membership*, *neutral membership*, and *negative membership* of x in P, respectively. $\pi_P(x)$ is the *degree of refusal membership* of x in P. In particular, the triplet $\langle \mu, \eta, \nu\rangle$ satisfying that $\mu, \eta, \nu \in [0,1]$

and $\mu+\eta+\nu \leq 1$ is called a *picture fuzzy number* (PFN). For convenience, a PFN α is denoted by $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$.

Let $\mathscr{P}$ denote the set of all PFNs, i.e., $\mathscr{P} = \{\langle \mu, \eta, \nu \rangle \mid \mu, \eta, \nu \in [0,1] \text{ and } \mu+\eta+\nu \leq 1\}$. If $\eta_P(x) = 0$, then the PFS reduces to the Atanassov's intuitionistic fuzzy set. Thus, the Atanassov's intuitionistic fuzzy sets are special form of the PFSs.

The following inclusion relation was introduced in [8].

Definition 9. [8] For two PFNs α and β, $\alpha \subseteq \beta$ if and only if $\mu_\alpha \leq \mu_\beta$, $\eta_\alpha \leq \eta_\beta$, and $\nu_\alpha \geq \nu_\beta$.

It can be verified that '$\subseteq$' is a partial order on $\mathscr{P}$. However, the poset $(\mathscr{P}, \subseteq)$ does not form a lattice, since the two PFNs $\langle 1,0,0 \rangle$ and $\langle 0,1,0 \rangle$ does not have a supremum in $\mathscr{P}$.

With a view to comparing two PFNs, in [48] the authors introduced the following comparison laws.

Definition 10. [48, Definition 3.3], [47, Definition 4.2.2] Given a picture fuzzy number $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$, let the *score function* $S(\alpha)$ and the *accuracy function* $H(\alpha)$ of α be defined by $S(\alpha) = \mu_\alpha - \nu_\alpha$ and $H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$. In this case, for any two PFNs α and β,

- if $S(\alpha) < S(\beta)$, then $\alpha \prec \beta$;
- if $S(\alpha) = S(\beta)$, then
  - if $H(\alpha) = H(\beta)$, then $\alpha \sim \beta$;
  - if $H(\alpha) < H(\beta)$, then $\alpha \prec \beta$.

If $\alpha \prec \beta$ or $\alpha \sim \beta$, we denote it by $\alpha \preccurlyeq \beta$.

Notice that the order $\preccurlyeq$ given by Definition 10 is not a linear order on $\mathscr{P}$. Moreover, it is not a partial order on $\mathscr{P}$ since the antisymmetry property is violated. We observe that the degrees of the positive membership and the negative membership are more important than the degree of the neutral membership for the PFNs. Thus, the larger (as the linear order of Atanassov's intuitionistic fuzzy numbers in [57, Definition 3.1]) the degrees of the positive membership and the negative membership, the better the PFN. In the case where the degrees of the positive membership and the negative membership are respectively equal, the higher the degree of the neutral membership, the better the PFN since more comprehensive information is obtained. Based on this idea, we introduce the following linear order for the PFNs on $\mathscr{P}$.

Definition 11. Given a picture fuzzy number $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$, let the *score function* $S(\alpha)$, the *first accuracy function* $H_1(\alpha)$, and the *second accuracy function* $H_2(\alpha)$ of α be defined by $S(\alpha) = \mu_\alpha - \nu_\alpha$, $H_1(\alpha) = \mu_\alpha + \nu_\alpha$, and $H_2(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$, respectively. In this case, for any two PFNs α and β,

- if $S(\alpha) < S(\beta)$, then $\alpha \prec_{\mathrm{W}} \beta$;
- if $S(\alpha) = S(\beta)$, then
  - if $H_1(\alpha) < H_1(\beta)$, then $\alpha \prec_{\mathrm{W}} \beta$;
  - if $H_1(\alpha) = H_1(\beta)$, then
    - if $H_2(\alpha) = H_2(\beta)$, then $\alpha = \beta$;
    - if $H_2(\alpha) < H_2(\beta)$, then $\alpha \prec_{\mathrm{W}} \beta$.

If $\alpha \prec_{\mathrm{W}} \beta$ or $\alpha = \beta$, we denote it by $\alpha \preccurlyeq_{\mathrm{W}} \beta$.

3. Some basic properties of linear order $\preccurlyeq_{\mathrm{W}}$

Example 1. Consider $\alpha = \langle 0.2, 0.2, 0.1 \rangle$ and $\beta = \langle 0.3, 0, 0.2 \rangle$.

(1) If we use the order $\preccurlyeq$ defined in Definition 10, then $S(\alpha) = S(\beta) = 0.1$ and $H(\alpha) = H(\beta) = 0.5$, and thus $\alpha \sim \beta$, which means that we cannot distinguish between α and β by using the order $\preccurlyeq$.

(2) If we use the order $\preccurlyeq_{\mathrm{W}}$ defined in Definition 11, then $S(\alpha) = S(\beta) = 0.1$ and $H_1(\alpha) = 0.3 < 0.5 = H_1(\beta)$, and thus $\alpha \prec_{\mathrm{W}} \beta$. Clearly, this is reasonable.

Theorem 1. *The order $\preccurlyeq_{\mathrm{W}}$ defined in Definition 11 is a total order.*

Proof. Now, we show that, for any given two PFNs $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$ and $\beta = \langle \mu_\beta, \eta_\beta, \nu_\beta \rangle$, if we have the equations: $S(\alpha) - S(\beta) = 0$, $H_1(\alpha) - H_1(\beta) = 0$, and $H_2(\alpha) - H_2(\beta) = 0$, then we must have $\alpha = \beta$. Namely, if the following equalities hold,

$$\begin{cases} \mu_\alpha - \nu_\alpha = \mu_\beta - \nu_\beta, \\ \mu_\alpha + \nu_\alpha = \mu_\beta + \nu_\beta, \\ \mu_\alpha + \eta_\alpha + \nu_\alpha = \mu_\beta + \eta_\beta + \nu_\beta, \end{cases}$$

i.e.,

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_\alpha - \mu_\beta \\ \eta_\alpha - \eta_\beta \\ \nu_\alpha - \nu_\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

then $\mu_\alpha = \mu_\beta$, $\eta_\alpha = \eta_\beta$, and $\nu_\alpha = \nu_\beta$. This holds trivially since $\begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2$.

This, together with the fact that $\preccurlyeq_{\mathrm{W}}$ is a partial order on $\mathscr{P}$, implies the conclusion. □

Remark 2. (1) It is easy to see that the bottom and the top elements of $\mathscr{P}$ are, respectively, $\langle 0, 0, 1 \rangle$ and $\langle 1, 0, 0 \rangle$ under the order $\preccurlyeq_{\mathrm{W}}$.

(2) To our knowledge, until now there is no common methodology that ranks any two arbitrary PFNs.

The concept of admissible order for $L([0,1])$ was introduced in [6], where $L([0,1])$ is the set of all closed subinterval of $[0,1]$. According to them, a binary relation $\preccurlyeq$ on $L([0,1])$ is an *admissible order* if it is a linear order, which refines the usual order on $L([0,1])$, i.e., $[a,b] \preccurlyeq [c,d]$ whenever $a \leq c$ and $b \leq d$ (see [6, Definition 3.1]). Motivated by this, we introduce the following admissible order for the PFNs.

Definition 12. An order $\preccurlyeq$ on $\mathscr{P}$ is said to be an *admissible order* if it is a total order and refines the order $\subseteq$ defined in Definition 9, i.e., it is a total order satisfying that for any $\alpha, \beta \in \mathscr{P}$, $\alpha \subseteq \beta$ implies $\alpha \preccurlyeq \beta$.

Theorem 2. *The order $\preccurlyeq_{\mathrm{W}}$ defined in Definition 11 is an admissible order.*

Proof. It follows from Theorem 1 that $\preccurlyeq_{\mathrm{W}}$ is a total order on $\mathscr{P}$. For two PFNs $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$ and $\beta = \langle \mu_\beta, \eta_\beta, \nu_\beta \rangle$, let $\alpha \subseteq \beta$. By Definition 9, there holds $\mu_\alpha - \mu_\beta \leq 0$, $\eta_\alpha - \eta_\beta \leq 0$, and $\nu_\alpha - \nu_\beta \geq 0$. Then, we have

$$S(\alpha) - S(\beta) = (\mu_\alpha - \nu_\alpha) - (\mu_\beta - \nu_\beta) = (\mu_\alpha - \mu_\beta) + (\nu_\beta - \nu_\alpha) \leq 0.$$

Case 1. For $S(\alpha) - S(\beta) < 0$, we have $S(\alpha) < S(\beta)$. From Definition 11, it follows that $\alpha \prec_{\mathrm{W}} \beta$.

Case 2. For $S(\alpha) - S(\beta) = 0$, we have $(\mu_\alpha - \mu_\beta) + (\nu_\beta - \nu_\alpha) = 0$. This, together with $\mu_\alpha \leq \mu_\beta$ and $\nu_\alpha \geq \nu_\beta$, implies that $\mu_\alpha = \mu_\beta$ and $\nu_\alpha = \nu_\beta$, and thus $H_1(\alpha) = \mu_\alpha + \nu_\alpha = \mu_\beta + \nu_\beta = H_1(\beta)$. By $\eta_\alpha \leq \eta_\beta$, we have $H_2(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha \leq \mu_\beta + \eta_\beta + \nu_\beta = H_2(\beta)$. Therefore, by Definition 11, we have $\alpha \preccurlyeq_{\mathrm{W}} \beta$. □

Theorem 3. *$(\mathscr{P}, \preccurlyeq_{\mathrm{W}})$ is a complete lattice.*

Proof. By the proof of [55, Theorem 4.1], we conclude that this is true. We omit its proof here. □

4. Limitations in existing picture fuzzy aggregation operators

After PFSs were introduced in [9], various operational laws for the PFSs and PFNs were proposed. However, many of these laws violate the condition that the sum of the three degrees should be in the interval $[0,1]$. Thus, they are not closed in $\mathscr{P}$.

4.1. **Operations in [2,14].** By applying the additive generators, the author of [14] introduced the following operational laws for the PFNs and obtained some basic properties.

Definition 13. [14, Definition 4] Let τ be an additive generator of a t-norm T and ζ be an additive generator of the dual t-conorm of T, i.e., $\zeta(x) = \tau(1-x)$. For $\alpha_1 = \langle \mu_1, \eta_1, \nu_1 \rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2 \rangle$, $\alpha = \langle \mu, \eta, \nu \rangle \in \mathscr{P}$, define

(i) $\alpha_1 \oplus \alpha_2 = \left\langle \zeta^{-1}(\zeta(\mu_1) + \zeta(\mu_2)), \tau^{-1}(\tau(\eta_1) + \tau(\eta_2)), \tau^{-1}(\tau(\nu_1) + \tau(\nu_2)) \right\rangle$;
(ii) $\alpha_1 \otimes \alpha_2 = \left\langle \tau^{-1}(\tau(\mu_1) + \tau(\mu_2)), \zeta^{-1}(\zeta(\eta_1) + \zeta(\eta_2)), \zeta^{-1}(\zeta(\nu_1) + \zeta(\nu_2)) \right\rangle$;
(iii) $\lambda \cdot \alpha = \left\langle \zeta^{-1}(\lambda\zeta(\mu)), \tau^{-1}(\lambda\tau(\eta)), \tau^{-1}(\lambda\tau(\nu)) \right\rangle$, $\lambda > 0$;
(iv) $\alpha^{\lambda} = \left\langle \tau^{-1}(\lambda\tau(\mu)), \zeta^{-1}(\lambda\zeta(\eta)), \zeta^{-1}(\lambda\zeta(\nu)) \right\rangle$, $\lambda > 0$.

Theorem 4. *[14, Theorem 1] Let α_1, α_2, and α be three PFNs. Then, $\alpha_1 \oplus \alpha_2$, $\alpha_1 \otimes \alpha_2$, $\lambda \cdot \alpha$, and α^{λ} $(\lambda > 0)$ are PFNs.*

The following example demonstrates that none of the operational laws defined in Definition 13 [14, Definition 4] is closed in $\mathscr{P}$. Consequently, Theorem 4 [14, Theorem 1] does not hold.

Example 2. Define $\tau : [0,1] \longrightarrow [0, +\infty]$ as follows:

$$\tau(x) = \begin{cases} +\infty, & x = 0, \\ -\log_8 x - \frac{1}{3}, & x \in (0, \frac{1}{8}], \\ -\frac{8}{3}x + 1, & x \in [\frac{1}{8}, \frac{1}{4}], \\ -\frac{1}{3}x + \frac{5}{12}, & x \in [\frac{1}{4}, \frac{1}{2}], \\ -\frac{1}{2}(x-1), & x \in [\frac{1}{2}, 1], \end{cases}$$

$\zeta(x) = \tau(1-x)$, and take $\alpha_1 = \alpha_2 = \langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \rangle \in \mathscr{P}$. Clearly, τ is an additive generator of some strict t-norm by Lemma 1.

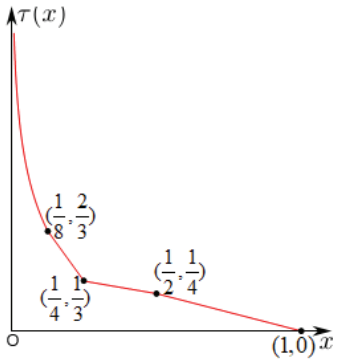


FIGURE 1. Graph of τ.

Applying Definition 13 and by direct calculation, we have

$$\begin{aligned} 2\alpha_1 &= \alpha_1 \oplus \alpha_2 \\ &= \left\langle \zeta^{-1}\left(\zeta\left(\frac{1}{2}\right) + \zeta\left(\frac{1}{2}\right)\right), \tau^{-1}\left(\tau\left(\frac{1}{4}\right) + \tau\left(\frac{1}{4}\right)\right), \tau^{-1}\left(\tau\left(\frac{1}{4}\right) + \tau\left(\frac{1}{4}\right)\right) \right\rangle \\ &= \left\langle \zeta^{-1}\left(2\zeta\left(\frac{1}{2}\right)\right), \tau^{-1}\left(2\tau\left(\frac{1}{4}\right)\right), \tau^{-1}\left(2\tau\left(\frac{1}{4}\right)\right) \right\rangle \\ &= \left\langle \zeta^{-1}\left(\frac{1}{2}\right), \tau^{-1}\left(\frac{2}{3}\right), \tau^{-1}\left(\frac{2}{3}\right) \right\rangle \\ &= \left\langle \zeta^{-1}\left(\frac{1}{2}\right), \frac{1}{8}, \frac{1}{8} \right\rangle = \left\langle \frac{13}{16}, \frac{1}{8}, \frac{1}{8} \right\rangle, \end{aligned}$$

and

$$\begin{aligned}(\alpha_1)^2 &= \alpha_1 \otimes \alpha_2 \\ &= \left\langle \tau^{-1}\left(\tau\left(\frac{1}{2}\right)+\tau\left(\frac{1}{2}\right)\right), \zeta^{-1}\left(\zeta\left(\frac{1}{4}\right)+\zeta\left(\frac{1}{4}\right)\right), \zeta^{-1}\left(\zeta\left(\frac{1}{4}\right)+\zeta\left(\frac{1}{4}\right)\right)\right\rangle \\ &= \left\langle \tau^{-1}\left(2\tau\left(\frac{1}{2}\right)\right), \zeta^{-1}\left(2\zeta\left(\frac{1}{4}\right)\right), \zeta^{-1}\left(2\zeta\left(\frac{1}{4}\right)\right)\right\rangle \\ &= \left\langle \tau^{-1}\left(\frac{1}{2}\right), \zeta^{-1}\left(\frac{1}{4}\right), \zeta^{-1}\left(\frac{1}{4}\right)\right\rangle \\ &= \left\langle \frac{3}{16}, \frac{1}{2}, \frac{1}{2}\right\rangle.\end{aligned}$$

Clearly, $2\alpha_1 = \alpha_1 \oplus \alpha_2 = \left\langle \frac{13}{16}, \frac{1}{8}, \frac{1}{8}\right\rangle \notin \mathscr{P}$ and $(\alpha_1)^2 = \alpha_1 \otimes \alpha_2 = \left\langle \frac{3}{16}, \frac{1}{2}, \frac{1}{2}\right\rangle \notin \mathscr{P}$ since $\frac{13}{16}+\frac{1}{8}+\frac{1}{8} = \frac{17}{16} > 1$ and $\frac{3}{16}+\frac{1}{2}+\frac{1}{2} = \frac{19}{16} > 1$.

The above example demonstrates that $\lambda \cdot \alpha$ and α^λ are not closed in $\mathscr{P}$ for $\lambda > 1$. The following example illustrates that they are also not closed in $\mathscr{P}$ for $\lambda \in (0,1)$. This observation means that we cannot use the operations in Definition 13 to construct aggregation operators for the PFNs.

Example 3. Continuation of Example 2

Given τ and ζ as in Example 2, we choose $\alpha = \langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\rangle \in \mathscr{P}$, and $\beta = \langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\rangle \in \mathscr{P}$. By direct calculation, we have

$$\begin{aligned}\frac{1}{2}\alpha &= \left\langle \zeta^{-1}\left(\frac{1}{2}\zeta\left(\frac{1}{2}\right)\right), \tau^{-1}\left(\frac{1}{2}\tau\left(\frac{1}{4}\right)\right), \tau^{-1}\left(\frac{1}{2}\tau\left(\frac{1}{4}\right)\right)\right\rangle \\ &= \left\langle \zeta^{-1}\left(\frac{1}{8}\right), \tau^{-1}\left(\frac{1}{6}\right), \tau^{-1}\left(\frac{1}{6}\right)\right\rangle \\ &= \left\langle \frac{1}{4}, \frac{2}{3}, \frac{2}{3}\right\rangle,\end{aligned}$$

and

$$\begin{aligned}\beta^{\frac{1}{2}} &= \left\langle \tau^{-1}\left(\frac{1}{2}\tau\left(\frac{1}{4}\right)\right), \zeta^{-1}\left(\frac{1}{2}\zeta\left(\frac{1}{2}\right)\right), \zeta^{-1}\left(\frac{1}{2}\zeta\left(\frac{1}{4}\right)\right)\right\rangle \\ &= \left\langle \tau^{-1}\left(\frac{1}{6}\right), \zeta^{-1}\left(\frac{1}{8}\right), \zeta^{-1}\left(\frac{1}{16}\right)\right\rangle \\ &= \left\langle \frac{2}{3}, \frac{1}{4}, \frac{1}{8}\right\rangle.\end{aligned}$$

Clearly, $\frac{1}{2}\alpha = \left\langle \frac{1}{4}, \frac{2}{3}, \frac{2}{3}\right\rangle \notin \mathscr{P}$ and $(\beta)^{\frac{1}{2}} = \left\langle \frac{2}{3}, \frac{1}{4}, \frac{1}{8}\right\rangle \notin \mathscr{P}$, since $\frac{1}{4}+\frac{2}{3}+\frac{2}{3} = \frac{19}{12} > 1$ and $\frac{2}{3}+\frac{1}{4}+\frac{1}{8} = \frac{25}{24} > 1$.

In [2], the authors introduced the similar operational laws for the PFNs by using the Archimedean t-(co)norm.

Definition 14. [2, Definition 9] Let τ be an additive generator of an Archimedean t-norm, and ζ be an additive generator of the dual t-conorm of T, i.e., $\zeta(x) = \tau(1-x)$. For $\alpha_1 = \langle \mu_1, \eta_1, \nu_1\rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2\rangle$, $\alpha = \langle \mu, \eta, \nu\rangle \in \mathscr{P}$, define

(i) $\alpha_1 \otimes_{\text{As}} \alpha_2 = \left\langle \tau^{-1}(\tau(\mu_1)+\tau(\mu_2)), \tau^{-1}(\tau(\eta_1)+\tau(\eta_2)), \zeta^{-1}(\zeta(\nu_1)+\zeta(\nu_2))\right\rangle$;

(ii) $\alpha^{\lambda_{\text{As}}} = \left\langle \tau^{-1}(\lambda\tau(\mu)), \tau^{-1}(\lambda\tau(\eta)), \zeta^{-1}(\lambda\zeta(\nu))\right\rangle$, $\lambda > 0$.

Example 4. Given τ and ζ as in Example 2, we choose $\alpha = \langle \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \rangle \in \mathscr{P}$. Clearly, the t-norm with the additive generator τ is strict; and thus, it is Archimedean by Lemma 1. By direct calculation, we have that

$$\begin{aligned}
\alpha \otimes_{\text{As}} \alpha &= \left\langle \tau^{-1}\left(\tau\left(\frac{1}{4}\right)+\tau\left(\frac{1}{4}\right)\right), \tau^{-1}\left(\tau\left(\frac{1}{4}\right)+\tau\left(\frac{1}{4}\right)\right), \zeta^{-1}\left(\zeta\left(\frac{1}{2}\right)+\zeta\left(\frac{1}{2}\right)\right)\right\rangle \\
&= \left\langle \tau^{-1}\left(2\tau\left(\frac{1}{4}\right)\right), \tau^{-1}\left(2\tau\left(\frac{1}{4}\right)\right), \zeta^{-1}\left(2\zeta\left(\frac{1}{2}\right)\right)\right\rangle \\
&= \left\langle \tau^{-1}\left(\frac{2}{3}\right), \tau^{-1}\left(\frac{2}{3}\right), \zeta^{-1}\left(\frac{1}{2}\right)\right\rangle \\
&= \left\langle \frac{1}{8}, \frac{1}{8}, \zeta^{-1}\left(\frac{1}{2}\right)\right\rangle = \left\langle \frac{1}{8}, \frac{1}{8}, \frac{13}{16}\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\alpha^{(\frac{1}{2})_{\text{As}}} &= \left\langle \tau^{-1}\left(\frac{1}{2}\tau\left(\frac{1}{4}\right)\right), \tau^{-1}\left(\frac{1}{2}\tau\left(\frac{1}{4}\right)\right), \zeta^{-1}\left(\frac{1}{2}\zeta\left(\frac{1}{2}\right)\right)\right\rangle \\
&= \left\langle \tau^{-1}\left(\frac{1}{6}\right), \tau^{-1}\left(\frac{1}{6}\right), \zeta^{-1}\left(\frac{1}{8}\right)\right\rangle \\
&= \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{4}\right\rangle,
\end{aligned}$$

Clearly, $\alpha \otimes_{\text{As}} \alpha = \langle \frac{1}{8}, \frac{1}{8}, \frac{13}{16} \rangle \notin \mathscr{P}$ and $\alpha^{(\frac{1}{2})_{\text{As}}} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{4} \rangle \notin \mathscr{P}$ since $\frac{1}{8} + \frac{1}{8} + \frac{13}{16} = \frac{17}{16} > 1$ and $\frac{2}{3} + \frac{2}{3} + \frac{1}{4} = \frac{19}{12} > 1$. This example implies that neither of the operational laws defined in Definition 14 [2, Definition 9] is closed in $\mathscr{P}$.

4.2. **Operations in [21].**

Definition 15. [21, Definition 12] Let $\alpha_1 = \langle \mu_1, \eta_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \eta_2, \nu_2 \rangle$ be two PFNs, $\gamma \geq 1$, and $\lambda > 0$. Define

(i) $\alpha_1 \oplus \alpha_2 = \left\langle 1 - \frac{1}{1+\left[\left(\frac{\mu_1}{1-\mu_1}\right)^\gamma + \left(\frac{\mu_2}{1-\mu_2}\right)^\gamma\right]^{1/\gamma}}, \frac{1}{1+\left[\left(\frac{1-\eta_1}{\eta_1}\right)^\gamma + \left(\frac{1-\eta_2}{\eta_2}\right)^\gamma\right]^{1/\gamma}}, \frac{1}{1+\left[\left(\frac{1-\nu_1}{\nu_1}\right)^\gamma + \left(\frac{1-\nu_2}{\nu_2}\right)^\gamma\right]^{1/\gamma}} \right\rangle$;

(ii) $\alpha_1 \otimes \alpha_2 = \left\langle \frac{1}{1+\left[\left(\frac{1-\mu_1}{\mu_1}\right)^\gamma + \left(\frac{1-\mu_2}{\mu_2}\right)^\gamma\right]^{1/\gamma}}, 1 - \frac{1}{1+\left[\left(\frac{\eta_1}{1-\eta_1}\right)^\gamma + \left(\frac{\eta_2}{1-\eta_2}\right)^\gamma\right]^{1/\gamma}}, 1 - \frac{1}{1+\left[\left(\frac{\nu_1}{1-\nu_1}\right)^\gamma + \left(\frac{\nu_2}{1-\nu_2}\right)^\gamma\right]^{1/\gamma}} \right\rangle$;

(iii) $\lambda \cdot \alpha_1 = \left\langle 1 - \frac{1}{1+\left[\lambda\left(\frac{\mu_1}{1-\mu_1}\right)^\gamma\right]^{1/\gamma}}, \frac{1}{1+\left[\lambda\left(\frac{1-\eta_1}{\eta_1}\right)^\gamma\right]^{1/\gamma}}, \frac{1}{1+\left[\lambda\left(\frac{1-\nu_1}{\nu_1}\right)^\gamma\right]^{1/\gamma}} \right\rangle$;

(iv) $(\alpha_1)^\lambda = \left\langle \frac{1}{1+\left[\lambda\left(\frac{1-\mu_1}{\mu_1}\right)^\gamma\right]^{1/\gamma}}, 1 - \frac{1}{1+\left[\lambda\left(\frac{\eta_1}{1-\eta_1}\right)^\gamma\right]^{1/\gamma}}, 1 - \frac{1}{1+\left[\lambda\left(\frac{\nu_1}{1-\nu_1}\right)^\gamma\right]^{1/\gamma}} \right\rangle$.

Remark 3. (i) Notice that Dombi t-norm $T_\gamma^{\mathbf{D}}(x,y) = \frac{1}{1+\left[\left(\frac{1-x}{x}\right)^\gamma \left(\frac{1-y}{y}\right)^\gamma\right]^{1/\gamma}}$ $(\gamma \in (0,+\infty))$ is a strict t-norm, its dual t-conorm $S_\gamma^{\mathbf{D}}(x,y) = 1 - \frac{1}{1+\left[\left(\frac{x}{1-x}\right)^\gamma \left(\frac{y}{1-y}\right)^\gamma\right]^{1/\gamma}}$, and $\alpha_1 \otimes \alpha_2 = \langle T_\gamma^{\mathbf{D}}(\mu_1,\mu_2), S_\gamma^{\mathbf{D}}(\eta_1,\eta_2), S_\gamma^{\mathbf{D}}(\nu_1,\nu_2) \rangle$. Then, we have $T_\gamma^{\mathbf{D}}(\mu_1,\mu_2) + S_\gamma^{\mathbf{D}}(\eta_1,\eta_2) + S_\gamma^{\mathbf{D}}(\nu_1,\nu_2) \geq S_\gamma^{\mathbf{D}}(\eta_1,\eta_2) + S_\gamma^{\mathbf{D}}(\nu_1,\nu_2) \geq \max\{\eta_1,\eta_2\} + \max\{\nu_1,\nu_2\}$, which can exceed one (see Example 5). Thus, the operation $\otimes$ defined in Definition 15 [21, Definition 12] is not closed in $\mathscr{P}$.

(ii) For any $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle \in \mathscr{P}$ with $0 < \mu_\alpha, \eta_\alpha, \nu_\alpha < 1$, by Definition 15, we have $\lim_{\lambda \to 0^+} \lambda \cdot \alpha = \langle 0,1,1 \rangle \notin \mathscr{P}$ and $\lim_{\lambda \to +\infty} \alpha^\lambda = \langle 0,1,1 \rangle \notin \mathscr{P}$, implying that there are λ_1, $\lambda_2 > 0$ such that (1) for any $0 < \lambda < \lambda_1$, $\lambda \cdot \alpha \notin \mathscr{P}$; (2) for any $\lambda > \lambda_2$, $\alpha^\lambda \notin \mathscr{P}$. Thus, neither of the operations $\lambda \cdot \alpha$ and α^λ in Definition 15 [21, Definition 12] is closed in $\mathscr{P}$.

Example 5. Let $\alpha_1 = \langle \frac{1}{4}, \frac{3}{4}, 0\rangle$ and $\alpha_2 = \langle \frac{1}{4}, 0, \frac{3}{4}\rangle$. By Definition 15, we have $\alpha_1 \otimes \alpha_2 = \left\langle \frac{1}{1+3\sqrt[3]{2}}, \frac{3}{4}, \frac{3}{4}\right\rangle$. Clearly, $\alpha_1 \otimes \alpha_2$ is not a PFN since $\frac{1}{1+3\sqrt[3]{2}} + \frac{3}{4} + \frac{3}{4} \geq \frac{3}{2} > 1$ holds for any $\gamma > 0$. Thus, the operation $\otimes$ in Definition 15 is not closed in $\mathscr{P}$.

4.3. **Operations in [23].** The same problem as Remark 3 may occur in [23, Definition 8].

Definition 16. [23, Definition 8] For $\alpha_1 = \langle \mu_1, \eta_1, \nu_1\rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2\rangle \in \mathscr{P}$, define

(i) $\alpha_1 \oplus_\varepsilon \alpha_2 = \left\langle \frac{\mu_1+\mu_2}{1+\mu_1\mu_2}, \frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)}, \frac{\nu_1\nu_2}{1+(1-\nu_1)(1-\nu_2)}\right\rangle$;

(ii) $\alpha_1 \otimes_\varepsilon \alpha_2 = \left\langle \frac{\mu_1\mu_2}{1+(1-\mu_1)(1-\mu_2)}, \frac{\eta_1+\eta_2}{1+\eta_1\eta_2}, \frac{\nu_1+\nu_2}{1+\nu_1\nu_2}\right\rangle$;

(iii) $\lambda_{\cdot\varepsilon}\alpha_1 = \left\langle \frac{(1+\mu_1)^\lambda-(1-\mu_1)^\lambda}{(1+\mu_1)^\lambda+(1-\mu_1)^\lambda}, \frac{2(\eta_1)^\lambda}{(2-\eta_1)^\lambda+(\eta_1)^\lambda}, \frac{2(\nu_1)^\lambda}{(2-\nu_1)^\lambda+(\nu_1)^\lambda}\right\rangle$;

(iv) $(\alpha_1)^\lambda = \left\langle \frac{2(\mu_1)^\lambda}{(2-\mu_1)^\lambda+(\mu_1)^\lambda}, \frac{(1+\eta_1)^\lambda-(1-\eta_1)^\lambda}{(1+\eta_1)^\lambda+(1-\eta_1)^\lambda}, \frac{(1+\nu_1)^\lambda-(1-\nu_1)^\lambda}{(1+\nu_1)^\lambda+(1-\nu_1)^\lambda}\right\rangle$.

Remark 4. (i) Notice that Einstein product $T^{\mathbf{E}}(x,y) = \frac{xy}{1+(1-x)(1-y)}$ is a strict t-norm, its dual t-conorm $S^{\mathbf{E}}(x,y) = \frac{x+y}{1+xy}$, and $\alpha_1 \otimes_\varepsilon \alpha_2 = \langle T^{\mathbf{E}}(\mu_1,\mu_2), S^{\mathbf{E}}(\eta_1,\eta_2), S^{\mathbf{E}}(\nu_1,\nu_2)\rangle$. Then, we have $T^{\mathbf{E}}(\mu_1,\mu_2) + S^{\mathbf{E}}(\eta_1,\eta_2) + S^{\mathbf{E}}(\nu_1,\nu_2) \geq S^{\mathbf{E}}(\eta_1,\eta_2) + S^{\mathbf{E}}(\nu_1,\nu_2) \geq \max\{\eta_1,\eta_2\} + \max\{\nu_1,\nu_2\}$, which can exceed one (see Example 6). Thus, the operation $\otimes_\varepsilon$ defined in Definition 16 is not closed in $\mathscr{P}$.

(ii) Consider an arbitrary $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha\rangle \in \mathscr{P}$ with $0 < \mu_\alpha, \eta_\alpha, \nu_\alpha < 1$. Then, by Definition 16, we have $\lim_{\lambda\to 0^+} \lambda_{\cdot\varepsilon}\alpha = \langle 0,1,1\rangle \notin \mathscr{P}$ and $\lim_{\lambda\to+\infty} \alpha^\lambda = \langle 0,1,1\rangle \notin \mathscr{P}$. These facts imply that there are $\lambda_1, \lambda_2 > 0$ such that (1) for any $0 < \lambda < \lambda_1$, $\lambda_{\cdot\varepsilon}\alpha \notin \mathscr{P}$; (2) for any $\lambda > \lambda_2$, $\alpha^\lambda \notin \mathscr{P}$. Hence, neither of the operations $\lambda \cdot \alpha$ and α^λ in Definition 16 [23, Definition 8] is closed in $\mathscr{P}$.

Example 6. Let $\alpha_1 = \langle \frac{1}{4}, \frac{3}{4}, 0\rangle$ and $\alpha_2 = \langle \frac{1}{4}, 0, \frac{3}{4}\rangle$. By Definition 16, we have $\alpha_1 \otimes_\varepsilon \alpha_2 = \left\langle \frac{1}{25}, \frac{3}{4}, \frac{3}{4}\right\rangle$. Clearly, $\alpha_1 \otimes \alpha_2$ is not a PFN duing to $\frac{1}{25} + \frac{3}{4} + \frac{3}{4} \geq \frac{3}{2} > 1$. Thus, the operation $\otimes_\varepsilon$ in Definition 16 is not closed in $\mathscr{P}$.

Proposition 2. *[23, Corollary 1] Let α be a PFN. Then, for any $\lambda \in \mathbb{R}^+$, $\lambda_{\cdot\varepsilon}\alpha$ is a PFN.*

By Remark 4 (ii), we observe that [23, Corollary 1] does not hold. So, Theorem 2 and 3 that are given [23] are also not valid. Considering $\alpha = \langle 0, \frac{1}{2}, \frac{1}{2}\rangle$, it follows from Definition 16 that, for any $\lambda > 0$, $\lambda_{\cdot\varepsilon}\alpha = \left\langle 0, \frac{2\cdot\left(\frac{1}{2}\right)^\lambda}{\left(\frac{3}{2}\right)^\lambda+\left(\frac{1}{2}\right)^\lambda}, \frac{2\cdot\left(\frac{1}{2}\right)^\lambda}{\left(\frac{3}{2}\right)^\lambda+\left(\frac{1}{2}\right)^\lambda}\right\rangle$. In particular, when taking $\lambda = \frac{1}{2}$, we obtain $\left(\frac{1}{2}\right)_{\cdot\varepsilon}\alpha = \left\langle 0, \frac{2}{\sqrt{3}+1}, \frac{2}{\sqrt{3}+1}\right\rangle$. Since $0 + \frac{2}{\sqrt{3}+1} + \frac{2}{\sqrt{3}+1} = \frac{4}{\sqrt{3}+1} > 1$, it is not a PFN.

4.4. **Operations in [51, 52].** Following the operations for the IFSs [57, 58], the author of the paper [51] introduced the following operational laws of the PFNs.

Definition 17. [51, Definition 8] For $\alpha_1 = \langle \mu_1, \eta_1, \nu_1\rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2\rangle \in \mathscr{P}$, define

(1) $\alpha_1 \wedge \alpha_2 = \langle \min\{\mu_1,\mu_2\}, \max\{\eta_1,\eta_2\}, \max\{\nu_1,\nu_2\}\rangle$;
(2) $\alpha_1 \vee \alpha_2 = \langle \max\{\mu_1,\mu_2\}, \min\{\eta_1,\eta_2\}, \min\{\nu_1,\nu_2\}\rangle$;
(3) $\alpha_1 \oplus \alpha_2 = \langle \mu_1+\mu_2-\mu_1\mu_2, \eta_1\eta_2, \nu_1\nu_2\rangle$;
(4) $\alpha_1 \otimes \alpha_2 = \langle \mu_1\mu_2, \eta_1+\eta_2-\eta_1\eta_2, \nu_1+\nu_2-\nu_1\nu_2\rangle$;
(5) $\lambda \cdot \alpha_1 = \langle 1-(1-\mu_1)^\lambda, \eta_1^\lambda, \nu_1^\lambda\rangle$, $\lambda > 0$;
(6) $(\alpha_1)^\lambda = \langle \mu_1^\lambda, 1-(1-\eta_1)^\lambda, 1-(1-\nu_1)^\lambda\rangle$, $\lambda > 0$.

In [44], the authors showed that the operation $\otimes$ defined in Definition 20 is not closed in $\mathscr{P}$ (also see [33, Example 2]). Clearly, the operation $\wedge$ in Definition 20 is also not closed in $\mathscr{P}$. More precisely, for two PFNs $\langle 0,1,0\rangle$, $\langle 0,0,1\rangle$, we get that $\langle 0,1,0\rangle \wedge \langle 0,0,1\rangle = \langle 0,1,1\rangle$. It is easy to see that $\langle 0,1,1\rangle$ is not a PFN. The following example reveals that neither of the operations $\lambda \cdot \alpha$ and α^λ is closed in $\mathscr{P}$. So, they are not a PFN. Therefore, many aggregation operators induced by the operations in Definition 20 do not work (see Examples 8, 10 and 12).

Example 7. Consider a PFN $\alpha = \langle 0, 0.5, 0.5\rangle$. By Definition 20, it can be verified that $\frac{1}{2}\alpha = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$ and $\alpha^2 = \left\langle 0, \frac{3}{4}, \frac{3}{4}\right\rangle$. Clearly, $\frac{1}{2}\alpha \notin \mathscr{P}$ and $\alpha^2 \notin \mathscr{P}$, since $0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 1$ and $0 + \frac{3}{4} + \frac{3}{4} = \frac{3}{2} > 1$.

Definition 18. [51, Definition 9] Let $\mathbf{\Omega} = (\omega_1, \ldots, \omega_n)^\top$ be the weight vector such that $\omega_j \in (0,1]$ and $\sum\limits_{j=1}^{n} \omega_j = 1$ and $\alpha_j \in \mathscr{P}$ $(j = 1, \ldots, n)$. Define the *picture fuzzy weighted average operator* (PFWA) $\mathrm{PFWA}_{\mathbf{\Omega}} : \mathscr{P}^n \longrightarrow \mathscr{P}$ by

$$\mathrm{PFWA}_{\mathbf{\Omega}}(\alpha_1, \ldots, \alpha_n) = \bigoplus_{j=1}^{n} (\omega_j \cdot \alpha_j).$$

It is easy check that the operator described in Definition 18 is not well-defined since $\omega_j \cdot \alpha_j$ does not need to be a PFN for $\omega_j \in (0,1)$.

Definition 19. [51, Definition 13] Let $\mathbf{\Omega} = (\omega_1, \ldots, \omega_n)^\top$ be the weight vector such that $\omega_j \in (0,1]$ and $\sum\limits_{j=1}^{n} \omega_j = 1$ and $\alpha_j \in \mathscr{P}$ $(j = 1, \ldots, n)$. Define the *picture fuzzy weighted geometric operator* (PFWG) $\mathrm{PFWG}_{\mathbf{\Omega}} : \mathscr{P}^n \longrightarrow \mathscr{P}$ by

$$\mathrm{PFWG}_{\mathbf{\Omega}}(\alpha_1, \ldots, \alpha_n) = \bigotimes_{j=1}^{n} (\alpha_j)^{\omega_j}.$$

Based on Definition 19, in [51] the author obtained the following formula for the operator $\mathrm{PFWG}_{\mathbf{\Omega}}$.

Theorem 5. *[51, Theorem 11] Let $\mathbf{\Omega} = (\omega_1, \ldots, \omega_n)^\top$ be the weight vector such that $\omega_j \in (0,1]$ and $\sum\limits_{j=1}^{n} \omega_j = 1$ and $\alpha_j \in \mathscr{P}$ $(j = 1, \ldots, n)$. Then,* $\mathrm{PFWG}_{\mathbf{\Omega}}(\alpha_1, \ldots, \alpha_n) \in \mathscr{P}$ *and*

$$\begin{aligned} &\mathrm{PFWG}_{\mathbf{\Omega}}(\alpha_1, \ldots, \alpha_n) \\ &= \left\langle \prod_{j=1}^{n} (\mu_j)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\eta_j)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\nu_j)^{\omega_j} \right\rangle. \end{aligned} \tag{2}$$

Example 8. Take $\alpha_1 = \langle 0, 0.9, 0.1\rangle$, $\alpha_2 = \langle 0, 0.1, 0.9\rangle \in \mathscr{P}$, and $\mathbf{\Omega} = (0.5, 0.5)^\top$. By formula (2), we have $\mathrm{PFWG}_{\mathbf{\Omega}}(\alpha_1, \alpha_2) = \langle 0, 1 - \sqrt{1-0.9}\sqrt{1-0.1}, 1 - \sqrt{1-0.1}\sqrt{1-0.9}\rangle = \langle 0, 0.7, 0.7\rangle \notin \mathscr{P}$. This means that Theorem 11 that is given in [51] does not hold.

Based on the Hamacher product and the Hamacher sum, in [52] the author introduced Hamacher operations for the PFNs.

Definition 20. [52, Definition 8] Let $\alpha_1 = \langle \mu_1, \eta_1, \nu_1\rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2\rangle \in \mathscr{P}$. For $\gamma > 0$, define

(1) $\alpha_1 \oplus \alpha_2 = \left\langle \frac{\mu_1+\mu_2-\mu_1\mu_2-(1-\gamma)\mu_1\mu_2}{1-(1-\gamma)\mu_1\mu_2}, \frac{\eta_1\eta_2}{\gamma+(1-\gamma)(\eta_1+\eta_2-\eta_1\eta_2)}, \frac{\nu_1\nu_2}{\gamma+(1-\gamma)(\nu_1+\nu_2-\nu_1\nu_2)} \right\rangle$;

(2) $\alpha_1 \otimes \alpha_2 = \left\langle \frac{\mu_1\mu_2}{\gamma+(1-\gamma)(\mu_1+\mu_2-\mu_1\mu_2)}, \frac{\eta_1+\eta_2-\eta_1\eta_2-(1-\gamma)\eta_1\eta_2}{1-(1-\gamma)\eta_1\eta_2}, \frac{\nu_1+\nu_2-\nu_1\nu_2-(1-\gamma)\nu_1\nu_2}{1-(1-\gamma)\nu_1\nu_2} \right\rangle$;

(3) $\lambda \cdot \alpha_1 = \left\langle \frac{[1+(\gamma-1)\mu_1]^\lambda-(1-\mu_1)^\lambda}{[1+(\gamma-1)\mu_1]^\lambda+(\gamma-1)(1-\mu_1)^\lambda}, \frac{\gamma\eta_1^\lambda}{[1+(\gamma-1)(1-\eta_1)]^\lambda+(\gamma-1)\eta_1^\lambda}, \frac{\gamma\nu_1^\lambda}{[1+(\gamma-1)(1-\nu_1)]^\lambda+(\gamma-1)\nu_1^\lambda} \right\rangle$, $\lambda > 0$;

(4) $(\alpha_1)^\lambda = \Big\langle \frac{\gamma\mu_1^\lambda}{[1+(\gamma-1)(1-\mu_1)]^\lambda+(\gamma-1)\mu_1^\lambda}, \frac{[1+(\gamma-1)\eta_1]^\lambda-(1-\eta_1)^\lambda}{[1+(\gamma-1)\eta_1]^\lambda+(\gamma-1)(1-\eta_1)^\lambda}, \frac{[1+(\gamma-1)\nu_1]^\lambda-(1-\nu_1)^\lambda}{[1+(\gamma-1)\nu_1]^\lambda+(\gamma-1)(1-\nu_1)^\lambda}\Big\rangle$, $\lambda > 0$.

Remark 5. (i) Notice that Hamacher t-norm $T_\gamma^{\mathbf{H}}(x,y) = \frac{xy}{\gamma+(1-\gamma)(x+y-xy)}$ $(\gamma \in (0,+\infty))$ is a strict t-norm, its dual t-conorm $S_\gamma^{\mathbf{H}}(x,y) = \frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}$, and $\alpha_1 \otimes \alpha_2 = \big\langle T_\gamma^{\mathbf{H}}(\mu_1,\mu_2), S_\gamma^{\mathbf{H}}(\eta_1,\eta_2), S_\gamma^{\mathbf{H}}(\nu_1,\nu_2)\big\rangle$ by Definition 20. Then, we have $T_\gamma^{\mathbf{H}}(\mu_1,\mu_2)+S_\gamma^{\mathbf{H}}(\eta_1,\eta_2)+S_\gamma^{\mathbf{H}}(\nu_1,\nu_2) \geq S_\gamma^{\mathbf{H}}(\eta_1,\eta_2)+S_\gamma^{\mathbf{H}}(\nu_1,\nu_2) \geq \max\{\eta_1,\eta_2\}+\max\{\nu_1,\nu_2\}$, which can exceed one. Thus, the operation $\otimes$ defined in Definition 20 is not closed in $\mathscr{P}$.

(ii) Consider an arbitrary $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha\rangle \in \mathscr{P}$ with $0 < \mu_\alpha, \eta_\alpha, \nu_\alpha < 1$. Then, by Definition 20, we have $\lim_{\lambda\to 0^+} \lambda\cdot\alpha = \langle 0,1,1\rangle \notin \mathscr{P}$ and $\lim_{\lambda\to+\infty} \alpha^\lambda = \langle 0,1,1\rangle \notin \mathscr{P}$. These facts imply that there are $\lambda_1, \lambda_2 > 0$ such that (1) for any $0 < \lambda < \lambda_1$, $\lambda\cdot\alpha \notin \mathscr{P}$; (2) for any $\lambda > \lambda_2$, $\alpha^\lambda \notin \mathscr{P}$. Thus, neither of the operations $\lambda\cdot\alpha$ and α^λ in Definition 20 [52, Definition 8] is closed in $\mathscr{P}$.

In view of Definition 20, the author of the paper [52] defined the PF Hamacher weighted averaging (PFHWA) operator, the PF Hamacher ordered weighted averaging (PFHOWA) operator, and the PF Hamacher hybrid average (PFHHA) operator. By Remark 5, we observe that these operators are not well-defined.

4.5. **Operations in [26].** To overcome shortcomings of operational laws in [51–53], in [26] the authors introduced the following interaction operational laws (IOLs) for the PFNs.

Definition 21. [26, Definition 10] For $\alpha_1 = \langle \mu_1,\eta_1,\nu_1\rangle$, $\alpha_2 = \langle \mu_2,\eta_2,\nu_2\rangle$, $\alpha = \langle \mu,\eta,\nu\rangle \in \mathscr{P}$, define the IOLs as follows:

(i) $\alpha_1 \oplus_{\text{Lin}} \alpha_2 = \big\langle \mu_1+\mu_2-\mu_1\mu_2, \eta_1+\eta_2-\eta_1\eta_2-\mu_1\eta_2-\eta_1\mu_2, \nu_1+\nu_2-\nu_1\nu_2-\mu_1\nu_2-\nu_1\mu_2\big\rangle$;
(ii) $\alpha_1 \otimes_{\text{Lin}} \alpha_2 = \big\langle \mu_1+\mu_2-\mu_1\mu_2-\mu_1\nu_2-\nu_1\mu_2, \eta_1+\eta_2-\eta_1\eta_2, \nu_1+\nu_2-\nu_1\nu_2\big\rangle$;
(iii) $\lambda_{\text{Lin}}\cdot\alpha = \big\langle 1-(1-\mu)^\lambda, (1-\mu)^\lambda-(1-\mu-\eta)^\lambda, (1-\mu)^\lambda-(1-\mu-\nu)^\lambda\big\rangle$;
(iv) $\alpha^{\lambda_{\text{Lin}}} = \big\langle (1-\nu)^\lambda-(1-\mu-\nu)^\lambda, 1-(1-\eta)^\lambda, 1-(1-\nu)^\lambda\big\rangle$;

where $\lambda \in \mathbb{N}$.

Theorem 6. *[26, Theorems 1 and 2] Let α_1 and α_2 be two PFNs. Then, $\alpha_1 \oplus_{\text{Lin}} \alpha_2$ and $\alpha_1 \otimes_{\text{Lin}} \alpha_2$ are PFNs.*

Example 9. Let $\alpha_1 = \alpha_2 = \big\langle 0, \frac{1}{2}, \frac{1}{2}\big\rangle$. For $\lambda \geq 2$, by Definition 21, we have $\alpha_1 \oplus_{\text{Lin}} \alpha_2 = \alpha_1 \otimes_{\text{Lin}} \alpha_2 = \big\langle 0, \frac{3}{4}, \frac{3}{4}\big\rangle$ and $\lambda_{\text{Lin}}\cdot\alpha_1 = (\alpha_1)^{\lambda_{\text{Lin}}} = \big\langle 0, 1-\frac{1}{2^\lambda}, 1-\frac{1}{2^\lambda}\big\rangle$. Clearly, $\alpha_1 \otimes_{\text{Lin}} \alpha_2 \notin \mathscr{P}$ and $\lambda_{\text{Lin}}\cdot\alpha_1 \notin \mathscr{P}$ since $0+\frac{3}{4}+\frac{3}{4} = \frac{3}{2} > 1$ and $0+(1-\frac{1}{2^\lambda})+(1-\frac{1}{2^\lambda}) \geq \frac{3}{2} > 1$ $(\lambda \geq 2)$. Thus, none of the operations in Definition 21 [26, Definition 10] is closed in $\mathscr{P}$. Therefore, Theorem 6 [26, Theorems 1 and 2] does not hold.

4.6. **PFMM operator in [7, 56].** Based on the Muirhead mean operator [31], in [56] the authors introduced the following PFMM operator.

Definition 22. [56, Definition 6] Let $\alpha_j \in \mathscr{P}$ $(j = 1,\dots,n)$ and $P = (p_1,\dots,p_n) \in \mathbb{R}^n$ be a vector of parameters. Define the *Picture fuzzy Muirhead mean* PFMM operator by:

$$\text{PFMM}^P(\alpha_1,\dots,\alpha_n) = \left(\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\bigotimes_{j=1}^{n}\alpha_{\vartheta(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}},$$

where $(\vartheta(1),\dots,\vartheta(n))$ is any permutation of $(1,\dots,n)$, S_n is the set of all permutations of $(1,\dots,n)$, and $\oplus$ and $\otimes$ are given by Definition 20.

According to Definition 22, the following formula for the PFMM was obtained by [56].

Theorem 7. *[56, Theorem 1] Let $\alpha_j \in \mathscr{P}$ $(j = 1,\dots,n)$ and $P = (p_1,\dots,p_n) \in \mathbb{R}^n$. Then,* $\text{PFMM}^P(\alpha_1,\dots,\alpha_n) \in \mathscr{P}$ *and*

$$
\begin{aligned}
&\mathrm{PFMM}^P(\alpha_1,\ldots,\alpha_n)\\
&=\left\langle \left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(j)}^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\right.\\
&\qquad 1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\eta_{\vartheta(j)})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\\
&\qquad \left. 1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\nu_{\vartheta(j)})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right\rangle.
\end{aligned}
\tag{3}
$$

Example 10. Take $\alpha_1=\langle 0,0.9,0.1\rangle$, $\alpha_2=\langle 0,0.1,0.9\rangle\in\mathscr{P}$, and $P=(0.5,0.5)\in\mathbb{R}^2$. By formula (3), we have

$$
\begin{aligned}
&\mathrm{PFMM}^P(\alpha_1,\alpha_2)\\
=&\left\langle 1-\sqrt{(1-\sqrt{\mu_1\mu_2})\,(1-\sqrt{\mu_2\mu_1})},\right.\\
&\quad \sqrt{\left(1-\sqrt{(1-\eta_1)(1-\eta_2)}\right)\cdot\left(1-\sqrt{(1-\eta_2)(1-\eta_1)}\right)},\\
&\quad \left.\sqrt{\left(1-\sqrt{(1-\nu_1)(1-\nu_2)}\right)\cdot\left(1-\sqrt{(1-\nu_2)(1-\nu_1)}\right)}\right\rangle\\
=&\langle 0,0.7,0.7\rangle\notin\mathscr{P}.
\end{aligned}
$$

This means that Theorem 1 that was given in [56] does not hold.

Taking into account the criterion's weight, in [7] the authors introduced the PFWMM operator as follows.

Definition 23. [7, Definition 8] Let $\mathbf{\Omega}=(\omega_1,\ldots,\omega_n)^\top$ be the weight vector such that $\omega_j\in(0,1]$ and $\sum_{j=1}^{n}\omega_j=1$, $\alpha_j\in\mathscr{P}$ $(j=1,\ldots,n)$, and $P=(p_1,\ldots,p_n)\in\mathbb{R}^n$ be a vector of parameters. Define the *picture fuzzy weighted Muirhead mean* (PFWMM) operator by

$$
\mathrm{PFWMM}^P_{n\mathbf{\Omega}}(\alpha_1,\ldots,\alpha_n)=\left[\frac{1}{n!}\bigoplus_{\vartheta\in S_n}\bigotimes_{j=1}^{n}\left(n\omega_{\vartheta(j)}\alpha_{\vartheta(j)}\right)^{p_j}\right]^{\frac{1}{\sum_{j=1}^{n}p_j}},
$$

where $(\vartheta(1),\ldots,\vartheta(n))$ is any permutation of $(1,\ldots,n)$, S_n is the set of all permutations of $(1,\ldots,n)$, and $\oplus$ and $\otimes$ are given by Definition 20.

Example 11. Continuation of Example 10

Given α_1, α_2, and P as in Example 10, we choose $\mathbf{\Omega}=(0.5,0.5)^\top$. By direct calculation and Example 10, we have $\mathrm{PFWMM}^P_{n\mathbf{\Omega}}(\alpha_1,\alpha_2)=\mathrm{PFMM}^P(\alpha_1,\alpha_2)=\langle 0,0.7,0.7\rangle\notin\mathscr{P}$.

Remark 6. (1) Example 10 also shows that Theorem 1 that was given in [7] does not hold.

(2) Example 11 shows that Theorem 2 that was given in [7] does not hold.

4.7. **PFBM operator in [4].** In [4], the authors introduced the following Bonferroni mean operator for the PFNs.

Definition 24. [4, Definition 3.1] Let $\alpha_j \in \mathscr{P}$ $(j = 1, \ldots, n)$. Then, for any p, $q \geq 0$, define the *picture fuzzy Bonferroni mean* (PFBM) operator $\mathrm{PFB}^{p,q}$ by:

$$\mathrm{PFB}^{p,q}(\alpha_1, \ldots, \alpha_n) = \left[\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1,\\ i \neq j}}^{n} (\alpha_i^p \otimes \alpha_j^q) \right) \right]^{\frac{1}{p+q}},$$

where $\oplus$ and $\otimes$ are given by Definition 20.

According to Definition 24, the following formula for the PFBM was obtained by in [4].

Theorem 8. *[4, Theorem 1] Let $\alpha_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \mathscr{P}$ $(j = 1, \ldots, n)$ and p, $q \geq 0$. Then,* $\mathrm{PFBM}^{p,q}(\alpha_1, \ldots, \alpha_n) \in \mathscr{P}$ *and*

$$\begin{aligned} &\mathrm{PFBM}^{p,q}(\alpha_1, \ldots, \alpha_n) \\ &= \left\langle \left(1 - \prod_{\substack{i,j=1,\\ i \neq j}}^{n} (1 - \mu_i^p \mu_j^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \right. \\ &\quad 1 - \left(1 - \prod_{\substack{i,j=1,\\ i \neq j}}^{n} (1 - (1 - \eta_i)^p (1 - \eta_j)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \\ &\quad \left. 1 - \left(1 - \prod_{\substack{i,j=1,\\ i \neq j}}^{n} (1 - (1 - \nu_i)^p (1 - \nu_j)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right\rangle. \end{aligned} \tag{4}$$

Example 12. Take $\alpha_1 = \langle 0, 0.9, 0.1 \rangle$, $\alpha_2 = \langle 0, 0.1, 0.9 \rangle \in \mathscr{P}$, and $p = q = 0.5$. By formula (4), we have

$$\begin{aligned} &\mathrm{PFBM}^{p,q}(\alpha_1, \alpha_2) \\ &= \left\langle 1 - \sqrt{(1 - \sqrt{\mu_1 \mu_2})\,(1 - \sqrt{\mu_2 \mu_1})}, \right. \\ &\quad \sqrt{\left(1 - \sqrt{1 - \eta_1}\sqrt{1 - \eta_2}\right) \cdot \left(1 - \sqrt{1 - \eta_2}\sqrt{1 - \eta_1}\right)}, \\ &\quad \left. \sqrt{\left(1 - \sqrt{1 - \nu_1}\sqrt{1 - \nu_2}\right) \cdot \left(1 - \sqrt{1 - \nu_2}\sqrt{1 - \nu_1}\right)} \right\rangle \\ &= \langle 0, 0.7, 0.7 \rangle \notin \mathscr{P}. \end{aligned}$$

This means that Theorem 1 that was given in [4] does not hold.

4.8. **PFNWBM operator in [60].** Based on the operations in Definition 20, in [60] the authors introduced the following normalized weighted Bonferroni mean operator for the PFNs.

Definition 25. [60, Definition 8] Let $\alpha_j \in \mathscr{P}$ $(j = 1, \ldots, n)$, and ω_j $(j = 1, \ldots, n)$ be the associated weight of α_j such that $\omega_j \in (0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Then, for any p, $q > 0$, define the *picture fuzzy normalized weighted Bonferroni mean* (PFNWBM) operator $\mathrm{PFB}^{p,q}$ by:

$$\mathrm{PFNWBM}^{p,q}_{\mathbf{\Omega}}(\alpha_1,\ldots,\alpha_n)=\left[\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}(\alpha_i^p\otimes\alpha_j^q)\right)\right]^{\frac{1}{p+q}},$$

where the operators $\oplus$ and $\otimes$ are given by Definition 20.

According to Definition 25, the following formula for the PFBM was obtained in [60].

Theorem 9. *[60, Theorem 2] Let p, $q>0$ and $\alpha_j=\langle\mu_j,\eta_j,\nu_j\rangle\in\mathscr{P}$ $(j=1,\ldots,n)$, whose weight vector is $\mathbf{\Omega}=(\omega_1,\ldots,\omega_n)^\top$ such that $\omega_j\in(0,1]$ and $\sum\limits_{j=1}^{n}\omega_j=1$. Then,* $\mathrm{PFNWBM}^{p,q}_{\mathbf{\Omega}}(\alpha_1,\ldots,\alpha_n)\in\mathscr{P}$ *and*

$$\begin{aligned}&\mathrm{PFNWBM}^{p,q}_{\mathbf{\Omega}}(\alpha_1,\ldots,\alpha_n)\\ =&\left\langle\left(1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}(1-\mu_i^p\mu_j^q)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}},\right.\\ &1-\left(1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}(1-(1-\eta_i)^p(1-\eta_j)^q)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}},\\ &\left.1-\left(1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}(1-(1-\nu_i)^p(1-\nu_j)^q)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}\right\rangle.\end{aligned}\tag{5}$$

Example 13. Take $\alpha_1=\langle0,0.9,0.1\rangle$, $\alpha_2=\langle0,0.1,0.9\rangle\in\mathscr{P}$, $p=q=0.5$, and $\mathbf{\Omega}=(\omega_1,\omega_2)^\top=(0.5,0.5)^\top$. By formula (5), we have

$$\begin{aligned}&\mathrm{PFNWBM}^{p,q}_{\mathbf{\Omega}}(\alpha_1,\alpha_2)\\ =&\left\langle1-\sqrt{(1-\sqrt{\mu_1\mu_2})\,(1-\sqrt{\mu_2\mu_1})},\right.\\ &\sqrt{\left(1-\sqrt{1-\eta_1}\sqrt{1-\eta_2}\right)\cdot\left(1-\sqrt{1-\eta_2}\sqrt{1-\eta_1}\right)},\\ &\left.\sqrt{\left(1-\sqrt{1-\nu_1}\sqrt{1-\nu_2}\right)\cdot\left(1-\sqrt{1-\nu_2}\sqrt{1-\nu_1}\right)}\right\rangle\\ =&\langle0,0.7,0.7\rangle\notin\mathscr{P}.\end{aligned}$$

This means that Theorem 2 that was given in [60] does not hold.

5. Interactional operational laws for the PFNs

To overcome the limitation of non-closeness for operational laws mentioned in Section 4, if we only use simple standardization to existing operations in Definitions 13–17 and Definitions 20–21, the obtained standardization operations cannot satisfy the associative law and distributive law. Consequently, the PFWA and PFWG operators induced by these operations are not well-defined. Therefore, we cannot apply these simple standardization operations to define the operational laws for the PFNs. Inspired by the interactional operations proposed by the authors of the papers [48, Definition 3.1] and [22, Definition 4], we introduce the following interactional operational laws for the PFNs, which can fundamentally overcome the limitation of non-closeness.

Definition 26. Let τ be an additive generator of a strict t-norm T and ζ be an additive generator of the dual t-conorm S of T; i.e., $\zeta(x) = \tau(1-x)$. For $\alpha_1 = \langle \mu_1, \eta_1, \nu_1 \rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2 \rangle$, $\alpha = \langle \mu, \eta, \nu \rangle \in \mathscr{P}$, define

(i) $\alpha^{\complement} = \langle \nu, \eta, \mu \rangle$;
(ii) $\alpha_1 \oplus_T \alpha_2 = \langle S(\mu_1, \mu_2), T(\eta_1 + \nu_1, \eta_2 + \nu_2) - T(\nu_1, \nu_2), T(\nu_1, \nu_2) \rangle$;
(iii) $\alpha_1 \otimes_T \alpha_2 = \langle T(\mu_1, \mu_2), T(\eta_1 + \mu_1, \eta_2 + \mu_2) - T(\mu_1, \mu_2), S(\nu_1, \nu_2) \rangle$;
(iv) $\lambda_T \cdot \alpha = \langle \zeta^{-1}(\lambda \cdot \zeta(\mu)), \tau^{-1}(\lambda \cdot \tau(\eta + \nu)) - \tau^{-1}(\lambda \cdot \tau(\nu)), \tau^{-1}(\lambda \cdot \tau(\nu)) \rangle$, $\lambda > 0$;
(v) $\alpha^{\lambda_T} = \langle \tau^{-1}(\lambda \cdot \tau(\mu)), \tau^{-1}(\lambda \cdot \tau(\eta + \mu)) - \tau^{-1}(\lambda \cdot \tau(\mu)), \zeta^{-1}(\lambda \cdot \zeta(\nu)) \rangle$, $\lambda > 0$.

Remark 7. (i) When $\eta_1 = \eta_2 = 0$ and $T = T_{\mathbf{P}}$, the operational laws in Definition 26 are consistent with the usual operations on ordinary intuitionistic fuzzy numbers (see [3, Definition 1.1] and [58, Definition 1.2.2]).

(ii) When $T = T_{\mathbf{P}}$, the operational laws $\alpha_1 \oplus_{T_{\mathbf{P}}} \alpha_2$ and $\lambda_{T_{\mathbf{P}}} \cdot \alpha$ in Definition 26 reduce to the one in [33, Definition 8].

If we assign specific forms of t-norms to Definition 26, some interesting operations are obtained:

(1) If we take $T = T_{\mathbf{P}}$, i.e., $\tau(x) = -\log x$ and $\zeta(x) = -\log(1-x)$, then
  i) $\alpha_1 \oplus_{T_{\mathbf{P}}} \alpha_2 = \big\langle \mu_1 + \mu_2 - \mu_1\mu_2, \eta_1\eta_2 + \eta_1\nu_2 + \eta_2\nu_1, \nu_1\nu_2 \big\rangle$;
  ii) $\alpha_1 \otimes_{T_{\mathbf{P}}} \alpha_2 = \big\langle \mu_1\mu_2, \eta_1\eta_2 + \eta_1\mu_2 + \eta_2\mu_1, \nu_1 + \nu_2 - \nu_1\nu_2 \big\rangle$;
  iii) $\lambda_{T_{\mathbf{P}}} \cdot \alpha = \big\langle 1 - (1-\mu)^{\lambda}, (\eta + \nu)^{\lambda} - \nu^{\lambda}, \nu^{\lambda} \big\rangle$;
  iv) $\alpha^{\lambda_{T_{\mathbf{P}}}} = \big\langle \mu^{\lambda}, (\eta + \mu)^{\lambda} - \mu^{\lambda}, 1 - (1-\nu)^{\lambda} \big\rangle$.
(2) If we take T as Schweizer-Sklar t-norms, i.e., $T(x,y) = T_{\gamma}^{\mathbf{SS}}(x,y) = (\max\{x^{\gamma} + y^{\gamma} - 1, 0\})^{1/\gamma}$ $(\gamma \in (-\infty, 0))$, $\tau_{\gamma}^{\mathbf{SS}}(x) = \frac{1-x^{\gamma}}{\gamma}$, and $\zeta_{\gamma}^{\mathbf{SS}}(x) = \frac{1-(1-x)^{\gamma}}{\gamma}$, then
  i) $\alpha_1 \oplus_{T_{\gamma}^{\mathbf{SS}}} \alpha_2 = \Big\langle 1 - (\max\{(1-\mu_1)^{\gamma} + (1-\mu_2)^{\gamma} - 1, 0\})^{1/\gamma}, (\max\{(\eta_1 + \nu_1)^{\gamma} + (\eta_2 + \nu_2)^{\gamma} - 1, 0\})^{1/\gamma} - (\max\{\nu_1^{\gamma} + \nu_2^{\gamma} - 1, 0\})^{1/\gamma}, (\max\{\nu_1^{\gamma} + \nu_2^{\gamma} - 1, 0\})^{1/\gamma} \Big\rangle$;
  ii) $\alpha_1 \otimes_{T_{\gamma}^{\mathbf{SS}}} \alpha_2 = \Big\langle (\max\{\mu_1^{\gamma} + \mu_2^{\gamma} - 1, 0\})^{1/\gamma}, (\max\{(\eta_1 + \mu_1)^{\gamma} + (\eta_2 + \mu_2)^{\gamma} - 1, 0\})^{1/\gamma} - (\max\{\mu_1^{\gamma} + \mu_2^{\gamma} - 1, 0\})^{1/\gamma}, 1 - (\max\{(1-\nu_1)^{\gamma} + (1-\nu_2)^{\gamma} - 1, 0\})^{1/\gamma} \Big\rangle$;
  iii) $\lambda_{T_{\gamma}^{\mathbf{SS}}} \cdot \alpha = \Big\langle 1 - [1 - \lambda(1 - (1-\mu)^{\gamma})]^{1/\gamma}, [1 - \lambda(1 - (\eta + \nu)^{\gamma})^{1/\gamma}] - [1 - \lambda(1 - \nu^{\gamma})]^{1/\gamma}, [1 - \lambda(1 - \nu^{\gamma})]^{1/\gamma} \Big\rangle$;
  iv) $\alpha^{\lambda_{T_{\gamma}^{\mathbf{SS}}}} = \Big\langle [1 - \lambda(1 - \mu^{\gamma})]^{1/\gamma}, [1 - \lambda(1 - (\eta + \mu)^{\gamma})^{1/\gamma}] - [1 - \lambda(1 - \mu^{\gamma})]^{1/\gamma}, 1 - [1 - \lambda(1 - (1-\nu)^{\gamma})]^{1/\gamma} \Big\rangle$.
(3) If we take T as Hamacher t-norms, i.e., $T(x,y) = T_{\gamma}^{\mathbf{H}}(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)}$ $(\gamma \in (0, +\infty))$, $\tau_{\gamma}^{\mathbf{H}}(x) = \log\Big(\frac{\gamma + (1-\gamma)x}{x}\Big)$, and $\zeta_{\gamma}^{\mathbf{H}}(x) = \log\Big(\frac{\gamma + (1-\gamma)(1-x)}{1-x}\Big)$, then
  i) $\alpha_1 \oplus_{T_{\gamma}^{\mathbf{H}}} \alpha_2 = \Big\langle \frac{\mu_1 + \mu_2 - \mu_1\mu_2 - (1-\gamma)\mu_1\mu_2}{1 - (1-\gamma)\mu_1\mu_2}, \frac{(\eta_1 + \nu_1)(\eta_2 + \nu_2)}{\gamma + (1-\gamma)((\eta_1 + \nu_1) + (\eta_2 + \nu_2) - (\eta_1 + \nu_1)(\eta_2 + \nu_2))} - \frac{\nu_1\nu_2}{\gamma + (1-\gamma)(\nu_1 + \nu_2 - \nu_1\nu_2)}, \frac{\nu_1\nu_2}{\gamma + (1-\gamma)(\nu_1 + \nu_2 - \nu_1\nu_2)} \Big\rangle$;
  ii) $\alpha_1 \otimes_{T_{\gamma}^{\mathbf{H}}} \alpha_2 = \Big\langle \frac{\mu_1\mu_2}{\gamma + (1-\gamma)(\mu_1 + \mu_2 - \mu_1\mu_2)}, \frac{(\eta_1 + \mu_1)(\eta_2 + \mu_2)}{\gamma + (1-\gamma)((\eta_1 + \mu_1) + (\eta_2 + \mu_2) - (\eta_1 + \mu_1)(\eta_2 + \mu_2))} - \frac{\mu_1\mu_2}{\gamma + (1-\gamma)(\mu_1 + \mu_2 - \mu_1\mu_2)}, \frac{\nu_1 + \nu_2 - \nu_1\nu_2 - (1-\gamma)\nu_1\nu_2}{1 - (1-\gamma)\nu_1\nu_2} \Big\rangle$;

iii) $\lambda_{T_\gamma^{\mathbf{H}}}\cdot\alpha=\left\langle\dfrac{\left(\frac{\gamma+(1-\gamma)(1-\mu)}{1-\mu}\right)^{\lambda}-1}{\left(\frac{\gamma+(1-\gamma)(1-\mu)}{1-\mu}\right)^{\lambda}-1+\gamma},\dfrac{\gamma}{\left(\frac{\gamma+(1-\gamma)(\eta+\nu)}{\eta+\nu}\right)^{\lambda}-1+\gamma}-\dfrac{\gamma}{\left(\frac{\gamma+(1-\gamma)\nu}{\nu}\right)^{\lambda}-1+\gamma},\right.$
$\left.\dfrac{\gamma}{\left(\frac{\gamma+(1-\gamma)\nu}{\nu}\right)^{\lambda}-1+\gamma}\right\rangle;$

iv) $\alpha^{\lambda_{T_\gamma^{\mathbf{H}}}}=\left\langle\dfrac{\gamma}{\left(\frac{\gamma+(1-\gamma)\mu}{\mu}\right)^{\lambda}-1+\gamma},\dfrac{\gamma}{\left(\frac{\gamma+(1-\gamma)(\eta+\mu)}{\eta+\mu}\right)^{\lambda}-1+\gamma}-\dfrac{\gamma}{\left(\frac{\gamma+(1-\gamma)\mu}{\mu}\right)^{\lambda}-1+\gamma},\right.$
$\left.\dfrac{\left(\frac{\gamma+(1-\gamma)(1-\nu)}{1-\nu}\right)^{\lambda}-1}{\left(\frac{\gamma+(1-\gamma)(1-\nu)}{1-\nu}\right)^{\lambda}-1+\gamma}\right\rangle.$

In particular, when $\gamma=2$, $T_2^{\mathbf{H}}$ and $S_2^{\mathbf{H}}$ reduce to the Einstein product and Einstein sum, respectively.

(4) If we take T as Frank t-norms, i.e., $T(x,y)=T_\gamma^{\mathbf{F}}(x,y)=\log_\gamma\left(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1}\right)$ ($\gamma\in(0,1)\cup(1,+\infty)$), $\tau_\gamma^{\mathbf{F}}(x)=\log\left(\frac{\gamma-1}{\gamma^x-1}\right)$, and $\zeta_\gamma^{\mathbf{F}}(x)=\log\left(\frac{\gamma-1}{\gamma^{1-x}-1}\right)$, then

i) $\alpha_1\oplus_{T_\gamma^{\mathbf{F}}}\alpha_2=\left\langle 1-\log_\gamma\left[1+\dfrac{(\gamma^{1-\mu_1}-1)(\gamma^{1-\mu_2}-1)}{\gamma-1}\right],\log_\gamma\left[1+\dfrac{(\gamma^{\eta_1+\nu_1}-1)(\gamma^{\eta_2+\nu_2}-1)}{\gamma-1}\right]-\right.$
$\left.\log_\gamma\left[1+\dfrac{(\gamma^{\nu_1}-1)(\gamma^{\nu_2}-1)}{\gamma-1}\right],\log_\gamma\left[1+\dfrac{(\gamma^{\nu_1}-1)(\gamma^{\nu_2}-1)}{\gamma-1}\right]\right\rangle;$

ii) $\alpha_1\otimes_{T_\gamma^{\mathbf{F}}}\alpha_2=\left\langle\log_\gamma\left[1+\dfrac{(\gamma^{\mu_1}-1)(\gamma^{\mu_2}-1)}{\gamma-1}\right],\log_\gamma\left[1+\dfrac{(\gamma^{\eta_1+\mu_1}-1)(\gamma^{\eta_2+\mu_2}-1)}{\gamma-1}\right]-\right.$
$\left.\log_\gamma\left[1+\dfrac{(\gamma^{\mu_1}-1)(\gamma^{\mu_2}-1)}{\gamma-1}\right],1-\log_\gamma\left[1+\dfrac{(\gamma^{1-\nu_1}-1)(\gamma^{1-\nu_2}-1)}{\gamma-1}\right]\right\rangle;$

iii) $\lambda_{T_\gamma^{\mathbf{F}}}\cdot\alpha=\left\langle 1-\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{1-\mu}-1}\right)^{\lambda}}+1\right],\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{\eta+\nu}-1}\right)^{\lambda}}+1\right]-\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{\nu}-1}\right)^{\lambda}}+1\right],\right.$
$\left.\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{\nu}-1}\right)^{\lambda}}+1\right]\right\rangle;$

iv) $\alpha^{\lambda_{T_\gamma^{\mathbf{F}}}}=\left\langle\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{\mu}-1}\right)^{\lambda}}+1\right],\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{\eta+\mu}-1}\right)^{\lambda}}+1\right]-\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{\mu}-1}\right)^{\lambda}}+1\right],1-\right.$
$\left.\log_\gamma\left[\dfrac{\gamma-1}{\left(\frac{\gamma-1}{\gamma^{1-\nu}-1}\right)^{\lambda}}+1\right]\right\rangle.$

(5) If we take T as Dombi t-norms, i.e., $T(x,y)=T_\gamma^{\mathbf{D}}(x,y)=\dfrac{1}{1+\left(\left(\frac{1-x}{x}\right)^{\gamma}+\left(\frac{1-y}{y}\right)^{\gamma}\right)^{1/\gamma}}$ ($\gamma\in(0,+\infty)$), $\tau_\gamma^{\mathbf{D}}(x)=\left(\frac{1-x}{x}\right)^{\gamma}$, and $\zeta_\gamma^{\mathbf{D}}(x)=\left(\frac{x}{1-x}\right)^{\gamma}$, then

i) $\alpha_1\oplus_{T_\gamma^{\mathbf{D}}}\alpha_2=\left\langle 1-\dfrac{1}{1+\left(\left(\frac{\mu_1}{1-\mu_1}\right)^{\gamma}+\left(\frac{\mu_2}{1-\mu_2}\right)^{\gamma}\right)^{1/\gamma}},\dfrac{1}{1+\left(\left(\frac{1-(\eta_1+\nu_1)}{\eta_1+\nu_1}\right)^{\gamma}+\left(\frac{1-(\eta_2+\nu_2)}{\eta_2+\nu_2}\right)^{\gamma}\right)^{1/\gamma}}-\right.$
$\left.\dfrac{1}{1+\left(\left(\frac{1-\nu_1}{\nu_1}\right)^{\gamma}+\left(\frac{1-\nu_2}{\nu_2}\right)^{\gamma}\right)^{1/\gamma}},\dfrac{1}{1+\left(\left(\frac{1-\nu_1}{\nu_1}\right)^{\gamma}+\left(\frac{1-\nu_2}{\nu_2}\right)^{\gamma}\right)^{1/\gamma}}\right\rangle;$

ii) $\alpha_1 \otimes_{T_\gamma^{\mathbf{D}}} \alpha_2 = \left\langle \frac{1}{1+\left(\left(\frac{1-\mu_1}{\mu_1}\right)^\gamma + \left(\frac{1-\mu_2}{\mu_2}\right)^\gamma\right)^{1/\gamma}}, \frac{1}{1+\left(\left(\frac{1-(\eta_1+\mu_1)}{\eta_1+\mu_1}\right)^\gamma + \left(\frac{1-(\eta_2+\mu_2)}{\eta_2+\mu_2}\right)^\gamma\right)^{1/\gamma}} - \frac{1}{1+\left(\left(\frac{1-\mu_1}{\mu_1}\right)^\gamma + \left(\frac{1-\mu_2}{\mu_2}\right)^\gamma\right)^{1/\gamma}}, 1 - \frac{1}{1+\left(\left(\frac{\nu_1}{1-\nu_1}\right)^\gamma + \left(\frac{\nu_2}{1-\nu_2}\right)^\gamma\right)^{1/\gamma}} \right\rangle;$

iii) $\lambda_{T_\gamma^{\mathbf{D}}} \cdot \alpha = \left\langle \frac{\sqrt[\gamma]{\lambda}\mu}{(1-\mu)+\sqrt[\gamma]{\lambda}\mu}, \frac{\eta+\nu}{(\eta+\nu)+\sqrt[\gamma]{\lambda}(1-\eta-\nu)} - \frac{\nu}{\nu+\sqrt[\gamma]{\lambda}(1-\nu)}, \frac{\nu}{\nu+\sqrt[\gamma]{\lambda}(1-\nu)} \right\rangle;$

iv) $\alpha^{\lambda_{T_\gamma^{\mathbf{D}}}} = \left\langle \frac{\mu}{\mu+\sqrt[\gamma]{\lambda}(1-\mu)}, \frac{\eta+\mu}{(\eta+\mu)+\sqrt[\gamma]{\lambda}(1-\eta-\mu)} - \frac{\mu}{\mu+\sqrt[\gamma]{\lambda}(1-\mu)}, \frac{\sqrt[\gamma]{\lambda}\,\nu}{(1-\nu)+\sqrt[\gamma]{\lambda}\nu} \right\rangle.$

(6) If we take T as Aczél-Alsina t-norms, i.e., $T(x,y) = T_\gamma^{\mathbf{AA}}(x,y) = e^{-((-\ln x)^\gamma+(-\ln y)^\gamma)^{1/\gamma}}$ ($\gamma \in (0,+\infty)$), $\tau_\gamma^{\mathbf{AA}}(x) = (-\ln x)^\gamma$, and $\zeta_\gamma^{\mathbf{AA}}(x) = (-\ln(1-x))^\gamma$, then

i) $\alpha_1 \oplus_{T_\gamma^{\mathbf{AA}}} \alpha_2 = \left\langle 1 - e^{-((-\ln(1-\mu_1))^\gamma+(-\ln(1-\mu_2))^\gamma)^{1/\gamma}}, e^{-((-\ln(\eta_1+\nu_1))^\gamma+(-\ln(\eta_2+\nu_2))^\gamma)^{1/\gamma}} - e^{-((-\ln \nu_1)^\gamma+(-\ln \nu_2)^\gamma)^{1/\gamma}}, e^{-((-\ln \nu_1)^\gamma+(-\ln \nu_2)^\gamma)^{1/\gamma}} \right\rangle;$

ii) $\alpha_1 \otimes_{T_\gamma^{\mathbf{AA}}} \alpha_2 = \left\langle e^{-((-\ln \mu_1)^\gamma+(-\ln \mu_2)^\gamma)^{1/\gamma}}, e^{-((-\ln(\eta_1+\mu_1))^\gamma+(-\ln(\eta_2+\mu_2))^\gamma)^{1/\gamma}} - e^{-((-\ln \mu_1)^\gamma+(-\ln \mu_2)^\gamma)^{1/\gamma}}, 1 - e^{-((-\ln(1-\nu_1))^\gamma+(-\ln(1-\nu_2))^\gamma)^{1/\gamma}} \right\rangle;$

iii) $\lambda_{T_\gamma^{\mathbf{AA}}} \cdot \alpha = \left\langle 1-(1-\mu)^{\sqrt[\gamma]{\lambda}}, (\eta+\nu)^{\sqrt[\gamma]{\lambda}} - \nu^{\sqrt[\gamma]{\lambda}}, \nu^{\sqrt[\gamma]{\lambda}} \right\rangle;$

iv) $\alpha^{\lambda_{T_\gamma^{\mathbf{AA}}}} = \left\langle \mu^{\sqrt[\gamma]{\lambda}}, (\eta+\mu)^{\sqrt[\gamma]{\lambda}} - \mu^{\sqrt[\gamma]{\lambda}}, 1-(1-\nu)^{\sqrt[\gamma]{\lambda}} \right\rangle.$

The following Theorem demonstrates that all operational laws defined in Definition 26 are closed in $\mathscr{P}$.

Theorem 10. *Let α_1, α_2, $\alpha \in \mathscr{P}$ and T be a strict t-norm. Then, for any $\lambda > 0$, $\alpha_1 \oplus_T \alpha_2$, $\alpha_1 \otimes_T \alpha_2$, $\lambda_T \cdot \alpha$, and α^{λ_T} are PFNs.*

Proof. (i) By Definition 26, we have $\alpha_1 \oplus_T \alpha_2 = \langle S(\mu_1,\mu_2), T(\eta_1+\nu_1,\eta_2+\nu_2) - T(\nu_1,\nu_2), T(\nu_1,\nu_2)\rangle$. This, together with $\mu_1 \le 1-(\eta_1+\nu_1)$ and $\mu_2 \le 1-(\eta_2+\nu_2)$, since S is increasing, implies that $S(\mu_1,\mu_2) + [T(\eta_1+\nu_1,\eta_2+\nu_2) - T(\nu_1,\nu_2)] + T(\nu_1,\nu_2) = S(\mu_1,\mu_2) + T(\eta_1+\nu_1,\eta_2+\nu_2) \le S(1-(\eta_1+\nu_1),1-(\eta_2+\nu_2)) + T(\eta_1+\nu_1,\eta_2+\nu_2) = 1$. Thus, $\alpha_1 \oplus_T \alpha_2 \in \mathscr{P}$. Similarly, we can prove that $\alpha_1 \otimes_T \alpha_2 \in \mathscr{P}$.

(ii) Fix $\lambda > 0$. By Definition 26, we have $\lambda_T \cdot \alpha = \langle \zeta^{-1}(\lambda \cdot \zeta(\mu)), \tau^{-1}(\lambda \cdot \tau(\eta+\nu)) - \tau^{-1}(\lambda \cdot \tau(\nu)), \tau^{-1}(\lambda \cdot \tau(\nu))\rangle$. Since τ is decreasing and $1-\mu \ge \eta+\nu$, we get $\zeta^{-1}(\lambda \cdot \zeta(\mu)) + \left[\tau^{-1}(\lambda \cdot \tau(\eta+\nu)) - \tau^{-1}(\lambda \cdot \tau(\nu))\right] + \tau^{-1}(\lambda \cdot \tau(\nu)) = \zeta^{-1}(\lambda \cdot \zeta(\mu)) + \tau^{-1}(\lambda \cdot \tau(\eta+\nu)) = 1 - \tau^{-1}(\lambda \cdot \tau(1-\mu)) + \tau^{-1}(\lambda \cdot \tau(\eta+\nu)) \le 1 - \tau^{-1}(\lambda \cdot \tau(1-\mu)) + \tau^{-1}(\lambda \cdot \tau(1-\mu)) = 1$. Thus, $\lambda_T \cdot \alpha \in \mathscr{P}$. Similarly, we can prove that $\alpha^{\lambda_T} \in \mathscr{P}$. □

Theorem 11. *Let α, β, $\gamma \in \mathscr{P}$ and T be a strict t-norm. Then, for any λ, $\xi > 0$, we have*

(i) $\alpha \oplus_T \beta = \beta \oplus_T \alpha$;
(ii) $\alpha \otimes_T \beta = \beta \otimes_T \alpha$;
(iii) $(\alpha \oplus_T \beta) \oplus_T \gamma = \alpha \oplus_T (\beta \oplus_T \gamma)$;
(iv) $(\alpha \otimes_T \beta) \otimes_T \gamma = \alpha \otimes_T (\gamma \otimes_T \beta)$;
(v) $(\xi_T \alpha) \oplus_T (\lambda_T \alpha) = (\xi+\lambda)_T \alpha$;
(vi) $(\alpha^{\xi_T}) \otimes_T (\alpha^{\lambda_T}) = \alpha^{(\xi+\lambda)_T}$;
(vii) $\lambda_T(\alpha \oplus_T \beta) = (\lambda_T \alpha) \oplus_T (\lambda_T \beta)$;
(viii) $(\alpha \otimes_T \beta)^{\lambda_T} = \alpha^{\lambda_T} \otimes_T \beta^{\lambda_T}$;
(ix) $\xi_T(\lambda_T \alpha) = (\lambda \cdot \xi)_T \alpha$;

(x) $(\alpha^{\lambda_T})^{\xi_T} = \alpha^{(\lambda\cdot\xi)_T}$.

Proof. Let S be the dual t-conorm of T. The statements (i)–(ii) follow directly from the commutativity of T and S.

(iii) By direct calculation, we have

$$\begin{aligned}
&(\alpha \oplus_T \beta) \oplus_T \gamma\\
=&\big\langle S(\mu_\alpha,\mu_\beta), T(\eta_\alpha+\nu_\alpha,\eta_\beta+\nu_\beta)-T(\nu_\alpha,\nu_\beta), T(\nu_\alpha,\nu_\beta)\big\rangle \oplus_T \gamma\\
=&\big\langle S(S(\mu_\alpha,\mu_\beta),\mu_\gamma), T(T(\eta_\alpha+\nu_\alpha,\eta_\beta+\nu_\beta),\eta_\gamma+\nu_\gamma)-T(T(\nu_\alpha,\nu_\beta),\nu_\gamma), T(T(\nu_\alpha,\nu_\beta),\nu_\gamma)\big\rangle\\
=&\big\langle S(\mu_\alpha,S(\mu_\beta,\mu_\gamma)), T(\eta_\alpha+\nu_\alpha,T(\eta_\beta+\nu_\beta,\eta_\gamma+\nu_\gamma))-T(\nu_\alpha,T(\nu_\beta,\nu_\gamma),T(\nu_\alpha,T(\nu_\beta,\nu_\gamma))\big\rangle\\
=&\alpha \oplus_T \big\langle S(\mu_\beta,\mu_\gamma), T(\eta_\beta+\nu_\beta,\eta_\gamma+\nu_\gamma)-T(\nu_\beta,\nu_\gamma), T(\nu_\beta,\nu_\gamma)\big\rangle\\
=&\alpha \oplus_T (\beta \oplus_T \gamma).
\end{aligned}$$

(v) Notice that $T(x,y)=\tau^{-1}(\tau(x)+\tau(y))$ and $S(x,y)=\zeta^{-1}(\zeta(x)+\zeta(y))$. Then, we get

$$\begin{aligned}
&(\xi_T\alpha) \oplus_T (\lambda_T\alpha)\\
=&\big\langle \zeta^{-1}(\xi\cdot\zeta(\mu_\alpha)), \tau^{-1}(\xi\cdot\tau(\eta_\alpha+\nu_\alpha))-\tau^{-1}(\xi\cdot\tau(\nu_\alpha)), \tau^{-1}(\xi\cdot\tau(\nu_\alpha))\big\rangle \oplus_T\\
&\big\langle \zeta^{-1}(\lambda\cdot\zeta(\mu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\eta_\alpha+\nu_\alpha))-\tau^{-1}(\lambda\cdot\tau(\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\nu_\alpha))\big\rangle\\
=&\big\langle S(\zeta^{-1}(\xi\cdot\zeta(\mu_\alpha)), \zeta^{-1}(\lambda\cdot\zeta(\mu_\alpha))),\\
&T(\tau^{-1}(\xi\cdot\tau(\eta_\alpha+\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\eta_\alpha+\nu_\alpha)))-T(\tau^{-1}(\xi\cdot\tau(\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\nu_\alpha))),\\
&T(\tau^{-1}(\xi\cdot\tau(\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\nu_\alpha)))\big\rangle\\
=&\big\langle \zeta^{-1}((\xi+\lambda)\cdot\zeta(\mu_\alpha)), \tau^{-1}((\xi+\lambda)\cdot\tau(\eta_\alpha+\nu_\alpha))-\tau^{-1}((\xi+\lambda)\cdot\tau(\nu_\alpha)), \tau^{-1}((\xi+\lambda)\cdot\tau(\nu_\alpha))\big\rangle\\
=&(\xi+\lambda)_T\alpha.
\end{aligned}$$

(vii) By direct calculation, we have

$$\begin{aligned}
&(\lambda_T\alpha) \oplus_T (\lambda_T\beta)\\
=&\big\langle \zeta^{-1}(\lambda\cdot\zeta(\mu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\eta_\alpha+\nu_\alpha))-\tau^{-1}(\lambda\cdot\tau(\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\nu_\alpha))\big\rangle \oplus_T\\
&\big\langle \zeta^{-1}(\lambda\cdot\zeta(\mu_\beta)), \tau^{-1}(\lambda\cdot\tau(\eta_\beta+\nu_\beta))-\tau^{-1}(\lambda\cdot\tau(\nu_\beta)), \tau^{-1}(\lambda\cdot\tau(\nu_\beta))\big\rangle\\
=&\big\langle S(\zeta^{-1}(\lambda\cdot\zeta(\mu_\alpha)), \zeta^{-1}(\lambda\cdot\zeta(\mu_\beta))),\\
&T(\tau^{-1}(\lambda\cdot\tau(\eta_\alpha+\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\eta_\beta+\nu_\beta)))-T(\tau^{-1}(\lambda\cdot\tau(\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\nu_\beta))),\\
&T(\tau^{-1}(\lambda\cdot\tau(\nu_\alpha)), \tau^{-1}(\lambda\cdot\tau(\nu_\beta)))\big\rangle\\
=&\big\langle \zeta^{-1}(\lambda\cdot(\zeta(\mu_\alpha)+\zeta(\mu_\beta))),\\
&\tau^{-1}(\lambda\cdot(\tau(\eta_\alpha+\nu_\alpha)+\tau(\eta_\beta+\nu_\beta)))-\tau^{-1}(\lambda\cdot(\tau(\nu_\alpha)+\tau(\nu_\beta))),\\
&\tau^{-1}(\lambda\cdot(\tau(\nu_\alpha)+\tau(\nu_\beta)))\big\rangle\\
=&\big\langle \zeta^{-1}(\lambda\cdot\zeta\circ\zeta^{-1}(\zeta(\mu_\alpha)+\zeta(\mu_\beta))),\\
&\tau^{-1}(\lambda\cdot\tau\circ\tau^{-1}(\tau(\eta_\alpha+\nu_\alpha)+\tau(\eta_\beta+\nu_\beta)))-\tau^{-1}(\lambda\cdot\tau\circ\tau^{-1}(\tau(\nu_\alpha)+\tau(\nu_\beta))),\\
&\tau^{-1}(\lambda\cdot\tau\circ\tau^{-1}(\tau(\nu_\alpha)+\tau(\nu_\beta)))\big\rangle\\
=&\big\langle \zeta^{-1}(\lambda\cdot\zeta(S(\mu_\alpha,\mu_\beta))),\\
&\tau^{-1}(\lambda\cdot\tau(T(\eta_\alpha+\nu_\alpha,\eta_\beta+\nu_\beta)))-\tau^{-1}(\lambda\cdot\tau(T(\nu_\alpha,\nu_\beta))),\\
&\tau^{-1}(\lambda\cdot\tau(T(\nu_\alpha,\nu_\beta)))\big\rangle\\
=&\lambda_T(\alpha \oplus_T \beta)
\end{aligned}$$

(ix) By direct calculation, we have

$$\begin{aligned}
&\xi_T(\lambda_T\alpha)\\
=&\xi_T\cdot\left\langle\zeta^{-1}(\lambda\zeta(\mu_\alpha)),\tau^{-1}(\lambda\tau(\eta_\alpha+\nu_\alpha))-\tau^{-1}(\lambda\tau(\nu_\alpha)),\tau^{-1}(\lambda\tau(\nu_\alpha))\right\rangle\\
=&\left\langle\zeta^{-1}((\xi\cdot\lambda)\cdot\zeta(\mu_\alpha)),\tau^{-1}((\xi\cdot\lambda)\cdot\tau(\eta_\alpha+\nu_\alpha))-\tau^{-1}((\xi\cdot\lambda)\cdot\tau(\nu_\alpha)),\tau^{-1}((\xi\cdot\lambda)\cdot\tau(\nu_\alpha))\right\rangle\\
=&(\lambda\cdot\xi)_T\alpha.
\end{aligned}$$

By using similar arguments to the proofs of (iii), (v), (vii), and (ix), it is verified the statements (iv), (vi), (viii), and (x). □

Remark 8. Due to the fact that the operational laws in Section 4, except for Definition 21, all consider μ, η, and ν independently, resulting in the limitation of non-closeness. According to Theorems 10 and 11, our proposed operational laws in Definition 26, due to considering the interaction, not only can fundamentally overcome the limitation of non-closeness, but also possess the desired properties.

Theorem 12. *Let $\alpha_j=\langle\mu_j,\eta_j,\nu_j\rangle\in\mathscr{P}$ $(j=1,\ldots,n)$ and T be a strict t-norm. Then,*

$$\begin{aligned}
&\alpha_1\oplus_T\alpha_2\oplus_T\cdots\oplus_T\alpha_n\\
=&\left\langle S^{(n)}(\mu_1,\ldots,\mu_n),T^{(n)}(\eta_1+\nu_1,\ldots,\eta_n+\nu_n)-T^{(n)}(\nu_1,\ldots,\nu_n),T^{(n)}(\nu_1,\ldots,\nu_n)\right\rangle,
\end{aligned}\tag{6}$$

and

$$\begin{aligned}
&\alpha_1\otimes_T\alpha_2\otimes_T\cdots\otimes_T\alpha_n\\
=&\left\langle T^{(n)}(\mu_1,\ldots,\mu_n),T^{(n)}(\eta_1+\mu_1,\ldots,\eta_n+\mu_n)-T^{(n)}(\mu_1,\ldots,\mu_n),S^{(n)}(\nu_1,\ldots,\nu_n)\right\rangle.
\end{aligned}\tag{7}$$

Proof. We only prove the formula (6) in view of the mathematical induction on n. The formula (7) is proved analogously.

(1) When $n=2$, it is easy to check that $\alpha_1\oplus_T\alpha_2=\langle S(\mu_1,\mu_2),T(\eta_1+\nu_1,\eta_2+\nu_2)-T(\nu_1,\nu_2),T(\nu_1,\nu_2)\rangle$.

(2) Suppose that the formula (6) holds when $n=k$; i.e.,

$$\begin{aligned}
&\alpha_1\oplus_T\alpha_2\oplus_T\cdots\oplus_T\alpha_k\\
=&\left\langle S^{(k)}(\mu_1,\mu_2,\ldots,\mu_k),T^{(k)}(\eta_1+\nu_1,\eta_2+\nu_2,\ldots,\eta_k+\nu_k)-T^{(k)}(\nu_1,\nu_2,\ldots,\nu_k),T^{(k)}(\nu_1,\nu_2,\ldots,\nu_k)\right\rangle,
\end{aligned}$$

When $n=k+1$, by Definition 26 and Theorem 11 (iii), we get

$$\begin{aligned}
&\alpha_1\oplus_T\alpha_2\oplus_T\cdots\oplus_T\alpha_k\oplus_T\alpha_{k+1}=(\alpha_1\oplus_T\alpha_2\oplus_T\cdots\oplus_T\alpha_k)\oplus_T\alpha_{k+1}\\
=&\Big\langle S^{(k)}(\mu_1,\mu_2,\ldots,\mu_k),\\
&\quad T^{(k)}(\eta_1+\nu_1,\eta_2+\nu_2,\ldots,\eta_k+\nu_k)-T^{(k)}(\nu_1,\nu_2,\ldots,\nu_k),\\
&\quad T^{(k)}(\nu_1,\nu_2,\ldots,\nu_k)\Big\rangle\oplus_T\left\langle\mu_{k+1},\eta_{k+1},\nu_{k+1}\right\rangle\\
=&\Big\langle S(S^{(k)}(\mu_1,\mu_2,\ldots,\mu_k),\mu_{k+1}),\\
&\quad T(T^{(k)}(\eta_1+\nu_1,\eta_2+\nu_2,\ldots,\eta_k+\nu_k),\eta_{k+1}+\nu_{k+1})-T(T^{(k)}(\nu_1,\nu_2,\ldots,\nu_k),\nu_{k+1}),\\
&\quad T(T^{(k)}(\nu_1,\nu_2,\ldots,\nu_k),\nu_{k+1})\Big\rangle,\\
=&\Big\langle S^{(k+1)}(\mu_1,\mu_2,\ldots,\mu_{k+1}),\\
&\quad T^{(k+1)}(\eta_1+\nu_1,\eta_2+\nu_2,\ldots,\eta_{k+1}+\nu_{k+1})-T^{(k+1)}(\nu_1,\nu_2,\ldots,\nu_{k+1}),\\
&\quad T^{(k+1)}(\nu_1,\nu_2,\ldots,\nu_{k+1})\Big\rangle,
\end{aligned}$$

and thus formula (6) holds when $n=k+1$.

Therefore, by (1) and (2), we obtain that formula (6) holds for all n. □

6. PF interactional aggregation operators

Based on the interactional operational laws defined in Section 5, this section is devoted to aggregating the PF information.

6.1. PF interactional weighted average (geometric) operator.

Definition 27. Let $\mathbf{\Omega} = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ be the weight vector such that $\omega_j \in (0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$ and T be a strict t-norm. Define the picture fuzzy interactional weighted average operator $\mathrm{PFIWA}_{T,\mathbf{\Omega}}$ and *picture fuzzy interactional weighted geometric operator* $\mathrm{PFIWG}_{T,\mathbf{\Omega}}$ induced by T as:

$$\begin{aligned} &\mathrm{PFIWA}_{T,\mathbf{\Omega}} : \mathscr{P}^n \longrightarrow \mathscr{P} \\ &(\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto (\omega_1)_T \alpha_1 \oplus_T (\omega_2)_T \alpha_2 \oplus_T \cdots \oplus_T (\omega_n)_T \alpha_n, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\mathrm{PFIWG}_{T,\mathbf{\Omega}} : \mathscr{P}^n \longrightarrow \mathscr{P} \\ &(\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto \alpha_1^{(\omega_1)_T} \otimes_T \alpha_2^{(\omega_2)_T} \otimes_T \cdots \otimes_T \alpha_n^{(\omega_n)_T}, \end{aligned} \tag{9}$$

respectively.

Theorem 13. *Let $\alpha_j = (\mu_j, \eta_j, \nu_j)$ $(j = 1, 2, \ldots, n)$ be a collection of the PFNs, $\mathbf{\Omega} = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ be the weight vector of α_j $(j = 1, 2, \ldots, n)$ such that $\omega_j \in (0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$, and T be a strict t-norm with an additive generator τ. Then,*

$$\begin{aligned} &\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\ &= \Big\langle \zeta^{-1}(\omega_1 \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \zeta(\mu_n)), \\ &\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \tau(\eta_n + \nu_n)) - \tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n)), \\ &\quad \tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n)) \Big\rangle \\ &= \left\langle \zeta^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \zeta(\mu_j)\right), \tau^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \tau(\eta_j + \nu_j)\right) - \tau^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \tau(\nu_j)\right), \tau^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \tau(\nu_j)\right) \right\rangle, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\ &= \Big\langle \tau^{-1}(\omega_1 \cdot \tau(\mu_1) + \cdots + \omega_n \cdot \tau(\mu_n)), \\ &\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_1 + \mu_1) + \cdots + \omega_n \cdot \tau(\eta_n + \mu_n)) - \tau^{-1}(\omega_1 \cdot \tau(\mu_1) + \cdots + \omega_n \cdot \tau(\mu_n)), \\ &\quad \zeta^{-1}(\omega_1 \cdot \zeta(\nu_1) + \cdots + \omega_n \cdot \zeta(\nu_n)) \Big\rangle \\ &= \left\langle \tau^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \tau(\mu_j)\right), \tau^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \tau(\eta_j + \mu_j)\right) - \tau^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \tau(\mu_j)\right), \zeta^{-1}\left(\sum_{j=1}^{n} \omega_j \cdot \zeta(\nu_j)\right) \right\rangle, \end{aligned}$$

where $\zeta(x) = \tau(1 - x)$.

Proof. By formulas (8) and (9), Theorem 12, and Definition 26, we have

$$
\begin{aligned}
&\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left\langle \zeta^{-1}(\omega_1\cdot\zeta(\mu_1)),\tau^{-1}(\omega_1\cdot\tau(\eta_1+\nu_1))-\tau^{-1}(\omega_1\cdot\tau(\nu_1)),\tau^{-1}(\omega_1\cdot\tau(\nu_1))\right\rangle\\
&\quad\oplus_T\cdots\oplus_T\left\langle \zeta^{-1}(\omega_n\cdot\zeta(\mu_n)),\tau^{-1}(\omega_n\cdot\tau(\eta_n+\nu_n))-\tau^{-1}(\omega_n\cdot\tau(\nu_n)),\tau^{-1}(\omega_n\cdot\tau(\nu_n))\right\rangle\\
&=\Big\langle S^{(n)}(\zeta^{-1}(\omega_1\cdot\zeta(\mu_1)),\ldots,\zeta^{-1}(\omega_n\cdot\zeta(\mu_n))),\\
&\quad T^{(n)}(\tau^{-1}(\omega_1\cdot\tau(\eta_1+\nu_1)),\ldots,\tau^{-1}(\omega_n\cdot\tau(\eta_n+\nu_n)))-T^{(n)}(\tau^{-1}(\omega_1\cdot\tau(\nu_1)),\ldots,\tau^{-1}(\omega_n\cdot\tau(\nu_n))),\\
&\quad T^{(n)}(\tau^{-1}(\omega_1\cdot\tau(\nu_1)),\ldots,\tau^{-1}(\omega_n\cdot\tau(\nu_n)))\Big\rangle\\
&=\Big\langle \zeta^{-1}(\omega_1\cdot\zeta(\mu_1)+\cdots+\omega_n\cdot\zeta(\mu_n)),\\
&\quad \tau^{-1}(\omega_1\cdot\tau(\eta_1+\nu_1)+\cdots+\omega_n\cdot\tau(\eta_n+\nu_n))-\tau^{-1}(\omega_1\cdot\tau(\nu_1)+\cdots+\omega_n\cdot\tau(\nu_n)),\\
&\quad \tau^{-1}(\omega_1\cdot\tau(\nu_1)+\cdots+\omega_n\cdot\tau(\nu_n))\Big\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left\langle \tau^{-1}(\omega_1\cdot\tau(\mu_1)),\tau^{-1}(\omega_1\cdot\tau(\eta_1+\mu_1))-\tau^{-1}(\omega_1\cdot\tau(\mu_1)),\zeta^{-1}(\omega_1\cdot\zeta(\nu_1))\right\rangle\\
&\quad\otimes_T\cdots\otimes_T\left\langle \tau^{-1}(\omega_n\cdot\tau(\mu_n)),\tau^{-1}(\omega_n\cdot\tau(\eta_n+\mu_n))-\tau^{-1}(\omega_n\cdot\tau(\mu_n)),\zeta^{-1}(\omega_n\cdot\tau(\nu_n))\right\rangle\\
&=\Big\langle T^{(n)}(\tau^{-1}(\omega_1\cdot\tau(\mu_1)),\ldots,\tau^{-1}(\omega_n\cdot\tau(\mu_n))),\\
&\quad T^{(n)}(\tau^{-1}(\omega_1\cdot\tau(\eta_1+\mu_1)),\ldots,\tau^{-1}(\omega_n\cdot\tau(\eta_n+\mu_n)))-T^{(n)}(\tau^{-1}(\omega_1\cdot\tau(\mu_1)),\ldots,\tau^{-1}(\omega_n\cdot\tau(\mu_n))),\\
&\quad S^{(n)}(\zeta^{-1}(\omega_1\cdot\zeta(\nu_1)),\ldots,\zeta^{-1}(\omega_n\cdot\zeta(\nu_n)))\Big\rangle\\
&=\Big\langle \tau^{-1}(\omega_1\cdot\tau(\mu_1)+\cdots+\omega_n\cdot\tau(\mu_n)),\\
&\quad \tau^{-1}(\omega_1\cdot\tau(\eta_1+\mu_1)+\cdots+\omega_n\cdot\tau(\eta_n+\mu_n))-\tau^{-1}(\omega_1\cdot\tau(\mu_1)+\cdots+\omega_n\cdot\tau(\mu_n)),\\
&\quad \zeta^{-1}(\omega_1\cdot\zeta(\nu_1)+\cdots+\omega_n\cdot\zeta(\nu_n))\Big\rangle.
\end{aligned}
$$

□

If we assign specific forms of t-norms to Theorem 13, we have the following results.

(1) If we take $T=T_{\mathbf{P}}$, then

i) $$\mathrm{PFIWA}_{T_{\mathbf{P}},\mathbf{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left\langle 1-\prod_{j=1}^{n}(1-\mu_j)^{\omega_j},\prod_{j=1}^{n}(\eta_j+\nu_j)^{\omega_j}-\prod_{j=1}^{n}\nu_j^{\omega_j},\prod_{j=1}^{n}\nu_j^{\omega_j}\right\rangle,$$

ii) $$\mathrm{PFIWG}_{T_{\mathbf{P}},\mathbf{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left\langle \prod_{j=1}^{n}\mu_j^{\omega_j},\prod_{j=1}^{n}(\eta_j+\mu_j)^{\omega_j}-\prod_{j=1}^{n}\mu_j^{\omega_j},1-\prod_{j=1}^{n}(1-\nu_j)^{\omega_j}\right\rangle.$$

(2) If we take T as Schweizer-Sklar t-norms $T_\gamma^{\mathbf{SS}}$ $(\gamma\in(-\infty,0))$, then

i) $$\mathrm{PFIWA}_{T_\gamma^{\mathbf{ss}},\mathbf{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left\langle 1-\left(\sum_{j=1}^{n}\omega_j(1-\mu_j)^{\gamma}\right)^{1/\gamma},\left(\sum_{j=1}^{n}\omega_j(\eta_j+\nu_j)^{\gamma}\right)^{1/\gamma}-\left(\sum_{j=1}^{n}\omega_j\nu_j^{\gamma}\right)^{1/\gamma},\left(\sum_{j=1}^{n}\omega_j\nu_j^{\gamma}\right)^{1/\gamma}\right\rangle,$$

ii) $\mathrm{PFIWG}_{T_\gamma^{\mathbf{ss}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\left\langle\left(\sum_{j=1}^{n}\omega_j\mu_j^\gamma\right)^{1/\gamma},\left(\sum_{j=1}^{n}\omega_j(\eta_j+\mu_j)^\gamma\right)^{1/\gamma}-\left(\sum_{j=1}^{n}\omega_j\mu_j^\gamma\right)^{1/\gamma},1-\left(\sum_{j=1}^{n}\omega_j(1-\nu_j)^\gamma\right)^{1/\gamma}\right\rangle.$$

(3) If we take T as Hamacher t-norms $T_\gamma^{\mathbf{H}}$ $(\gamma\in(0,+\infty))$, then

i) $\mathrm{PFIWA}_{T_\gamma^{\mathbf{H}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\left\langle\frac{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)(1-\mu_j)}{1-\mu_j}\right)^{\omega_j}-1}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)(1-\mu_j)}{1-\mu_j}\right)^{\omega_j}-1+\gamma},\right.$$

$$\frac{\gamma}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)(\eta_j+\nu_j)}{\eta_j+\nu_j}\right)^{\omega_j}-1+\gamma}-\frac{\gamma}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)\nu_j}{\nu_j}\right)^{\omega_j}-1+\gamma},$$

$$\left.\frac{\gamma}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)\nu_j}{\nu_j}\right)^{\omega_j}-1+\gamma}\right\rangle,$$

ii) $\mathrm{PFIWG}_{T_\gamma^{\mathbf{H}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\left\langle\frac{\gamma}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)\mu_j}{\mu_j}\right)^{\omega_j}-1+\gamma},\right.$$

$$\frac{\gamma}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)(\eta_j+\mu_j)}{\eta_j+\mu_j}\right)^{\omega_j}-1+\gamma}-\frac{\gamma}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)\mu_j}{\mu_j}\right)^{\omega_j}-1+\gamma},$$

$$\left.\frac{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)(1-\nu_j)}{1-\nu_j}\right)^{\omega_j}-1}{\prod_{j=1}^{n}\left(\frac{\gamma+(1-\gamma)(1-\nu_j)}{1-\nu_j}\right)^{\omega_j}-1+\gamma}\right\rangle.$$

(4) If we take T as Frank t-norms $T_\gamma^{\mathbf{F}}$ $(\gamma\in(0,1)\cup(1,+\infty))$, then

i) $\mathrm{PFIWA}_{T_\gamma^{\mathbf{F}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\left\langle 1-\log_\gamma\left(\prod_{j=1}^{n}(\gamma^{1-\mu_j}-1)^{\omega_j}+1\right),\right.$$

$$\log_\gamma\left(\prod_{j=1}^{n}\left(\gamma^{\eta_j+\nu_j}-1\right)^{\omega_j}+1\right)-\log_\gamma\left(\prod_{j=1}^{n}(\gamma^{\nu_j}-1)^{\omega_j}+1\right),$$

$$\left.\log_\gamma\left(\prod_{j=1}^{n}(\gamma^{\nu_j}-1)^{\omega_j}+1\right)\right\rangle,$$

ii) $\mathrm{PFIWG}_{T_\gamma^{\mathbf{F}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\left\langle\log_\gamma\left(\prod_{j=1}^{n}(\gamma^{\mu_j}-1)^{\omega_j}+1\right),\right.$$

$$\log_\gamma\left(\prod_{j=1}^{n}\left(\gamma^{\eta_j+\mu_j}-1\right)^{\omega_j}+1\right)-\log_\gamma\left(\prod_{j=1}^{n}\left(\gamma^{\mu_j}-1\right)^{\omega_j}+1\right),$$
$$1-\log_\gamma\left(\prod_{j=1}^{n}(\gamma^{1-\nu_j}-1)^{\omega_j}+1\right)\Bigg\rangle.$$

(5) If we take T as Dombi t-norms $T_\gamma^{\mathbf{D}}$ $(\gamma\in(0,+\infty))$, then

i) $\mathrm{PFIWA}_{T_\gamma^{\mathbf{D}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\Bigg\langle\frac{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{\mu_j}{1-\mu_j}\right)^\gamma}}{1+\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{\mu_j}{1-\mu_j}\right)^\gamma}},\ \frac{1}{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{1-\eta_j-\nu_j}{\eta_j+\nu_j}\right)^\gamma}+1}$$
$$-\frac{1}{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{1-\nu_j}{\nu_j}\right)^\gamma}+1},\ \frac{1}{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{1-\nu_j}{\nu_j}\right)^\gamma}+1}\Bigg\rangle,$$

ii) $\mathrm{PFIWG}_{T_\gamma^{\mathbf{D}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\Bigg\langle\frac{1}{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{1-\mu_j}{\mu_j}\right)^\gamma}+1},\ \frac{1}{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{1-\eta_j-\mu_j}{\eta_j+\mu_j}\right)^\gamma}+1}$$
$$-\frac{1}{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{1-\mu_j}{\mu_j}\right)^\gamma}+1},\ \frac{\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{\nu_j}{1-\nu_j}\right)^\gamma}}{1+\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(\frac{\nu_j}{1-\nu_j}\right)^\gamma}}\Bigg\rangle.$$

(6) If we take T as Aczél-Alsina t-norms $T_\gamma^{\mathbf{AA}}$ $(\gamma\in(0,+\infty))$, then

i) $\mathrm{PFIWA}_{T_\gamma^{\mathbf{AA}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\Bigg\langle1-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log(1-\mu_j))^\gamma}},$$
$$e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log(\eta_j+\nu_j))^\gamma}}-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log\nu_j)^\gamma}},$$
$$e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log\nu_j)^\gamma}}\Bigg\rangle,$$

ii) $\mathrm{PFIWG}_{T_\gamma^{\mathbf{AA}},\boldsymbol{\Omega}}(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$=\Bigg\langle e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log\mu_j)^\gamma}},$$
$$e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log(\eta_j+\mu_j))^\gamma}}-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log\mu_j)^\gamma}},$$
$$1-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\log(1-\nu_j))^\gamma}}\Bigg\rangle.$$

The operators $\mathrm{PFIWA}_{T,\boldsymbol{\Omega}}$ and $\mathrm{PFIWG}_{T,\boldsymbol{\Omega}}$ have the following desirable properties.

Theorem 14. (Monotonicity) *Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle$ $(j = 1, 2, \ldots, n)$ and $\beta_j = \langle \mu_{\beta_j}, \eta_{\beta_j}, \nu_{\beta_j} \rangle$ $(j = 1, 2, \ldots, n)$ be two collections of the PFNs such that $\mu_{\alpha_j} \leq \mu_{\beta_j}$, $\eta_{\alpha_j} \leq \eta_{\beta_j}$, and $\nu_{\alpha_j} \geq \nu_{\beta_j}$, i.e., $\alpha_j \subseteq \beta_j$. Then*

$$\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \preccurlyeq_{\mathrm{W}} \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\beta_1, \beta_2, \ldots, \beta_n).$$

Proof. By Theorem 13, we have

$$\begin{aligned}
&\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
=&\Big\langle \zeta^{-1}(\omega_1 \cdot \zeta(\mu_{\alpha_1}) + \cdots + \omega_n \cdot \zeta(\mu_{\alpha_n})),\\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_{\alpha_1} + \nu_{\alpha_1}) + \cdots + \omega_n \cdot \tau(\eta_{\alpha_n} + \nu_{\alpha_n})) - \tau^{-1}(\omega_1 \cdot \tau(\nu_{\alpha_1}) + \cdots + \omega_n \cdot \tau(\nu_{\alpha_n})),\\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\nu_{\alpha_1}) + \cdots + \omega_n \cdot \tau(\nu_{\alpha_n}))\Big\rangle\\
\triangleq& \langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\beta_1, \beta_2, \ldots, \beta_n)\\
=&\Big\langle \zeta^{-1}(\omega_1 \cdot \zeta(\mu_{\beta_1}) + \cdots + \omega_n \cdot \zeta(\mu_{\beta_n})),\\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_{\beta_1} + \nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\eta_{\beta_n} + \nu_{\beta_n})) - \tau^{-1}(\omega_1 \cdot \tau(\nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\nu_{\beta_n})),\\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\nu_{\beta_n}))\Big\rangle\\
\triangleq& \langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle.
\end{aligned}$$

Since ζ is increasing, from $\mu_{\alpha_j} \leq \mu_{\beta_j}$, it follows that $\tilde{\mu} \leq \hat{\mu}$. Similarly, since τ is decreasing, by $\nu_{\alpha_j} \geq \nu_{\beta_j}$, we have $\tilde{\nu} \geq \hat{\nu}$. To prove $\langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle \preccurlyeq_{\mathrm{W}} \langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle$, we consider the following three cases:

(1) If there exists $1 \leq j_0 \leq n$ such that $\mu_{\alpha_{j_0}} < \mu_{\beta_{j_0}}$, noting that ζ is strictly increasing and $\zeta(1) = +\infty$, from $\mu_{\alpha_j} \leq \mu_{\beta_j}$ $(j = 1, 2, \ldots, n)$, it follows that $\tilde{\mu} = \zeta^{-1}(\omega_1 \cdot \zeta(\mu_{\alpha_1}) + \cdots + \omega_n \cdot \zeta(\mu_{\alpha_n})) < \zeta^{-1}(\omega_1 \cdot \zeta(\mu_{\beta_1}) + \cdots + \omega_n \cdot \zeta(\mu_{\beta_n})) = \hat{\mu}$. This, together with $\tilde{\nu} \geq \hat{\nu}$, implies that $S(\langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle) = \tilde{\mu} - \tilde{\nu} < \hat{\mu} - \tilde{\nu} \leq \hat{\mu} - \hat{\nu} = S(\langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle)$, and thus

$$\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle \prec_{\mathrm{W}} \langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle = \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\beta_1, \beta_2, \ldots, \beta_n).$$

(2) If there exists $1 \leq j_0 \leq n$ such that $\nu_{\alpha_{j_0}} > \nu_{\beta_{j_0}}$, noting that τ is strictly decreasing and $\tau(0) = +\infty$, from $\nu_{\alpha_j} \geq \nu_{\beta_j}$ $(j = 1, 2, \ldots, n)$, it follows that $\tilde{\nu} = \tau^{-1}(\omega_1 \cdot \tau(\nu_{\alpha_1}) + \cdots + \omega_n \cdot \tau(\nu_{\alpha_n})) > \tau^{-1}(\omega_1 \cdot \tau(\nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\nu_{\beta_n})) = \hat{\nu}$. This, together with $\tilde{\mu} \leq \hat{\mu}$, implies that $S(\langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle) = \tilde{\mu} - \tilde{\nu} < \tilde{\mu} - \hat{\nu} \leq \hat{\mu} - \hat{\nu} = S(\langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle)$, and thus

$$\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle \prec_{\mathrm{W}} \langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle = \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\beta_1, \beta_2, \ldots, \beta_n).$$

(3) If $\mu_{\alpha_j} = \mu_{\beta_j}$ and $\nu_{\alpha_j} = \nu_{\beta_j}$ for all $1 \leq j \leq n$, then $\tilde{\mu} = \hat{\mu}$ and $\tilde{\nu} = \hat{\nu}$, and thus $S(\langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle) = S(\langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle)$ and $H_1(\langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle) = H_1(\langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle)$. Since τ is strictly decreasing, by $\eta_{\alpha_j} \leq \eta_{\beta_j}$ $(j = 1, 2, \ldots, n)$, we have

$$\begin{aligned}
\tilde{\eta} &= \tau^{-1}(\omega_1 \cdot \tau(\eta_{\alpha_1} + \nu_{\alpha_1}) + \cdots + \omega_n \cdot \tau(\eta_{\alpha_n} + \nu_{\alpha_n})) - \tau^{-1}(\omega_1 \cdot \tau(\nu_{\alpha_1}) + \cdots + \omega_n \cdot \tau(\nu_{\alpha_n}))\\
&= \tau^{-1}(\omega_1 \cdot \tau(\eta_{\alpha_1} + \nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\eta_{\alpha_n} + \nu_{\beta_n})) - \tau^{-1}(\omega_1 \cdot \tau(\nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\nu_{\beta_n}))\\
&\leq \tau^{-1}(\omega_1 \cdot \tau(\eta_{\beta_1} + \nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\eta_{\beta_n} + \nu_{\beta_n})) - \tau^{-1}(\omega_1 \cdot \tau(\nu_{\beta_1}) + \cdots + \omega_n \cdot \tau(\nu_{\beta_n})) = \hat{\eta},
\end{aligned}$$

implying that

$$H_2(\langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle) = \tilde{\mu} + \tilde{\eta} + \tilde{\nu} \leq \hat{\mu} + \hat{\eta} + \hat{\nu} = H_2(\langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle).$$

Therefore, $\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \tilde{\mu}, \tilde{\eta}, \tilde{\nu} \rangle \preccurlyeq_{\mathrm{W}} \langle \hat{\mu}, \hat{\eta}, \hat{\nu} \rangle = \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\beta_1, \beta_2, \ldots, \beta_n)$ by Definition 11. □

Theorem 15. (Idempotency) *If $\alpha_j = \langle \mu, \eta, \nu \rangle \in \mathscr{P}$ for all $j = 1, 2, \ldots, n$, then*

$$\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \mu, \eta, \nu \rangle.$$

Proof. By Theorem 13, we have

$$\begin{aligned}
&\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \Big\langle \zeta^{-1}(\omega_1 \cdot \zeta(\mu) + \cdots + \omega_n \cdot \zeta(\mu)), \\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\eta + \nu) + \cdots + \omega_n \cdot \tau(\eta + \nu)) - \tau^{-1}(\omega_1 \cdot \tau(\nu) + \cdots + \omega_n \cdot \tau(\nu)), \\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\nu) + \cdots + \omega_n \cdot \tau(\nu)) \Big\rangle \\
&= \langle \mu, \eta, \nu \rangle.
\end{aligned}$$

□

Remark 9. In [33], Theorems 2-5 are direct corollaries of our Theorems 13 –15.

Theorem 16. (Boundedness) *Let $\alpha_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$, then*

$$\begin{aligned}
&\left\langle \min_{1 \le j \le n} \{\mu_j\}, \min_{1 \le j \le n} \{\eta_j\}, \max_{1 \le j \le n} \{\nu_j\} \right\rangle \preccurlyeq_{\mathrm{W}} \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&\preccurlyeq_{\mathrm{W}} \left\langle \max_{1 \le j \le n} \{\mu_j\}, 1 - \left(\max_{1 \le j \le n} \{\mu_j\} + \min_{1 \le j \le n} \{\nu_j\} \right), \min_{1 \le j \le n} \{\nu_j\} \right\rangle.
\end{aligned}$$

Proof. For convenience, denote $\left\langle \min_{1 \le j \le n}\{\mu_j\}, \min_{1 \le j \le n}\{\eta_j\}, \max_{1 \le j \le n}\{\nu_j\} \right\rangle = \langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1 \rangle$, $\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \mu, \eta, \nu \rangle$, and $\left\langle \max_{1 \le j \le n}\{\mu_j\}, 1 - (\max_{1 \le j \le n}\{\mu_j\} + \min_{1 \le j \le n}\{\nu_j\}), \min_{1 \le j \le n}\{\nu_j\} \right\rangle = \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle$. Clearly, $\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1 \rangle, \langle \mu, \eta, \nu \rangle$, and $\langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle$ are PFNs. Noting that $\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1 \rangle \subseteq \alpha_j$ holds for all $1 \le j \le n$, by Theorems 14 and 15, we have $\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1 \rangle = \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1 \rangle) \preccurlyeq_{\mathrm{W}} \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Clearly, $\mu_j \le \tilde{\mu}_2$ and $\nu_j \ge \tilde{\nu}_2$ holds for all $1 \le j \le n$. To prove $\langle \mu, \eta, \nu \rangle \preccurlyeq_{\mathrm{W}} \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle$, we consider the following three cases:

(1) If there exists $1 \le j_0 \le n$ such that $\mu_{j_0} < \tilde{\mu}_2$, since ζ is a strictly increasing function with $\zeta(1) = +\infty$ and $\mu_j \le \tilde{\mu}_2$ $(j = 1, 2, \ldots, n)$, by Theorem 13, we have

$$\mu = \zeta^{-1}(\omega_1 \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \zeta(\mu_n)) < \zeta^{-1}(\omega_1 \cdot \zeta(\tilde{\mu}_2) + \cdots + \omega_n \cdot \zeta(\tilde{\mu}_2)) = \tilde{\mu}_2. \tag{11}$$

Meanwhile, since τ is a strictly decreasing function with $\tau(0) = +\infty$, by Theorem 13 and $\nu_j \ge \tilde{\nu}_2$, we have

$$\nu = \tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n)) \ge \tau^{-1}(\omega_1 \cdot \tau(\tilde{\nu}_2) + \cdots + \omega_n \cdot \tau(\tilde{\nu}_2)) = \tilde{\nu}_2.$$

This, together with formula (11), implies that $S(\langle \mu, \eta, \nu \rangle) = \mu - \nu < \tilde{\mu}_2 - \nu \le \tilde{\mu}_2 - \tilde{\nu}_2 = S(\langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle)$, and thus $\langle \mu, \eta, \nu \rangle \prec_{\mathrm{W}} \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle$.

(2) If there exists $1 \le j_0 \le n$ such that $\nu_{j_0} > \tilde{\nu}_2$, similarly to the above proof, it can be verified that $S(\langle \mu, \eta, \nu \rangle) < S(\langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle)$, and thus $\langle \mu, \eta, \nu \rangle \prec_{\mathrm{W}} \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle$.

(3) If $\mu_j = \tilde{\mu}_2$ and $\nu_j = \tilde{\nu}_2$ holds for all $1 \le j \le n$, i.e., $\tilde{\mu}_2 = \mu_j$ and $\tilde{\nu}_2 = \nu_j$ $(j = 1, 2, \ldots, n)$, from $\mu_j + \eta_j + \nu_j \le 1$, it follows that $\max_{1 \le j \le n}\{\eta_j\} \le 1 - (\tilde{\mu}_2 + \tilde{\nu}_2) = \tilde{\eta}_2$. This means that $\alpha_j \subseteq \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle$ holds for all $1 \le j \le n$. By Theorems 14 and 15, we have $\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \preccurlyeq_{\mathrm{W}} \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle) = \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2 \rangle$. □

Remark 10. Many scholars have erroneously derived the boundedness property for various PF aggregation operators using the following two incorrect forms:

(1) In [23,51–53], the lower bound and upperbound are stated as $\min_{\le j \le n}\{\alpha_j\}$ and $\max_{\le j \le n}\{\alpha_j\}$, respectively. However, since there is no total order for the PFNs, the minimum and maximum of $\alpha_1, \alpha_2, \ldots, \alpha_n$ do not necessarily exist.

(2) In [4, Theorems 5, 10, and 15], [14, Property 3], and [21, Theorems 4, 8, 13, and 17], the lower bound is given as $\big\langle \min_{1\le j\le n}\{\mu_{\alpha_j}\}, \max_{1\le j\le n}\{\eta_{\alpha_j}\}, \max_{1\le j\le n}\{\nu_{\alpha_j}\}\big\rangle$. However, this does not necessarily form a PFN since $\min_{1\le j\le n}\{\mu_{\alpha_j}\} + \max_{1\le j\le n}\{\eta_{\alpha_j}\} + \max_{1\le j\le n}\{\nu_{\alpha_j}\}$ can be larger than 1.

Theorem 17. (Shift Invariance) *Let* $\alpha_j = \langle \mu_j, \eta_j, \nu_j\rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$ *and* $\beta = \langle \mu_\beta, \eta_\beta, \nu_\beta\rangle \in \mathscr{P}$. *Then,*

$$\mathrm{PFIWA}_{T,\boldsymbol{\Omega}}(\alpha_1 \oplus_T \beta, \alpha_2 \oplus_T \beta, \ldots, \alpha_n \oplus_T \beta) = \mathrm{PFIWA}_{T,\boldsymbol{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus_T \beta.$$

Proof. By Definition 26, we have

$$\begin{aligned}\alpha_j \oplus_T \beta &= \langle S(\mu_j, \mu_\beta), T(\eta_j + \nu_j, \eta_\beta + \nu_\beta) - T(\nu_j, \nu_\beta), T(\nu_j, \nu_\beta)\rangle \\ &= \big\langle \zeta^{-1}(\zeta(\mu_j) + \zeta(\mu_\beta)), \tau^{-1}(\tau(\eta_j + \nu_j) + \tau(\eta_\beta + \nu_\beta)) \\ &\qquad -\tau^{-1}(\tau(\nu_j) + \tau(\nu_\beta)), \tau^{-1}(\tau(\nu_j) + \tau(\nu_\beta))\big\rangle.\end{aligned}$$

This, together with Theorem 13, implies that

$$\begin{aligned}&\mathrm{PFIWA}_{T,\boldsymbol{\Omega}}(\alpha_1 \oplus_T \beta, \alpha_2 \oplus_T \beta, \ldots, \alpha_n \oplus_T \beta) \\ &= \Big\langle \zeta^{-1}(\omega_1 \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \zeta(\mu_n) + \zeta(\mu_\beta)), \\ &\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \tau(\eta_n + \nu_n) + \tau(\eta_\beta + \nu_\beta)) \\ &\quad - \tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n) + \tau(\nu_\beta)), \\ &\quad \tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n) + \tau(\nu_\beta))\Big\rangle \\ &= \Big\langle \zeta^{-1}(\zeta(\zeta^{-1}(\omega_1 \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \zeta(\mu_n))) + \zeta(\mu_\beta)), \\ &\quad \tau^{-1}(\tau(\tau^{-1}(\omega_1 \cdot \tau(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \tau(\eta_n + \nu_n))) + \tau(\eta_\beta + \nu_\beta)) \\ &\quad - \tau^{-1}(\tau(\tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n))) + \tau(\nu_\beta)), \\ &\quad \tau^{-1}(\tau(\tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n))) + \tau(\nu_\beta))\Big\rangle \\ &= \Big\langle S(\zeta^{-1}(\omega_1 \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \zeta(\mu_n)), \mu_\beta), \\ &\quad T(\tau^{-1}(\omega_1 \cdot \tau(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \tau(\eta_n + \nu_n)), \eta_\beta + \nu_\beta) \\ &\quad - T(\tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n)), \nu_\beta), \\ &\quad T(\tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n)), \nu_\beta)\Big\rangle \\ &= \mathrm{PFIWA}_{T,\boldsymbol{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus_T \beta \quad \text{(by formula (10))}.\end{aligned}$$

□

Theorem 18. (Homogeneity) *Let* $\alpha_j = \langle \mu_j, \eta_j, \nu_j\rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$ *and* $\lambda > 0$. *Then,*

$$\mathrm{PFIWA}_{T,\boldsymbol{\Omega}}(\lambda_T \cdot \alpha_1, \lambda_T \cdot \alpha_2, \ldots, \lambda_T \cdot \alpha_n) = \lambda_T \cdot \mathrm{PFIWA}_{T,\boldsymbol{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Proof. Noting that $\lambda_T \cdot \alpha_j = \big\langle \zeta^{-1}(\lambda \cdot \zeta(\mu_j)), \tau^{-1}(\lambda \cdot \tau(\eta_j + \nu_j)) - \tau^{-1}(\lambda \cdot \tau(\nu_j)), \tau^{-1}(\lambda \cdot \tau(\nu_j))\big\rangle$ $(j = 1, 2, \ldots, n)$, by Definition 26 and Theorem 13, we have

$$\begin{aligned}&\mathrm{PFIWA}_{T,\boldsymbol{\Omega}}(\lambda_T \cdot \alpha_1, \lambda_T \cdot \alpha_2, \ldots, \lambda_T \cdot \alpha_n) \\ &= \Big\langle \zeta^{-1}(\omega_1 \cdot \lambda \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \lambda \cdot \zeta(\mu_n)), \\ &\quad \tau^{-1}(\omega_1 \cdot \lambda \cdot \tau(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \lambda \cdot \tau(\eta_n + \nu_n)) - \tau^{-1}(\omega_1 \cdot \lambda \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \lambda \cdot \tau(\nu_n)), \\ &\quad \tau^{-1}(\omega_1 \cdot \lambda \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \lambda \cdot \tau(\nu_n))\Big\rangle,\end{aligned}$$

and

$$\begin{aligned}
&\lambda_{_T} \cdot \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \lambda_{_T} \cdot \Big\langle \zeta^{-1}(\omega_1 \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \zeta(\mu_n)), \\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \tau(\eta_n + \nu_n)) - \tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n)), \\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n)) \Big\rangle \\
&= \Big\langle \zeta^{-1}(\lambda(\omega_1 \cdot \zeta(\mu_1) + \cdots + \omega_n \cdot \zeta(\mu_n))), \\
&\quad \tau^{-1}(\lambda(\omega_1 \cdot \tau(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \tau(\eta_n + \nu_n))) - \tau^{-1}(\lambda(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n))), \\
&\quad \tau^{-1}(\lambda(\omega_1 \cdot \tau(\nu_1) + \cdots + \omega_n \cdot \tau(\nu_n))) \Big\rangle,
\end{aligned}$$

implying that $\mathrm{PFIWA}_{T,\mathbf{\Omega}}(\lambda_{_T} \cdot \alpha_1, \lambda_{_T} \cdot \alpha_2, \ldots, \lambda_{_T} \cdot \alpha_n) = \lambda_{_T} \cdot \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. □

Similarly, we have the following results.

Theorem 19. (Monotonicity) *Let* $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle$ $(j = 1, 2, \ldots, n)$ *and* $\beta_j = \langle \mu_{\beta_j}, \eta_{\beta_j}, \nu_{\beta_j} \rangle$ $(j = 1, 2, \ldots, n)$ *be two collections of the PFNs such that* $\mu_{\alpha_j} \leq \mu_{\beta_j}$, $\eta_{\alpha_j} \leq \eta_{\beta_j}$, *and* $\nu_{\alpha_j} \geq \nu_{\beta_j}$, *i.e.,* $\alpha_j \subseteq \beta_j$. *Then,*

$$\mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \preccurlyeq_{\mathrm{W}} \mathrm{PFIWG}_{T,\mathbf{\Omega}}(\beta_1, \beta_2, \ldots, \beta_n).$$

Theorem 20. (Idempotency) *If* $\alpha_j = \langle \mu, \eta, \nu \rangle \in \mathscr{P}$ *for all* $j = 1, 2, \ldots, n$, *then*

$$\mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \mu, \eta, \nu \rangle.$$

Theorem 21. (Boundedness) *Let* $\alpha_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$, *then*

$$\begin{aligned}
&\left\langle \min_{1 \leq j \leq n} \{\mu_j\}, \min_{1 \leq j \leq n} \{\eta_j\}, \max_{1 \leq j \leq n} \{\nu_j\} \right\rangle \preccurlyeq_{\mathrm{W}} \mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&\preccurlyeq_{\mathrm{W}} \left\langle \max_{1 \leq j \leq n} \{\mu_j\}, 1 - \left(\max_{1 \leq j \leq n} \{\mu_j\} + \min_{1 \leq j \leq n} \{\nu_j\} \right), \min_{1 \leq j \leq n} \{\nu_j\} \right\rangle.
\end{aligned}$$

Theorem 22. (Shift Invariance) *Let* $\alpha_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$ *and* $\beta = \langle \mu_\beta, \eta_\beta, \nu_\beta \rangle \in \mathscr{P}$. *Then,*

$$\mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1 \otimes_{_T} \beta, \alpha_2 \otimes_{_T} \beta, \ldots, \alpha_n \otimes_{_T} \beta) = \mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \otimes_{_T} \beta.$$

Theorem 23. (Homogeneity) *Let* $\alpha_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$ *and* $\lambda > 0$. *Then,*

$$\mathrm{PFIWG}_{T,\mathbf{\Omega}}((\alpha_1)^{\lambda_{_T}}, (\alpha_2)^{\lambda_{_T}}, \ldots, (\alpha_n)^{\lambda_{_T}}) = \left(\mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n)\right)^{\lambda_{_T}}.$$

6.2. **PF interactional ordered weighted average (geometric) operator.**

Definition 28. Let $\mathbf{\Omega} = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ be the weight vector such that $\omega_j \in (0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$ and T be a strict t-norm. Define the *picture fuzzy interactional ordered weighted average operator* $\mathrm{PFIOWA}_{T,\mathbf{\Omega}}$ and *picture fuzzy interactional ordered weighted geometric operator* $\mathrm{PFIOWG}_{T,\mathbf{\Omega}}$ induced by T as:

$$\begin{aligned}
&\mathrm{PFIOWA}_{T,\mathbf{\Omega}} : \mathscr{P}^n \longrightarrow \mathscr{P} \\
&\quad (\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto (\omega_1)_{_T}\alpha_{\sigma(1)} \oplus_{_T} (\omega_2)_{_T}\alpha_{\sigma(2)} \oplus_{_T} \cdots \oplus_{_T} (\omega_n)_{_T}\alpha_{\sigma(n)},
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{PFIOWG}_{T,\mathbf{\Omega}} : \mathscr{P}^n \longrightarrow \mathscr{P} \\
&\quad (\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto \alpha_{\sigma(1)}^{(\omega_1)_T} \otimes_{_T} \alpha_{\sigma(2)}^{(\omega_2)_T} \otimes_{_T} \cdots \otimes_{_T} \alpha_{\sigma(n)}^{(\omega_n)_T},
\end{aligned}$$

respectively, where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\alpha_{\sigma(j)} \preccurlyeq_{\mathrm{W}} \alpha_{\sigma(j-1)}$ for all $2 \leq j \leq n$, i.e., $\alpha_{\sigma(1)} \succcurlyeq_{\mathrm{W}} \alpha_{\sigma(2)} \succcurlyeq_{\mathrm{W}} \cdots \succcurlyeq_{\mathrm{W}} \alpha_{\sigma(n)}$.

Similarly to Theorem 13, we have the following result.

Theorem 24. *Let $\alpha_j = (\mu_j, \eta_j, \nu_j)$ $(j = 1, 2, \ldots, n)$ be a collection of the PFNs, $\mathbf{\Omega} = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ be the weight vector of α_j $(j = 1, 2, \ldots, n)$ such that $\omega_j \in (0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, and T be a strict t-norm with an additive generator τ. Then,*

$$\begin{aligned}
&\mathrm{PFIOWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \Big\langle \zeta^{-1}(\omega_1 \cdot \zeta(\mu_{\sigma(1)}) + \cdots + \omega_n \cdot \zeta(\mu_{\sigma(n)})),\\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_{\sigma(1)} + \nu_{\sigma(1)}) + \cdots + \omega_n \cdot \tau(\eta_{\sigma(n)} + \nu_{\sigma(n)})) - \tau^{-1}(\omega_1 \cdot \tau(\nu_{\sigma(1)}) + \cdots + \omega_n \cdot \tau(\nu_{\sigma(n)})),\\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\nu_{\sigma(1)}) + \cdots + \omega_n \cdot \tau(\nu_{\sigma(n)}))\Big\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{PFIOWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \Big\langle \tau^{-1}(\omega_1 \cdot \tau(\mu_{\sigma(1)}) + \cdots + \omega_n \cdot \tau(\mu_{\sigma(n)})),\\
&\quad \tau^{-1}(\omega_1 \cdot \tau(\eta_{\sigma(1)} + \mu_{\sigma(1)}) + \cdots + \omega_n \cdot \tau(\eta_{\sigma(n)} + \mu_{\sigma(n)})) - \tau^{-1}(\omega_1 \cdot \tau(\mu_{\sigma(1)}) + \cdots + \omega_n \cdot \tau(\mu_{\sigma(n)})),\\
&\quad \zeta^{-1}(\omega_1 \cdot \zeta(\nu_{\sigma(1)}) + \cdots + \omega_n \cdot \zeta(\nu_{\sigma(n)}))\Big\rangle,
\end{aligned}$$

where $\zeta(x) = \tau(1 - x)$.

Meanwhile, it can be verified that the operators $\mathrm{PFIOWA}_{T,\mathbf{\Omega}}$ and $\mathrm{PFIOWG}_{T,\mathbf{\Omega}}$ have idempotency and boundedness. Besides, we can obtain that they have commutativity.

Theorem 25. (Commutativity) *Let $\alpha_j = \langle \mu_j, \eta_j, \nu_j \rangle$ $(j = 1, 2, \ldots, n)$ and $\alpha'_j = \langle \mu'_j, \eta'_j, \nu'_j \rangle$ $(j = 1, 2, \ldots, n)$ be two collections of the PFNs such that $(\alpha'_1, \alpha'_2, \ldots, \alpha'_n)$ is any permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then*

$$\mathrm{PFIOWA}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{PFIOWA}_{T,\mathbf{\Omega}}(\alpha'_1, \alpha'_2, \ldots, \alpha'_n),$$

and

$$\mathrm{PFIOWG}_{T,\mathbf{\Omega}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{PFIOWG}_{T,\mathbf{\Omega}}(\alpha'_1, \alpha'_2, \ldots, \alpha'_n).$$

6.3. **PF mean interactional weighted average and geometric (PFMIWAG) operator.** To the best of our knowledge, all existing closed PF aggregation operators are only found in [22, 33, 44, 48]. However, those operators have the following disadvantage: If the rating of an alternative on some criterion is $\langle 1, 0, 0 \rangle$ or $\langle 0, 0, 1 \rangle$, regardless of the ratings of this alternative on other criteria, the overall rating of this alternative always is $\langle 1, 0, 0 \rangle$ or $\langle 0, 0, 1 \rangle$ (see Example 14). This is impractical. To overcome this disadvantage, we propose the following aggregation operator for the MCDM problems under the PF environment.

Definition 29. Let $\mathbf{\Omega} = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ be the weight vector such that $\omega_j \in (0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$ and T be a strict t-norm. Define the *PF mean interactional weighted average and geometric (PFMIWAG) operator* $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$ induced by T as:

$$\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}(\alpha_1, \ldots, \alpha_n) = \left\langle \frac{\mu_A + \mu_G}{2}, \frac{\eta_A + \eta_G}{2}, \frac{\nu_A + \nu_G}{2} \right\rangle, \tag{12}$$

where $\langle \mu_A, \eta_A, \nu_A \rangle = \mathrm{PFIWA}_{T,\mathbf{\Omega}}(\alpha_1, \ldots, \alpha_n)$ and $\langle \mu_G, \eta_G, \nu_G \rangle = \mathrm{PFIWG}_{T,\mathbf{\Omega}}(\alpha_1, \ldots, \alpha_n)$.

Remark 11. (1) Because $\langle \mu_A, \eta_A, \nu_A \rangle$ and $\langle \mu_G, \eta_G, \nu_G \rangle$ are PFNs (by Theorem 13), $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$ $(\alpha_1, \ldots, \alpha_n)$ is a PFN. Thus, the aggregation operator $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$ in Definition 29 is closed.

(2) By Definition 27, it can be verified that (a) $\frac{\mu_A+\mu_G}{2}=1$ if and only if $\mu_{\alpha_1}=\cdots=\mu_{\alpha_n}=1$; (b) $\frac{\nu_A+\nu_G}{2}=1$ if and only if $\nu_{\alpha_1}=\cdots=\mu_{\alpha_n}=1$. This overcomes the disadvantage mentioned above.

(3) By Theorems 14–16 and Theorems 19–21, it can be shown that the operator $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$ is idempotent, bounded, and monotonous.

(4) If $T=T_{\mathbf{P}}$, by direct calculation, we have

$$\begin{aligned}&\mathrm{PFMIWAG}_{T_{\mathbf{P}},\mathbf{\Omega}}(\alpha_1,\ldots,\alpha_n)\\&=\left\langle\frac{\langle 1-\prod_{j=1}^{n}(1-\mu_j)^{\omega_j}+\prod_{j=1}^{n}\mu_j^{\omega_j}}{2},\right.\\&\quad\frac{\prod_{j=1}^{n}(\eta_j+\nu_j)^{\omega_j}-\prod_{j=1}^{n}\nu_j^{\omega_j}+\prod_{j=1}^{n}(\eta_j+\mu_j)^{\omega_j}-\prod_{j=1}^{n}\mu_j^{\omega_j}}{2},\\&\quad\left.\frac{\prod_{j=1}^{n}\nu_j^{\omega_j}+1-\prod_{j=1}^{n}(1-\nu_j)^{\omega_j}}{2}\right\rangle.\end{aligned}$$

Example 14. Let $\alpha_1=\langle\mu_{\alpha_1},\eta_{\alpha_1},\nu_{\alpha_1}\rangle$, $\alpha_2=\langle 0,0,1\rangle$, and $\mathbf{\Omega}=(\omega_1,\omega_2)^\top$ be the weight vector with $0<\omega_1,\ \omega_2\le 1$ and $\omega_1+\omega_2=1$.

Regardless of the values of α_1 and $\mathbf{\Omega}$, (1) by applying the PF aggregation operators in [33, Definition 9] for $\alpha_1^{\complement}$ and $\alpha_2^{\complement}$, we obtain $\langle 1,0,0\rangle$; (2) by applying the PF aggregation operators in [22, Theorem 2], [44, Theorem 3], and [48, Theorem 4.2] for α_1 and α_2, we obtain $\langle 0,0,1\rangle$. Clearly, all of these are unreasonable.

Taking T as a strict t-norm with an additive generator τ, by direct calculation, it follows from Eq. (12) that

$$\begin{aligned}&\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}(\alpha_1,\alpha_2)\\&=\left\langle\frac{\zeta^{-1}(\omega_1\cdot\zeta(\mu_{\alpha_1}))}{2},\frac{\tau^{-1}(\omega_1\cdot\tau(\eta_{\alpha_1}+\nu_{\alpha_1}))-\tau^{-1}(\omega_1\cdot\tau(\nu_{\alpha_1}))}{2},\frac{\tau^{-1}(\omega_1\cdot\tau(\nu_{\alpha_1}))+1}{2}\right\rangle.\end{aligned}$$

It can be verified that $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}(\alpha_1,\alpha_2)=\langle 0,0,1\rangle$ if and only if $\alpha_1=\langle 0,0,1\rangle$, demonstrating that our proposed operator $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$ is more reasonable.

7. A novel PF MCDM method and a numerical example

7.1. **A novel PF MCDM method.** Consider an MCDM problem under the PF environment, suppose $A=\{A_1,A_2,\ldots,A_m\}$ is a set of alternatives to be selected, and $G=\{G_1,G_2,\ldots,G_n\}$ is a set of criteria to be evaluated, whose weight vector is $\mathbf{\Omega}=(\omega_1,\omega_2,\ldots,\omega_n)^\top$ such that $\omega_j\in(0,1]$ and $\sum_{j=1}^{n}\omega_j=1$. Assume that the rating of an alternative A_i $(i=1,\ldots,m)$ based on the criterion G_j $(j=1,\ldots,n)$ is assessed by the decision-maker, forming PFNs $\alpha_{ij}=\langle\mu_{ij},\eta_{ij},\nu_{ij}\rangle$, where μ_{ij} and η_{ij} are the degrees of the positive and the neutral memberships, respectively, and ν_{ij} is the degree that the alternative A_i does not satisfy the criterion G_j. To rank alternatives, the following steps are given:

Step 1: (Construct the PF decision matrix) The decision-maker provides their preference in the form of PFNs $\alpha_{ij}=\langle\mu_{ij},\eta_{ij},\nu_{ij}\rangle$ towards the alternative A_i and the criterion G_j. Hence, a PF decision matrix $D=(\alpha_{ij})_{m\times n}$ is constructed as shown below:

TABLE 2. The PF decision matrix D.

	G_1	G_2	$\cdots$	G_n
A_1	$\langle \mu_{11}, \eta_{11}, \nu_{11} \rangle$	$\langle \mu_{12}, \eta_{12}, \nu_{12} \rangle$	$\cdots$	$\langle \mu_{1n}, \eta_{1n}, \nu_{1n} \rangle$
A_2	$\langle \mu_{21}, \eta_{21}, \nu_{21} \rangle$	$\langle \mu_{22}, \eta_{22}, \nu_{22} \rangle$	$\cdots$	$\langle \mu_{2n}, \eta_{2n}, \nu_{2n} \rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_m	$\langle \mu_{m1}, \eta_{m1}, \nu_{m1} \rangle$	$\langle \mu_{m2}, \eta_{m2}, \nu_{m2} \rangle$	$\cdots$	$\langle \mu_{mn}, \eta_{mn}, \nu_{mn} \rangle$

Step 2: (Normalize the PF decision matrix) The matrix $D = (\alpha_{ij})_{m\times n}$ is transformed into the normalized matrix $R = (r_{ij})_{m\times n}$ for the PF decision as below:

$$r_{ij} = \begin{cases} \alpha_{ij}, & \text{for benefit criteria } G_j, \\ \alpha_{ij}^{\complement}, & \text{for cost criteria } G_j, \end{cases}$$

where $\alpha_{ij}^{\complement}$ is the complement of α_{ij}.

Step 3: (Compute the aggregated value) Based on the normalized PF decision matrix R obtained from Step 2, the overall aggregated value of every alternative A_i $(i = 1, \ldots, m)$, under the different criteria $G_1, G_2, \ldots, G_n$, is obtained by the PFMIWAG operator $\mathrm{PFMIWAG}_{T,\mathbf{\Omega}}$ for some strict t-norm T. Hence, the collective value r_i for each alternative A_i is computed.

Step 4: (Rank the alternatives) The alternatives $A_1, A_2, \ldots, A_m$ are ranked according to the total order given by Definition 11, and the most desirable one is selected.

7.2. **An illustrative example.** In an effort to increase incomes, a commercial company in China is planning its financial strategy and investments for the upcoming year. Based on the company's strategic plan, they have the following six alternative schemes after the strict screening: (1) A_1: to invest in "Hunan Province"; (2) A_2: to invest in "Guangdong Province"; (3) A_3: to invest in "Hubei Province"; (4) A_4: to invest in "Sichuan Province"; (5) A_5: to invest in "Hebei Province"; (6) A_6: to invest in "Anhui Province". These six alternatives are assessed according to four criteria: (1) the "growth analysis" denoted by G_1; (2) the "risk analysis" denoted by G_2; (3) the "social-political impact analysis", denoted by G_3; (4) the "environmental impact analysis", denoted by G_4; where the weight vector is $\mathbf{\Omega} = (0.2, 0.3, 0.1, 0.4)^{\top}$.

Step 1: The normalized decision matrix $R = (r_{ij})_{6\times 4}$, transformed by the decision matrix from the expert, is listed in Table 3.

TABLE 3. The normalized PF decision matrix R.

	G_1	G_2	G_3	G_4
A_1	$\langle 0.6, 0.1, 0.2 \rangle$	$\langle 0.5, 0.3, 0.1 \rangle$	$\langle 0.5, 0.1, 0.3 \rangle$	$\langle 0.2, 0.3, 0.4 \rangle$
A_2	$\langle 0.4, 0.4, 0.1 \rangle$	$\langle 0.6, 0.3, 0.1 \rangle$	$\langle 0.5, 0.2, 0.2 \rangle$	$\langle 0.7, 0.1, 0.2 \rangle$
A_3	$\langle 0.2, 0.2, 0.3 \rangle$	$\langle 0.6, 0.2, 0.1 \rangle$	$\langle 0.4, 0.1, 0.3 \rangle$	$\langle 0.4, 0.3, 0.3 \rangle$
A_4	$\langle 0.5, 0.1, 0.2 \rangle$	$\langle 0.4, 0.2, 0.1 \rangle$	$\langle 0.2, 0.2, 0.5 \rangle$	$\langle 0.3, 0.2, 0.2 \rangle$
A_5	$\langle 0.2, 0.2, 0.2 \rangle$	$\langle 0.5, 0.2, 0.1 \rangle$	$\langle 0.3, 0.2, 0.3 \rangle$	$\langle 0.6, 0.2, 0.1 \rangle$
A_6	$\langle 0.6, 0.1, 0.3 \rangle$	$\langle 0.1, 0.2, 0.6 \rangle$	$\langle 0.1, 0.3, 0.5 \rangle$	$\langle 0.2, 0.3, 0.2 \rangle$

Step 2: If we take $T = T_{\mathbf{P}}$, by the PFMIWAG operator $\mathrm{PFMIWAG}_{T_{\mathbf{P}},\mathbf{\Omega}}$, the overall rating value of each alternative A_i is as below: $r_1 = \langle 0.3912, 0.2584, 0.2472 \rangle$, $r_2 = \langle 0.5912, 0.2357, 0.1464 \rangle$, $r_3 = \langle 0.4153, 0.2361, 0.2305 \rangle$, $r_4 = \langle 0.3573, 0.1881, 0.1937 \rangle$, $r_5 = \langle 0.4530, 0.2096, 0.1355 \rangle$, and $r_6 = \langle 0.2294, 0.2532, 0.3634 \rangle$.

Step 3: By direct calculation, the score values of the aggregated values $r_1, r_2, \ldots, r_6$ are $r_1, r_2, \ldots, r_6$ are $S(r_1) = 0.1440$, $S(r_2) = 0.4448$, $S(r_3) = 0.1848$, $S(r_4) = 0.1636$, $S(r_5) = 0.3175$, and $S(r_6) = -0.1340$. Thus, we have $r_2 \succ_{\mathrm{W}} r_5 \succ_{\mathrm{W}} r_3 \succ_{\mathrm{W}} r_4 \succ_{\mathrm{W}} r_1 \succ_{\mathrm{W}} r_6$ by Definition 11. It means that the best financial strategy is investing in "Guangdong Province"; i.e., A_2.

Sensitivity analysis. To explore the changing tendencies of the scores and the rankings for A_1, A_2, A_3, A_4, A_5, and A_6 with the variation of the t-norm T and the parameter γ, we utilize the following figures to illustrate these topics.

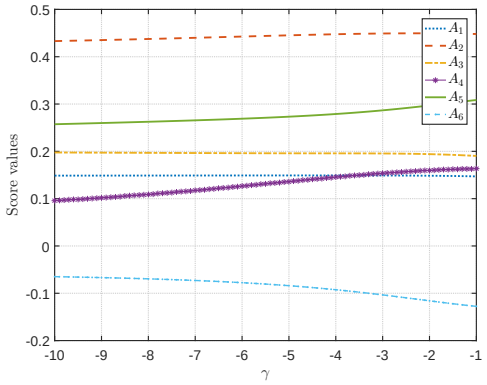


FIGURE 2. Scores for alternatives derived by the $\text{PFMIWAG}_{T_\gamma^{\mathbf{SS}},\mathbf{\Omega}}$.

Fig. 2 gives the scores of the alternatives derived by the operator $\text{PFMIWAG}_{T_\gamma^{\mathbf{SS}},\mathbf{\Omega}}$, induced by Schweizer-Sklar t-norm $T_\gamma^{\mathbf{SS}}$, as the parameter γ changes from -10 to -1. Then, we observe that (1) for $\gamma \in [-10, -3.5]$, the ranking of A_1–A_6 is $A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4 \succ A_6$; (2) for $\gamma \in [-3.5, -1]$, the ranking of A_1–A_6 is $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$; (3) A_2 is consistently the best financial strategy.

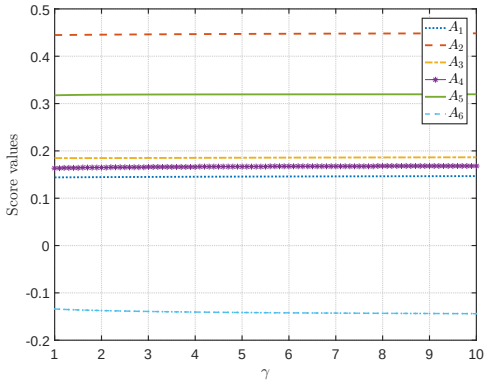


FIGURE 3. Scores for alternatives derived by the $\text{PFMIWAG}_{T_\gamma^{\mathbf{H}},\mathbf{\Omega}}$.

Fig. 3 gives the scores of the alternatives derived by the operator $\text{PFMIWAG}_{T_\gamma^{\mathbf{H}},\mathbf{\Omega}}$, induced by Hamacher t-norm $T_\gamma^{\mathbf{H}}$, as the parameter γ changes from 1 to 10. Then, we observe that (1) the ranking of A_1–A_6 is always $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$; (2) A_2 is consistently the best financial strategy.

Fig. 4 gives the scores of the alternatives derived by the operator $\text{PFMIWAG}_{T_\gamma^{\mathbf{F}},\mathbf{\Omega}}$, induced by Frank t-norm $T_\gamma^{\mathbf{F}}$, as the parameter γ changes from 1 to 10. Then, we observe that (1) the ranking of A_1–A_6 is always $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$; (2) A_2 is consistently the best financial strategy.

Fig. 5 gives the scores of the alternatives derived by the operator $\text{PFIWA}_{T_\gamma^{\mathbf{D}},\mathbf{\Omega}}$, induced by Dombi t-norm $T_\gamma^{\mathbf{D}}$, as the parameter γ changes from 1 to 10. Then, we observe that (1) for $\gamma \in [1, 2]$, the ranking of A_1–A_6 is $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$; (2) for $\gamma \in [2, 10]$, the

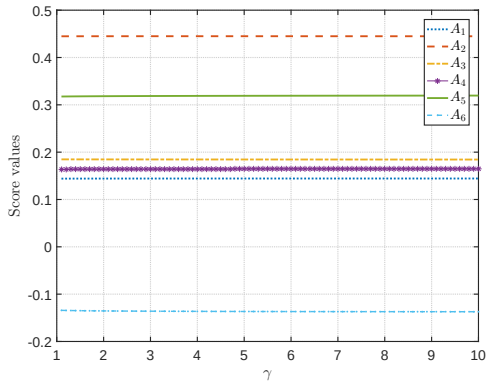


FIGURE 4. Scores for alternatives derived by the PFMIWAG$_{T_\gamma^{\mathbf{F}},\mathbf{\Omega}}$.

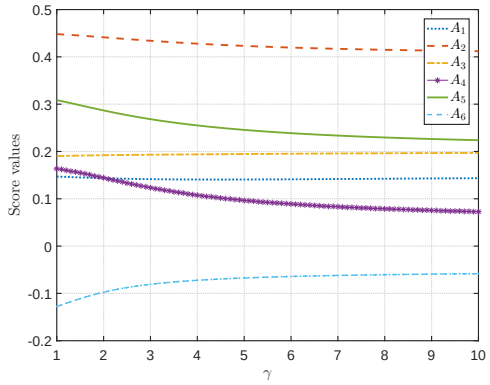


FIGURE 5. Scores for alternatives derived by the PFMIWAG$_{T_\gamma^{\mathbf{D}},\mathbf{\Omega}}$.

ranking of A_1–A_6 is $A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4 \succ A_6$; (3) A_2 is consistently the best financial strategy.

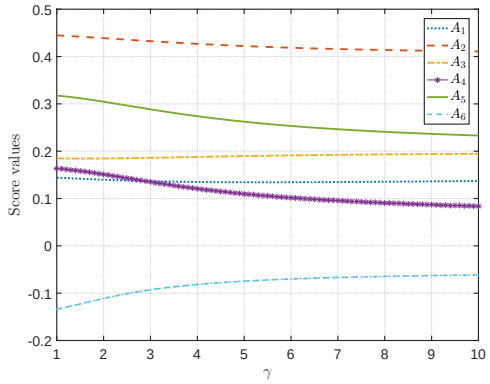


FIGURE 6. Scores for alternatives derived by PFMIWAG$_{T_\gamma^{\mathbf{AA}},\mathbf{\Omega}}$.

Fig. 6 gives the scores of the alternatives derived by the operator PFMIWAG$_{T_\gamma^{\mathbf{AA}},\mathbf{\Omega}}$, induced by the Aczél-Alsina t-norm $T_\gamma^{\mathbf{AA}}$, as the parameter γ changes from 1 to 10. Then, we observe that (1) for $\gamma \in [1, 2.5]$, the ranking of A_1–A_6 is $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$; (2) for

$\gamma \in [2.5, 10]$, the ranking of A_1–A_6 is $A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4 \succ A_6$; (3) A_2 is consistently the best financial strategy.

From the above analysis, the parameter γ can be regarded as a reflection of the decision-makers' preferences. As the parameter γ changes within a certain range, the scores and rankings of the alternatives vary. However, the best financial strategy is consistently A_2. Therefore, we have sufficient reasons to conclude that this company's best financial strategy is to invest in "Guangdong Province".

7.3. **Comparative analysis.** This subsection conducts comparative analysis with the PF MCDM methods in [22,33,44,48], which are all based on closed operational laws. The comparative results are summarized in Table 6.

From Table 6, it can be seen that (1) except for a slight difference in the ranking obtained the operations that were given in [33], the ranking obtained by other operators is $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$; (2) the best financial strategy is consistently A_2. This indicates that our method is effective.

Furthermore, we consider other PF decision matrices as shown in Tables 4 and 5.

TABLE 4. The normalized PF decision matrix R.

	G_1	G_2	G_3	G_4
A_1	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.5, 0.3, 0.1\rangle$	$\langle 0.5, 0.1, 0.3\rangle$	$\langle 1, 0, 0\rangle$
A_2	$\langle 0.4, 0.4, 0.1\rangle$	$\langle 0.6, 0.3, 0.1\rangle$	$\langle 0.5, 0.2, 0.2\rangle$	$\langle 1, 0, 0\rangle$
A_3	$\langle 0.2, 0.2, 0.3\rangle$	$\langle 0.6, 0.2, 0.1\rangle$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 1, 0, 0\rangle$
A_4	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.4, 0.2, 0.1\rangle$	$\langle 0.2, 0.2, 0.5\rangle$	$\langle 1, 0, 0\rangle$
A_5	$\langle 0.2, 0.2, 0.2\rangle$	$\langle 0.5, 0.2, 0.1\rangle$	$\langle 0.3, 0.2, 0.3\rangle$	$\langle 1, 0, 0\rangle$
A_6	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.1, 0.2, 0.6\rangle$	$\langle 0.1, 0.3, 0.5\rangle$	$\langle 1, 0, 0\rangle$

TABLE 5. The normalized PF decision matrix R.

	G_1	G_2	G_3	G_4
A_1	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.5, 0.3, 0.1\rangle$	$\langle 0.5, 0.1, 0.3\rangle$	$\langle 0, 0, 1\rangle$
A_2	$\langle 0.4, 0.4, 0.1\rangle$	$\langle 0.6, 0.3, 0.1\rangle$	$\langle 0.5, 0.2, 0.2\rangle$	$\langle 0, 0, 1\rangle$
A_3	$\langle 0.2, 0.2, 0.3\rangle$	$\langle 0.6, 0.2, 0.1\rangle$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0, 0, 1\rangle$
A_4	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.4, 0.2, 0.1\rangle$	$\langle 0.2, 0.2, 0.5\rangle$	$\langle 0, 0, 1\rangle$
A_5	$\langle 0.2, 0.2, 0.2\rangle$	$\langle 0.5, 0.2, 0.1\rangle$	$\langle 0.3, 0.2, 0.3\rangle$	$\langle 0, 0, 1\rangle$
A_6	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.1, 0.2, 0.6\rangle$	$\langle 0.1, 0.3, 0.5\rangle$	$\langle 0, 0, 1\rangle$

By fixing the weight vector $\mathbf{\Omega} = (0.2, 0.3, 0.1, 0.4)^\top$, we carried out a comparative analysis with the operational laws presented in [22, 33, 44, 48]. The results are shown in Table 7 and Table 8.

Observing from Table 7 and Table 8, we can conclude that, except for our proposed MCDM method, other MCDM methods in [22, 33, 44, 48] cannot distinguish the ranking order between any two alternatives.

To conclude, the proposed MCDM method not only demonstrates effectiveness, but also comprehensively utilizes t-norms and the proposed PFMIWAG operator to aggregate the PF information. This capability enables it to completely overcome the limitations of the methods outlined in [22, 33, 44, 48]. Therefore, the overall reliability of the proposed MCDM methods surpasses that of the MCDM methods outlined in [22, 33, 44, 48].

TABLE 6. A comparison of the ranking orders with the decision matrix shown in Table 3 for different operational laws.

Methods	γ	Scores						Ranking
		A_1	A_2	A_3	A_4	A_5	A_6	
Ju et al.'s operations in [22]	\	0.0978	0.4638	0.2700	0.1475	0.3394	-0.0371	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Pan et al.'s operations in [33]	\	0.1997	0.4632	0.2215	0.1887	0.3522	-0.0605	$A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4 \succ A_6$
Tian et al.'s operations in [44]	\	0.0978	0.4638	0.2700	0.1475	0.3394	-0.0371	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Wang et al.'s operations in [48]	\	0.1402	0.4697	0.1653	0.1522	0.2956	-0.1609	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Our method via PFMIWAG$_{T_\gamma^{\mathbf{SS}},\mathbf{\Omega}}$	-1	0.1470	0.4480	0.1903	0.1634	0.3085	-0.1277	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Our method via PFMIWAG$_{T_\gamma^{\mathbf{H}},\mathbf{\Omega}}$	1	0.1440	0.4448	0.1848	0.1636	0.3175	-0.1340	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Our method via PFMIWAG$_{T_\gamma^{\mathbf{H}},\mathbf{\Omega}}$	1.5	0.1443	0.4452	0.1848	0.1644	0.3183	-0.1360	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Our method via PFMIWAG$_{T_\gamma^{\mathbf{F}},\mathbf{\Omega}}$	1.5	0.1441	0.4449	0.1847	0.1639	0.3179	-0.1349	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Our method via PFMIWAG$_{T_\gamma^{\mathbf{D}},\mathbf{\Omega}}$	1.5	0.1451	0.4450	0.1913	0.1547	0.2977	-0.1111	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
Our method via PFMIWAG$_{T_\gamma^{\mathbf{AA}},\mathbf{\Omega}}$	1.5	0.1418	0.4419	0.1844	0.1581	0.3119	-0.1227	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$

TABLE 7. A comparison of the ranking orders with the decision matrix shown in Table 4 for different operational laws.

Methods	A_1	A_2	A_3	A_4	A_5	A_6	Ranking
Pan et al. [33]	$\langle 1,0,0\rangle$	$\langle 1,0,0\rangle$	$\langle 1,0,0\rangle$	$\langle 1,0,0\rangle$	$\langle 1,0,0\rangle$	$\langle 1,0,0\rangle$	$A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_5 \sim A_6$
	1	1	1	1	1	1	
Proposed method	$\langle 0.84,0.07,0.05\rangle$	$\langle 0.83,0.11,0.04\rangle$	$\langle 0.78,0.08,0.06\rangle$	$\langle 0.78,0.07,0.07\rangle$	$\langle 0.76,0.09,0.05\rangle$	$\langle 0.68,0.12,0.17\rangle$	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5 \succ A_6$
	0.7892	0.7971	0.7190	0.7138	0.7080	0.5097	

TABLE 8. A comparison of the ranking orders with the decision matrix shown in Table 5 for different operational laws.

Methods	A_1	A_2	A_3	A_4	A_5	A_6	Ranking
Ju et al. [22]	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_5 \sim A_6$
	-1	-1	-1	-1	-1	-1	
Tian et al. [44]	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_5 \sim A_6$
	-1	-1	-1	-1	-1	-1	
Wang et al. [48]	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,1\rangle$	$A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_5 \sim A_6$
	-1	-1	-1	-1	-1	-1	
Proposed method	$\langle 0.18,0.11,0.66\rangle$	$\langle 0.18,0.18,0.63\rangle$	$\langle 0.15,0.10,0.67\rangle$	$\langle 0.13,0.09,0.67\rangle$	$\langle 0.13,0.11,0.66\rangle$	$\langle 0.10,0.07,0.81\rangle$	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5 \succ A_6$
	-0.4765	-0.4546	-0.5198	-0.5347	-0.5358	-0.7137	

8. Conclusion

In summary, we introduced an total order on $\mathscr{P}$, denoted by $\preccurlyeq_{\mathrm{W}}$, by using the score function, and the first and the second accuracy functions. We also showed that $\preccurlyeq_{\mathrm{W}}$ is an admissible order, with which $\mathscr{P}$ has the structure of a complete lattice. Moreover, we discussed the operational laws in [2, 4, 7, 14, 21, 23, 26, 51, 52, 56]. We demonstrated that these operational laws might violate the closedness property on $\mathscr{P}$, which implies that that decision procedure might not be convergent. To clear up the problems of these operational laws, we proposed new interactive operational laws for the PFNs via strict t-norms. If assigning specific forms of t-norms to our laws, some attracted operators were obtained. It should be pointed out that our proposed PF operators satisfy the closedness property in comparison with the above-mentioned ones. We also described the main characteristics of these operators. Through the instrument of new operators, we defined the PF interactional weighted average (PFIWA) operator, the PF interactional weighted geometric (PFIWG) operator, and the PF mean interactional weighted average and geometric (PFMIWAG) operator. All of them get possession of monotonicity, idempotency, and boundedness. Furthermore, by applying the PFMIWAG operator, a novel MCDM method was developed with the PF environment. We finally provided a practical example of an investment plan that illustrates the efficiency and superiority of the developed method.

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Xinxing Wu - received his B.S. and Ph.D. degrees in Mathematics from the School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, China, in 2010 and 2015, respectively. He is currently a Professor of Guizhou University of Finance and Economics, Guiyang, China. His research interests include dynamical system, machine learning, fuzzy analysis, and decision analysis.



Zhiyi Zhu - received her B.S. and M.Sc. degrees in Mathematics from the School of Sciences, Southwest Petroleum University, Chengdu, China, in 2020 and 2024, respectively. She is currently pursuing a Ph.D. in Management at the Business School, Sichuan University, Chengdu, China. Her research interests include machine learning, fuzzy analysis, and decision analysis.



Tao Wang - received his B.S and M.Sc. degrees in mathematics from the School of Mathematics & Information, China West Normal University, Nanchong, China, in 2016 and 2019, respectively. He received the Ph.D. degree in Mathematics from the School of Sciences, Southwest Petroleum University, Chengdu, China, in 2023. He is currently a lecturer of Chengdu Normal University, Chengdu, China. His research interests include optimization algorithm, fuzzy mathematics, and decision analysis.



Xu Zhang - received his Ph.D. degree in Mathematics from the Department of Mathematics, Michigan State University, Michigan, USA, in 2016. He is currently a Professor of Shandong University (Weihai), Weihai, China. His research interests include differential equations, dynamical systems, and fuzzy set theory.