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# A SOCIAL MEDIA RUMOR SPREADING MODEL WITH CONTROL MECHANISM BY CONSIDERING DEBUNKING STRATEGIES AND TRIPARTITE GAME

Q. GAO[1,2,*], J.-N. LIN[2], Q. ZHANG[2], Y. XU[2]

Abstract. In this paper, we consider the collaborative governance model of the two main entities for social media rumors, which is among the initial efforts to analyze the positive performance of opinion leaders in rumor control. To this end, we use a mathematical method to model the interplay, mechanism between rumor and anti-rumor information, and apply the game theory to analyze the decision-making results of social media and opinion leaders in rumor evolution system. More precisely, we propose a new Susceptible-Exposed-Infected-Debunker-Recovered (SEIDR) rumor spreading-debunking model with two rumor-control main entities. The novel idea in the proposed model is to identify opinion leader as a main entity in rumor management and to introduce the probability coefficients of official media debunking rumor and opinion leader blocking rumor. Using the basic reproduction number, the local and global stabilities in both the rumor-free and rumor-existence equilibria are derived, along side efficient rumor-epidemic control strategies for rumor attacks. Considering the spreading process of rumor information flow, the opinion leader group is stratified into the upstream and downstream groups, which, along with the social media, constitute three players in the game. Then, we analyze the evolutionary game behaviors of the three players based on their own characteristics, and identify the optimal debunking efforts to reduce the speed and scale of rumor propagation. Meanwhile, some numerical simulations are carried out to illustrate the feasibility of our theoretical results and the effectiveness of the obtained control strategies.

Keywords: Online Social Media Rumor, SEIDR Spreading Model, Stability Analysis, Debunking Strategy, Tripartite Game.

AMS Subject Classification: 34D20, 91A22.

## 1. Introduction

In today's world, human society has been entering the information age, and meanwhile, online social media has emerged as the primary platform for its users to obtain, transmit, and share information. In comparison to traditional media, the mode, role, and impact of information dissemination have undergone significant changes. Social media users are not only recipients of information, but also possess the ability to post and disseminate information [41]. In the new media era, online information can travel faster [1], and opinion may be more asymptotically clustered [38]. The scale, speed, and real-time of the social Web and the increased possibilities for audiences play a crucial role to share and express themselves [31]. However, the low threshold of online information released on social media, although can facilitate the information propagation, reduces the cost of spreading false information. Also, because of the lack of response mechanisms to ensure the authenticity and security of information, social media has gradually evolved into a breeding ground for the online rumors. As a result, the online social media with information

[1]School of Mathematics, Nanjing Audit University, Nanjing, China
[2]School of Statistics and Data Science, Nanjing Audit University, Nanjing, China
*Corresponding author e-mail: qwgao@nau.edu.cn
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diffusion is a double-edged sword, which can be used to by authorities for effective information distribution and crisis management, or by malicious entities to spread rumors and fake news [19].

Rumor was originally defined as "a tall tale of explanations of events circulating from person to person and pertaining to an object, event, or issue in public concern" [35], which has always existed throughout history, and has been used to influence public opinion [32]. Afterwards, the author of the paper [22] characterized rumor as "a proposition for belief of topical reference disseminated without official verification", and in [15] the author identified rumor as "a claim whose truthfulness is in doubt and which often has no clear source even if its ideological or partisan origins and intents are clear". Especially in the digital age, massive digital misinformation was ranked first by the World Economic Forum among the top future global risks including water supply crisis, major systemic financial failures, and failure to adapt to climate change [16]. Recent studies have shown that social media sites are the major platforms to spread rumors [25], and the popularity and speed of rumor spreading on online social media are much higher than those on physical social networks, which easily leads to serious public panic and economic loss. For example, in 2011, the Fukushima nuclear leak in Japan caused some merchants to hoard salt and drive up its price [47]; and on April 23, 2013, a false rumor about a terror attack on the White House, in which President Obama was allegedly injured, provoked an immediate crash in the stock market [44]. Another is taken as that during Hurricane Harvey in 2017, a rumor, "Immigration status has to be checked before you are allowed to enter a shelter" [3], resulted in great confusion, panic, and anger among people in the affected area.

During extreme events such as man-made and natural disasters, the rumor propagation and its control mechanism have increasingly become urgent issues for social media platforms and government agencies. The spreading and debunking of online rumor is a dynamic and reciprocal process, which has been studied in multiple disciplines and analyzed from multiple aspects, involving rumor detection [49], prevention [17], dissemination [24], debunking and intervention [19, 40], and rumor source tracking [37]. As a rumor starts spreading on social media, the first job the stakeholders have to do is to debunk the rumor and reduce the loss as soon as possible, where human and financial resources are necessary for the information management. The efficient debunking strategy, as a potential tool, is explored to reduce the loss of massive digital misinformation, for example, by increasing the number of accounts to disseminate the anti-rumors, attracting more people to re-tweet the anti-rumor, or encouraging existing rumor spreaders to delete or clarify the rumors [19].

A certain specified user of social media may play an influential role in the information management [48]. Once an online rumor posts on social media, the official media, such as official websites, Twitter accounts, and TV news, will immediately release the anti-rumor information. Even as the spread of rumors is booming, multiple debunking strategies will be taken by social media [19]. Due to its authority, timeliness, and precision in debunking rumor, the social media is regard as a main entity of rumor guidance and control, which can be used to shape and lead the public opinion. However, the official media maintains a relatively low level of engagement with social media users, resulting in a somewhat superior yet significantly delayed information dissemination capability. As a counterpart, the opinion leaders including social media influencers have more fan followers owing to their real-time interaction with social media users, and are more proactive in conducting emotion transitions, which precisely compensates for the shortcomings of official media in real-time interaction. Unfortunately, online rumors often originate from opinion leaders distorting, exaggerating, and fabricating truth, intentionally creating gimmicks to capture traffic, and thus the opinion leaders have always been assumed in traditional studies to maintain a hostile game relationship with the official media. The point to make here is that the assumption is biased, and in reality for the rumor-control main entity, its decision-making behaviors depend on interest, right, and responsibility from the perspective of economics. So in this paper, we focus on the positive role of opinion leaders to guide and control rumor, and identify the opinion leader group as another main entity in rumor management.

Throughout the network of online social media, the dissemination of rumors and anti-rumors, similarly to infectious disease transmission, is a person-to-person information contagion process [8], it is natural to mathematically model rumor/anti-rumor spreading based on the theory of epidemics [26]. Particularly, because of the relative tractability of the underlying social network [18], the online rumor spreading would fit into the epidemic modeling even better than the oral spreading. The earliest references in dynamic rumor spreading are based on SIR (Susceptible-Infected-Recovered) epidemic model, which were introduced by the researchers in [20, 30]. The extension of the SIR-based rumor spreading models was revisited by many researchers in [10, 42, 43, 47], and references therein. The rumor spreading could be affected by some factors, such as spreading media [4], spreading network topology [14], information characteristics [34], event uncertainty [33], topic type [27], gossip networks [2, 5, 11, 12], and the interplay among them [19, 45]. And the key factors that influence debunking strategies include the current scope and the popularity of the rumor spreading, and the effectiveness of various debunking strategies [19]. Besides, considering that the main entities in rumor guiding and controlling consist of the official media and opinion leader, the probability coefficients of official media debunking rumor and opinion leader blocking rumor are two critical factors in the spreading and debunking processes of online rumor. Therefore, it is imperative to understand the underlying dynamics of the interplay between the two rumor-control main entities, and to design a general method of collaborative debunking-efforts to control the rumor attacks.

In this paper, we apply a mathematical method to model the interplay mechanism between rumor spreading and debunking process, and apply the dynamic game theory to analyze the decision-making results of social media and opinion leader in rumor evolution system. More specifically, we propose a new SEIDR rumor spreading-debunking competitive model with two main entities of rumor guidance and control. The key characteristics of the social network, rumor, and anti-rumor are parameterized in an ordinary differential equation system. The novel idea in the proposed model is to identify opinion leader as the main body in rumor management and to assign the probability coefficients of official media debunking rumor and opinion leader blocking rumor. Using the basic reproduction number, the local and global stabilities in both the rumor-free and rumor-existence equilibria are obtained along side efficient rumor-epidemic control strategies for the rumor attacks. Remark that the local asymptotic stability means that near a certain state point of a system, the system state will, regardless of the initial conditions, tend to its stable state point after a period of time, which deals only with the system behavior near the state point, not the whole state space; while the global asymptotic stability involves not only the stability of equilibrium points, but also the system long-term behavior under any initial conditions. Since the process of information dissemination is of obvious upstream and downstream feature, then the opinion leader group is stratified into the upstream and downstream groups, which, along with the social media, constitute three players in the game theory. The evolutionary game behaviors of the three players are analyzed based on their own characteristics, and the optimal debunking efforts are identified to reduce the speed and scale of rumor propagation. Meanwhile, numerical simulations are implemented to illustrate the feasibility of our theoretical results and the effectiveness of the obtained control strategies.

The remaining part of this paper is organized as follows: In Section 2, we propose a new SEIDR rumor spreading model with control mechanism, in Section 3 we present the stability analysis of the rumor-free and rumor-existence equilibria, in Section 4 we identify the game-based mechanism in debunking rumor, and in Section 5 we conclude the paper with future research directions. The related proofs and parameter settings of this study are available in Appendix.

## 2. Social media rumor spreading model with control mechanism

Due to the high similarity between their dynamic behaviors of internet information and biological viruses, in this paper we use the dynamic model to characterize the propagation mechanism of the online rumor throughout social media, where a new exposed/latent group is added to the classical SIR-model because of the heterogeneity of information dissemination. A rumor control mechanism is also designed to combat the prevalence of social media rumors, with the main entities to control rumor comprised of official mainstream media and online opinion leaders. Because of the real-time and openness of social media, the number of users varies with online websites or accounts login and logout, and thus two system-parameters are assigned as in-migration rate and out-migration rate. Note that after the online rumors appearing, the decision making by official medias and opinion leaders can affect users' emotion transitions and further drive the evolution of public opinion events. Hence, the probabilities of official media actively debunking rumors and opinion leaders driving the rumor propagation are introduced to explore the impacts of social media and opinion leaders on the rumor information dissemination.

The arguments mentioned-above motivate us to develop an SEIDR rumor spreading model with its action mechanism, namely that the outbreak of an extreme event could provoke the rumor and anti-rumor information, and then evolve gradually into a public opinion event after spreading through social media. According to whether they know the rumor, spread the rumor, and debunk the rumor, the population of social media users is divided into five groups as follows: Ignorant users (similar to the Susceptible ones and then denoted by $S$) refer to those who have not been exposed to or aware of the rumor and anti-rumor information; Latent/Exposed users (denoted by $E$) refer to those who have been exposed to and aware of the rumor and anti-rumor information, but have not yet decided to spread it due to the latent time requirement; Rumor Spreaders (similar to the Infected ones by virus and then denoted by $I$) refer to those who spread the rumor information and are contagious; Debunkers (denoted by $D$) refer to those who spread the anti-rumor information and are contagious; and Stiflers (similar to the Removed ones and then denoted by $R$) refer to those who have known the rumor and anti-rumor information but do not have contagiousness due to the information overload or the forgetting mechanism. The transition diagram among social media users in SEIDR model with its modelling parameters is shown in Figure 1, where the involved parameters are explained in Table 1, and the transition events with its rates are summarized in Table 2.

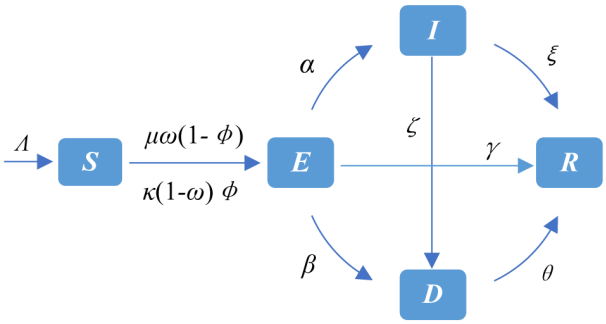



Figure 1. The transition diagram among social media users.

From Fig.1, Tables 1 and 2, the transitions of social media users in SEIDR model are detailed as follows:

1. Note that the rumor-control main entities involve official mainstream medias and online opinion leaders, with the probabilities $\phi$ and $\omega$ that the official medias control the rumor transmission and the opinion leaders drive the public opinion dissemination, respectively. Hence, when a rumor Spreader contacts an Ignorant user, the Ignorant may become an Exposed user

TABLE 1. Modelling parameters and their explanations.

| Parameters | Explanation |
|---|---|
| $N(t)$ | Total population of social media users at time $t$ |
| $S(t)$ | Population of Ignorant users at time $t$ |
| $E(t)$ | Population of Exposed users at time $t$ |
| $I(t)$ | Population of rumor Spreaders at time $t$ |
| $D(t)$ | Population of rumor Debunkers at time $t$ |
| $R(t)$ | Population of Stiflers at time $t$ |
| $\Lambda$ | In-migration rate of new users |
| $\mu$ | Contact rate of rumor |
| $\kappa$ | Contact rate of anti-rumor |
| $\omega$ | Probability of opinion leaders driving the rumor propagation |
| $\phi$ | Probability of official media actively debunking rumors |
| $\epsilon$ | Out-migration rate of existing users |
| $\alpha$ | Rumor spreading rate of exposed users |
| $\beta$ | Rumor debunking rate of exposed users |
| $\gamma$ | Transition rate from E to R |
| $\zeta$ | Transition rate from I to D |
| $\xi$ | Transition rate from I to R |
| $\theta$ | Transition rate from D to R |

TABLE 2. Transition paths and transition rates.

| Transition paths | Transition rates |
|---|---|
| Online users $\rightarrow S(t)$ | $\Lambda$ |
| $E(t) \rightarrow I(t)$ | $\alpha E$ |
| $E(t) \rightarrow D(t)$ | $\beta E$ |
| $S(t) \rightarrow E(t)$ | $\mu\omega(1-\phi)SI$ |
| $S(t) \rightarrow E(t)$ | $k(1-\omega)\phi SD$ |
| $I(t) \rightarrow D(t)$ | $\zeta I$ |
| $I(t) \rightarrow R(t)$ | $\xi I$ |
| $D(t) \rightarrow R(t)$ | $\theta D$ |
| $E(t) \rightarrow R(t)$ | $\gamma E$ |
| $S, E, I, D, R \rightarrow$ Offline users | $\epsilon$ |

with the rate $\mu\omega(1-\phi)$, and thus the population of users who transit from groups $S$ to $E$ is $\mu\omega(1-\phi)S(t)I(t)$. Likewise, when a Debunker contacts an Ignorant, the Ignorant may become an Exposed user with the rate $\kappa(1-\omega)\phi$, and thus the population of users transiting from groups $S$ to $E$ is $\kappa(1-\omega)\phi S(t)D(t)$.

2. The Exposed users in group $E$ transit into the Spreader and Debunker groups with rates $\alpha$ and $\beta$ when the Exposed ones begin to spread rumor and anti-rumor information throughout the online social media after an incubation period.

3. Because of the forgetting mechanism, the information overload or some sufficient control mechanisms, the Exposed user may choose not to repost the rumor and anti-rumor information, and directly becomes a Stifler with a rate $\gamma$, and the Spreaders and the Debunkers become a Stifler with rates $\xi$ and $\theta$, respectively. Hence, the population of users who transit from groups $E$ to $R$, $I$ to $R$, and $D$ to $R$, are $\gamma E(t)$, $\xi I(t)$ and $\theta D(t)$, respectively.

4. As a rumor Spreader see the anti-rumor information, the Spreader becomes a Debunker with a rate $\zeta$, and the population of users who transit from groups $I$ to $D$ is $\zeta I(t)$.

5. Considering the real-time and openness of the online social media, all the existing users in groups $S$, $E$, $I$, $D$, and $R$ may migrate out with a rate $\epsilon$. The in-migration of new users is done in $S$-group with its rate $\Lambda$.

To sum up, the SEIDR rumor spreading model with its control mechanism can be formulated by the system of non-linear ordinary differential equations as followings:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu\omega(1-\phi)S(t)I(t) - \kappa(1-\omega)\phi S(t)D(t) - \epsilon S(t), \\ \dfrac{dE(t)}{dt} = \mu\omega(1-\phi)S(t)I(t) + k(1-\omega)\phi S(t)D(t) - (\alpha+\beta+\gamma+\epsilon)E(t), \\ \dfrac{dI(t)}{dt} = \alpha E(t) - (\zeta+\xi+\epsilon)I(t), \\ \dfrac{dD(t)}{dt} = \beta E(t) + \zeta I(t) - (\theta+\epsilon)D(t), \\ \dfrac{dR(t)}{dt} = \gamma E(t) + \xi I(t) + \theta D(t) - \epsilon R(t). \end{cases} \tag{1}$$

The total population of social media users at time $t$ satisfies

$$N(t) = S(t) + E(t) + I(t) + D(t) + R(t).$$

Note that $N(t)$ varies over time since the users enter and leave the system through their online websites or accounts login and logout, which imbalances the inflows and outflows of a given system. A summation of all the equations in (1) gives

$$N'(t) = \Lambda - \epsilon N.$$

After a simple computation, one has

$$N(t) = \left(N(0) - \frac{\Lambda}{\epsilon}\right) e^{-\epsilon t} + \frac{\Lambda}{\epsilon},$$

and then

$$\lim_{t\to\infty} N(t) = \frac{\Lambda}{\epsilon},$$

which means that $N(t)$ approaches to the stable equilibrium point $\Lambda/\epsilon$ as $t \to \infty$. Denote the feasible region of system (1) by

$$\Omega = \left\{ (S,E,I,D,R) \in \mathbb{R}_+^5 \middle| 0 \le S+E+I+D+R \le \frac{\Lambda}{\epsilon} \right\}.$$

The non-negativity analysis is necessary to enrich the plausibility of the SEIDR model proposed by system (1). In fact, the first equation of (1) implies that

$$S'(t) \le \Lambda - \epsilon S,$$

and then

$$S(t) \le \frac{\Lambda}{\epsilon} + \left(S(0) - \frac{\Lambda}{\epsilon}\right) e^{-\epsilon t}.$$

For a long time, it follows that

$$S(t) \le \frac{\Lambda}{\epsilon} =: \nabla_1. \tag{2}$$

This, along with the second equation of (1), leads to

$$E'(t) \le (\mu\omega(1-\phi) + \kappa(1-\omega)\phi)\nabla_1^2 - (\alpha+\beta+\gamma+\epsilon)E,$$

and then

$$E(t) \le \frac{(\mu\omega(1-\phi)+\kappa(1-\omega)\phi)\nabla_1^2}{\alpha+\beta+\gamma+\epsilon} + \left(E(0) - \frac{(\mu\omega(1-\phi)+\kappa(1-\omega)\phi)\nabla_1^2}{\alpha+\beta+\gamma+\epsilon}\right) e^{-(\alpha+\beta+\gamma+\epsilon)t},$$

which, for a long time, yields that

$$E(t) \le \frac{(\mu\omega(1-\phi)+\kappa(1-\omega)\phi)\nabla_1^2}{\alpha+\beta+\gamma+\epsilon} =: \nabla_2.$$

Similarly to the above derivations, we obtain that

$$I(t) \le \frac{\alpha\nabla_2}{\zeta+\xi+\epsilon} =: \nabla_3,$$
$$D(t) \le \frac{\beta\nabla_2+\xi\nabla_3}{\theta+\epsilon} =: \nabla_4,$$
$$R(t) \le \frac{\gamma\nabla_2+\zeta\nabla_3+\theta\nabla_4}{\epsilon} =: \nabla_5.$$

Hence by the above discussion, the following proposition is derived as below.

**Proposition 1.** *The solutions of system (1) with initial conditions satisfy $S(t) > 0$, $E(t) > 0$, $I(t) > 0$, $D(t) > 0$, and $R(t) > 0$ for all $t > 0$, and the feasible region $\Omega$ is a positively invariant and attracting set with respect to system (1).*

Notice here that the positively invariant and attracting set $\Omega$ means that all feasible solutions of system (1) enter or remain in the region $\Omega$, and further the existence, uniqueness and continuation results hold for the system in this region.

## 3. Stability analysis

In this section, we conduct the stability analysis of the rumor-free and rumor-existence equilibria, which can be used to predict the rumor spreading tendency in a long run and to identify strategies for controlling rumor propagation.

3.1. **Equilibria and basic reproduction number.** The basic reproduction number, denoted by $R_0$, is a fundamental concept in epidemic dynamics of biological disease and online information, which refers to "the average number of secondary cases generated by one primary case at the start of the epidemic in a completely susceptible population" [9, 28]. And the number $R_0$ essentially determines the dynamic behavior of an epidemic and how intensive a policy will need to be to control the epidemic [28].

Firstly, we establish the rumor-free equilibrium, where the total number of online social media users who have known the rumor and anti-rumor information, namely, $E(t)+I(t)+D(t)$, tends to 0 as the time $t$ goes to infinity. Hence, we obtain that the system (1) always has the rumor-free equilibrium as:

$$P_0 = (S_0, E_0, I_0, D_0, R_0) = \left(\frac{\Lambda}{\epsilon}, 0, 0, 0, 0\right). \tag{3}$$

Next, we determine the expression of the basic reproduction number $R_0$ with the method of next generation matrix [9]. Let $X = (E, I, D, R)^T$, then $X' = F(X) - V(X)$, where

$$F(X) = \begin{pmatrix} \mu\omega(1-\phi)S(t)I(t) + \kappa(1-\omega)\phi S(t)D(t) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$V(X) = \begin{pmatrix} (\alpha+\beta+\gamma+\epsilon)E(t) \\ -\alpha E(t) + (\zeta+\xi+\epsilon)I(t) \\ -\beta E(t) - \xi I(t) + (\theta+\epsilon)D(t) \\ -\zeta I(t) - \theta D(t) - \gamma E(t) + \epsilon R(t) \end{pmatrix}.$$

At the rumor-free equilibrium $P_0$, the Jacobian matrices of $F(X)$ and $V(X)$ are, respectively,

$$F^*(P_0) = \begin{pmatrix} 0 & \mu\omega(1-\phi)\frac{\Lambda}{\epsilon} & \kappa(1-\omega)\phi\frac{\Lambda}{\epsilon} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V^*(P_0) = \begin{pmatrix} \alpha+\beta+\gamma+\epsilon & 0 & 0 & 0 \\ -\alpha & \zeta+\xi+\epsilon & 0 & 0 \\ -\beta & -\xi & \theta+\epsilon & 0 \\ -\gamma & -\zeta & -\theta & \epsilon \end{pmatrix}.$$

Set $B = F^*(P_0){V^*}^{-1}(P_0)$, its spectral radius (i.e. the basic reproduction number) $R_0$ is obtained as:

$$R_0 = \frac{\Lambda}{\epsilon}\frac{\Lambda\mu\omega(1-\phi)\alpha(\theta+\epsilon)+\Lambda(\xi+\zeta+\epsilon)k(1-\omega)\phi\beta+\alpha\xi\kappa(1-\omega)\phi\Lambda}{(\alpha+\beta+\gamma+\epsilon)(\theta+\epsilon)(\zeta+\xi+\epsilon)}. \tag{4}$$

To study the dynamic behavior of the SEIDR model described by system (1), we now present the following results on the rumor-free and rumor-existence equilibria.

**Theorem 1.** *When $R_0 \leq 1$, the rumor-free equilibrium $P_0$ is the only equilibrium in $\Omega$ of the form (3); when $R_0 > 1$, the unique rumor-existence equilibrium $P^* = (S^*, E^*, I^*, Q^*, R^*)$ exists in* $\mathrm{int}(\Omega)$ *with coordinates satisfying*

$$\begin{cases} S^* = \dfrac{(\alpha+\beta+\gamma+\epsilon)(\theta+\epsilon)(\xi+\zeta+\epsilon)}{\mu\omega(1-\phi)\alpha(\theta+\epsilon)+(\xi+\zeta+\epsilon)k(1-\omega)\phi\beta+\alpha\xi k(1-\omega)\phi}, \\ E^* = \dfrac{\Lambda-\epsilon S^*}{\alpha+\beta+\gamma+\epsilon}, \\ I^* = \dfrac{\alpha E^*}{\xi+\zeta+\epsilon}, \\ D^* = \dfrac{(\beta(\alpha+\xi+\epsilon)+\alpha\xi)E^*}{(\alpha+\xi+\epsilon)(\theta+\epsilon)}, \\ R^* = \dfrac{\zeta I^*+\theta D^*+\gamma S^*}{\epsilon}. \end{cases} \tag{5}$$

*Also, $N^* = S^* + E^* + I^* + D^* + R^*$.*

*Remark* 1. The rumor-existence equilibrium with its coordinates $(S^*, E^*, I^*, D^*, R^*)$ can give the final scales of the Ignorant, Exposed, Spreaders, Debunkers, and Stiflers, respectively, as the time $t$ tends to infinity. In particular, the final scale of users affected by the rumor, i.e., $E^* + I^* + D^*$, can be used to measure the level of the rumor popularity.

Note that the basic reproduction number $R_0$ is a sharp threshold parameter that determines the existence of the rumor-free and rumor-existence equilibria. For the proof of Theorem 1, refer to Appendix A.1.

3.2. **Stability of the rumor-free equilibrium.** In the subsection, we present two theorems on the dynamic behavior of the rumor-free equilibrium, whose proofs are deferred to Appendixes A.2 and A.3. The following theorem is concerned with the local and asymptotical stability.

**Theorem 2.** *When $R_0 \leq 1$, the rumor-free equilibrium $P_0$ is locally asymptotically stable with respect to $\Omega$.*

Further, we consider the global and asymptotical stability of the rumor-free equilibrium, which, along with Theorem 2, shows that the local and global stabilities are equivalent.

**Theorem 3.** *When $R_0 \leq 1$, the rumor-free equilibrium $P_0$ is globally asymptotically stable with respect to $\Omega$.*

*Remark* 2. In Theorems 2 and 3, the local and global stabilities of the rumor-free equilibrium $P_0$ mean that when $R_0 \leq 1$, the Ignorant users $S(t)$ approaches to its steady state $\frac{\Lambda}{\epsilon}$, and the users who are aware of the rumor and anti-rumor information all approach to 0, as the time $t$ goes to infinity, which implies that $R_0 \leq 1$ guarantees the rumor is vanishing in the end and then its propagation remains under control. Hence, in order to bring the rumor attack under

control, we adopt some feasible and effective strategies to reduce $R_0$ to be less than 1, where these strategies are the control strategies to reduce the intensity of rumor attack. From the arguments above, Theorems 2 and 3 are the theoretical basis of these control strategies.

Note that the basic reproduction number $R_0$ not only determines the local and global stabilities of the rumor-free equilibrium $P_0$, but also governs whether the affected users by rumor and anti-rumor in time locally and globally, respectively, which implies that the number $R_0$ will play an important role in the control of an outbreak of rumor throughout online social media. As shown in Fig.2, the tendency of the rumor propagation is depressive in a long run, and the final size of affected users will gradually disappear, which is consistent with the theoretic results in Theorems 2 and 3. See Table 7 in Appendix for the parameter values used in Fig.2.

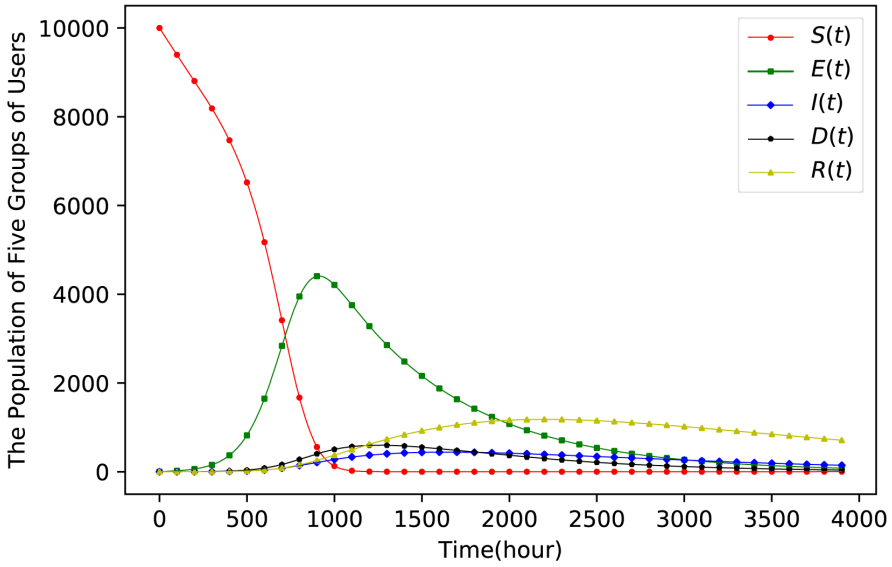



FIGURE 2. The simulated tendency of social media users when $R_0 < 1$.

3.3. **Stability of the rumor-existence equilibrium.** Firstly in this subsection, the uniform persistence of system (1) is given by the acyclicity theorem ( [46], p.18), and its proof is delayed to Appendix A.4.

**Definition 1.** [7] The system (1) is said to be uniformly persistent in $\Omega$, if there exists a constant $0 < C < 1$ such that any solution $(S(t), E(t), I(t), D(t), R(t))$ of the system (1) with initial value $(S(0), E(0), I(0), D(0), R(0)) \in \text{int}(\Omega)$ satisfies

$$\min\left\{\liminf_{t\to\infty} S(t), \liminf_{t\to\infty} E(t), \liminf_{t\to\infty} I(t), \liminf_{t\to\infty} D(t), \liminf_{t\to\infty} R(t)\right\} \geq C.$$

**Theorem 4.** *The system (1) is uniform persistent in $\Omega$ if and only if $R_0 > 1$.*

*Remark* 3. The uniform persistence of (1) in the bounded set $\Omega$ is equivalent to the existence of a compact set $K \subset \Omega$ that is absorbing for (1) (see [6]).

Observe that the first four equations of the system (1) do not contain the group $R$, the dynamics for the system (1) can be determined by groups $S$, $E$, $I$, and $D$. Thus, we use the

reduction method by considering a subsystem with only four equations, namely,

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu\omega(1-\phi)S(t)I(t) - k(1-\omega)\phi S(t)D(t) - \epsilon S(t), \\ \dfrac{dE(t)}{dt} = \mu\omega(1-\phi)S(t)I(t) + k(1-\omega)\phi S(t)D(t) - (\alpha+\beta+\gamma+\epsilon)E(t), \\ \dfrac{dI(t)}{dt} = \alpha E(t) - (\zeta+\xi+\epsilon)I(t), \\ \dfrac{dD(t)}{dt} = \beta E(t) + \zeta I(t) - (\theta+\epsilon)D(t). \end{cases} \tag{6}$$

Similarly to Proposition 1, an assertion regarding subsystem (6) is given as follows:

**Proposition 2.** *The feasible region* $\Theta = \{(S,E,I,D) \in \mathbb{R}^4_+ | 0 \leq S+E+I+D \leq \frac{\Lambda}{\epsilon}\}$ *is a positively invariant and attracting set with respect to system (6).*

Secondly, we analyze the asymptotical stability of the rumor-existence equilibrium $P^*$, which is of more importance to investigate than the rumor-free equilibrium $P_0$ from the epidemiological perspective. The corresponding proofs are presented in Appendixes A.5 and A.6.

**Theorem 5.** *When* $R_0 > 1$*, the rumor-existence equilibrium of subsystem (6),* $(S^*, E^*, I^*, D^*)$ *is locally asymptotically stable with respect to* $\Theta$*.*

In a real social media system, the local stability of rumor-existence equilibrium $P^*$ may be of no practical significance since it merely guarantees this stability relative to small perturbation of the initial state from the equilibrium. Thus in what follows, by the Li-Muldowney geometric approach [29], we aim to establish the global and asymptotical stability of the equilibrium $(S^*, E^*, I^*, D^*)$ for subsystem (6) when $R_0 > 1$. The global stability results relate to all possible displacements of the initial state in the feasible region. A combination with Definition 1 leads to the global stability of $P^*$ for the entire system (1).

**Theorem 6.** *When* $R_0 > 1$*, the rumor-existence equilibrium of subsystem (6),* $(S^*, E^*, I^*, D^*)$ *is globally asymptotically stable in* $\mathrm{int}(\Theta)$*.*

*Remark* 4. The global and asymptotical stability of $P^*$ in $\mathrm{int}(\Theta)$ means that the region $\Theta - \{(S,E,I,D)|I=0\}$ is a globally asymptotically stable region for the rumor-existence equilibrium.

*Remark* 5. The local and global stabilities of the rumor-existence equilibrium $P^*$ established in Theorems 5 and 6 illustrate that when $R_0 > 1$, all the states $S(t)$, $E(t)$, $I(t)$, $D(t)$, and $R(t)$ approach to their steady states $S^*$, $E^*$, $I^*$, $D^*$, and $R^*$, respectively, appearing in system (5), as the time $t$ goes to infinity. The results show that $R_0 > 1$ guarantees the online rumor is persistent and always circulates within a population, where we can make a prediction of the tendency of the rumor propagation in a long run, and assess the level of the rumor popularity through the final scale of the users affected by the rumor, that is $E^* + I^* + D^*$.

Obviously, the basic reproduction number $R_0$, as a threshold number, completely determines the local and global stabilities of the rumor-existence equilibrium $P^*$, and governs whether or not the malicious rumor is always persistent in time locally and globally throughout online social media.

Finally, a numerical experiment is conducted to predict the trend of rumor transmission in a long run. See the simulation results in Fig.3, which are consistent with the theoretic results in Theorems 5 and 6. For the setting of parameters in Fig.3, we refer to Table 8 in Appendix.

3.4. **Control strategies based on basic regeneration number.** In this subsection, the control strategies are provided by a combination of the basic reproduction number and the rumor-control main entities. During the rumor propagation modelling, we focus on the analysis

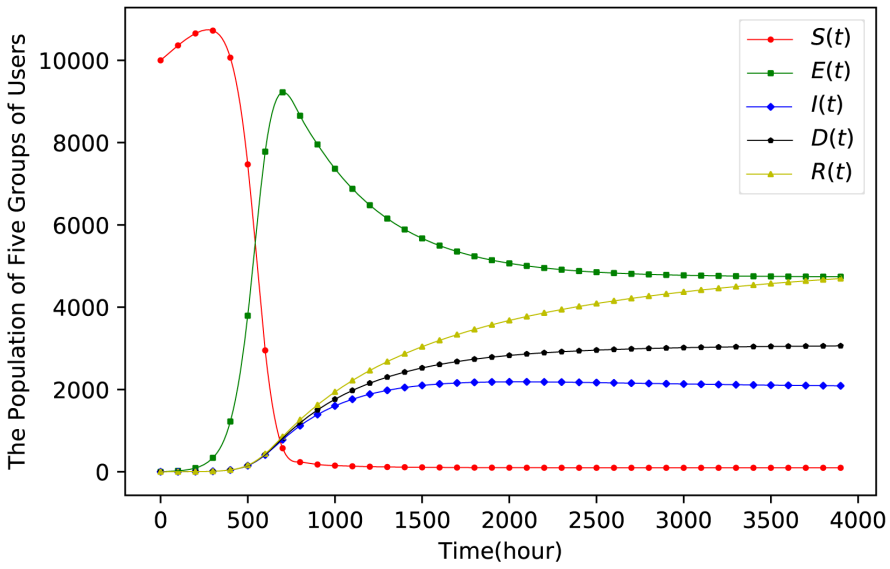
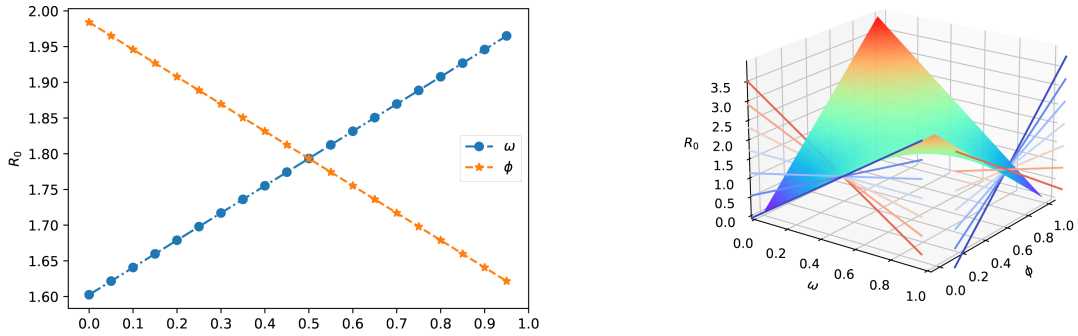



FIGURE 3. The simulated tendency of social media users when $R_0 > 1$.

of whether the official medias combat the rumor transmission and whether the opinion leaders drive the public opinion dissemination. As mentioned in Theorems 2 and 3 as well as Remark 2, the threshold value $R_0$ completely determines whether the rumor propagation remains under control, and its value is positively related to the scale of rumor dissemination. Therefore from Eqn. (4), we take $R_0$ as a function of $\phi$ and $\omega$ to explore how the function $R_0$ varies with different response behaviors of the rumor-control main entities to the rumor, where two variables $\phi$ and $\omega$ are the probabilities that the official medias actively debunk rumor and the opinion leaders actively drive the rumor spreading, respectively. Also, we suggest to Table 1 for their explanation.

As shown in Fig.4(a), we observe that the function $R_0$ decreases as $\phi$ increases or $\omega$ decreases, which indicates the debunking efforts of official mainstream media can avoid escalation of public opinion, while the promotion degree of opinion leaders can accelerate the emotion transformation of social media users and then expand the scale of rumor dissemination. And the 3D diagram in Fig.4(b) matches well with this indication. It should be noted that the minimum point on the saddle surface is not simply a choice of complete debunking by official media, nor that of complete blocking by opinion leaders. In other words, the minimum value of $R_0$ does not mechanically depend on the pure strategic decision of the rumor-control main entities, which demonstrates that a reasonable setting for the proportion of debunking efforts of official media and opinion leaders helps to obtain the minimum threshold for rumor dissemination, thereby eliminating the scale of public opinion. The theoretic result is of practical significance that in real scenarios, the investment and supervision of the social media cost is necessary to be further defined and assessed, and the game between official media and opinion leaders is a complex and dynamic process, therefore a crucial research issue is inspired as how integrate the strategies of both players and balance the rumor-debunking proportion, so as to achieve the maximal protection from rumor attacks with the minimal rumor-control costs.




(a) 2D diagram

(b) 3D diagram

FIGURE 4. The relationship between $R_0$ and $\omega$, $\phi$.

## 4. Game-based mechanism for the rumor control

In the rumor-debunking process, the official media and opinion leader are two main entities to guide and control the online rumor with different action mechanisms due to their own characteristics. More precisely, official medias are the public preference to disseminate information and play a directional role to shape public opinion because of its high authority. However, the official medias are of relatively low interaction with social media users and have a significant lag in information dissemination. As the counterpart, opinion leaders have more fan followers due to their real-time interaction with social media users, and are more proactive in conducting emotion transitions. Unfortunately, the online rumors often originate from opinion leaders distorting, exaggerating, and fabricating truth, intentionally creating gimmicks to capture traffic. So in this section, based on the proposed SEIDR model with the probability coefficients of official media debunking rumors and opinion leaders provoking public opinion, we apply the dynamic game theory to analyze the decision results of social media and opinion leaders in rumor evolution system under the assumption of optimal returns for the "Economist", and explore a general method of collaborative efforts between both to control the rumor attacks throughout online social media.

4.1. **Preliminaries.** Considering the dissemination process of information flow shown in Fig.5 with a relative lag of official media, the first rumor-control main entity during the outbreak of rumor upstream event are a group of opinion leaders driving the rumor spreading, denoted by $A$, while in the rumor downstream event, the rumor-debunking main entities consist of the official medias and the other opinion leaders (denoted by $B$). Because of the fluidity of information, the upstream group $A$ have priority decision-making rights. As the rumor starts circulating, the group $A$ should balance the traffic returns and government penalties which are both brought by driving the rumor propagation. If the traffic returns exceed the government penalties, then the group $A$ choose to drive the rumor spreading; otherwise, the group $A$ choose to block the rumor spreading. Therefore, the choice strategy of upstream opinion leader $A$ depends on the probabilities of driving and blocking the rumor, as well as the government penalties.

As for the downstream rumor-debunking groups that are affected by the lag of information flow, their efforts to control rumors will be greatly reduced if the upstream group $A$ choose to block rumor. So in this paper, we focus on the case in which the upstream opinion leaders $A$ only choose to drive the rumor, the social medias choose to debunk rumor or not to, and the downstream opinion leaders $B$ choose to debunk rumor or not to. A collaborative mechanism among opinion leaders $A$, official medias, and opinion leaders $B$ is established as follows: the

government will transfer the penalty paid by group $A$ who choose to drive the propagation of upstream rumor to group $B$ who actively debunk the rumors. Refer to Table 3 for the game parameters and their explanations in the tripartite game model.




FIGURE 5. Transmission diagram of game information flow.

TABLE 3. Game parameters and their explanations.

| Parameters | Explanation |
|---|---|
| $x$ | Probability of opinion leader $A$ choosing to block rumor |
| $y$ | Probability of social media choosing not to debunk rumor |
| $z$ | Probability of opinion leader $B$ choosing not to debunk rumor |
| $R_{A1}$ | Return of opinion leader $A$ choosing to block rumor |
| $C_{A1}$ | Cost of opinion leader $A$ choosing to drive rumor |
| $R_{A2}(> R_{A1})$ | Return of opinion leader $A$ choosing to drive rumor |
| $C_{A2}(> C_{A1})$ | Cost of opinion leader $A$ choosing to drive rumor |
| $P$ | Penalty of opinion leader $A$ choosing to drive rumor |
| $R_{G1}$ | Basic return of social media if opinion leader $A$ block rumor |
| $R_{B1}$ | Basic return of opinion leader $B$ if $A$ block rumor |
| $R_{G2}$ | Excess return of social media after $A$ blocking rumor |
| $R_{B2}$ | Excess return of opinion leader $B$ after $A$ blocking rumor |
| $C_{Gu}$ | Comprehensive loss of social media because $A$ block rumor |
| $C_{Gu}$ | Comprehensive loss of opinion leader $B$ because $A$ block rumor |
| $E_{G1}$ | Operation Basic cost of official media if $A$ block rumor |
| $\rho E_{B1}$ | Operation Basic cost of opinion leader $B$ if $A$ block rumor |
| $E_{G2}$ | Operation cost of official media if $A$ drive rumor |
| $\rho E_{B2}$ | Operation cost of opinion leader $B$ if $A$ drive rumor |

The logical correlation among some parameters setting in Table 3 is listed as follows:

1. $R_{G1} < R_{G2}, R_{B1} < R_{B2}, E_{G1} < E_{G2}, \rho E_{B1} < \rho E_{B2}$.

2. If the upstream opinion leaders $A$ decide to block rumors, their impact on the other rumor-control main entities requires a certain reaction time in view of the lag of information flow. Hence, the basic return and excess return are both considered for the social media and opinion leaders $B$ in rumor downstream.

3. Due to their respective roles and duties in controlling rumor, the control performance of social medias and opinion leaders $B$ is varying. Especially, the opinion leaders $B$, located downstream of official media, can guide and control the rumors by following the official media. To simplify the model, in this paper we assume that the information flow faced by the two rumor-control entities is homogeneous.

4. $E_Z = E_{G1} + E_{B1}$, $E_T = E_{G2} + E_{B2}$, where $E_Z$ and $E_T$ denote the total cost of debunking rumors downstream. Clearly, $E_Z < E_T$.

5. A coefficient $\rho \in (0, 1)$ is introduced to distinguish the capacity of official medias and opinion leaders to debunk rumors.

6. The parameter $P$ is composed of $P_G$ and $P_B$, namely, $P = P_G + P_B$, where $P_G$ denotes the penalty by the government, and $P_B$ denotes the rewards paid to the opinion leaders $B$ when debunking rumors.

4.2. **Game analysis.** Firstly in the subsection, we present the payoff matrix and the replicated dynamic equations of the tripartite game among opinion leaders $A$, social medias, and opinion

leaders $B$. Based on the assumptions as above, the payoff matrix is shown in Table 4, where $r$ is the returns of three game players under the different strategy combinations, with values as

$$
\begin{aligned}
&r_x^1 = r_x^2 = r_x^3 = r_x^4 = R_{A1} - C_{A1},\\
&r_x^5 = r_x^6 = r_x^7 = r_x^8 = R_{A1} + R_{A2} - C_{A2} - P,\\
&r_y^1 = r_y^2 = R_{G1}, r_y^3 = R_{G1} + R_{G2} - E_Z, r_y^4 = R_{G1} + R_{G2} - E_{G1},\\
&r_y^5 = r_y^6 = R_{G1} - C_{Gu} + P, r_y^7 = R_{G1} + R_{G2} - E_T + P,\\
&r_y^8 = R_{G1} + R_{G2} - E_{G2} + P_G, r_z^1 = R_{B1}, r_z^2 = R_{B1} + R_{B2} - \rho E_Z,\\
&r_z^3 = R_{B1} + R_{B2}, r_z^4 = R_{B1} + R_{B2} - \rho E_{B1}, r_z^5 = R_{B1} - C_{Bu},\\
&r_z^6 = R_{B1} + R_{B2} - \rho E_T, r_z^7 = R_{B1} + R_{B2}, r_z^8 = R_{B1} + R_{B2} - \rho E_{B2} + P_B.
\end{aligned}
$$

TABLE 4. The payoff matrix of tripartite game.

| | Strategy | Leader A | Leader A | Strategy | |
|---|---|---|---|---|---|
| | | Block($x$) | Drive($1-x$) | | |
| | Debunk($y$) | $r_x^1, r_y^1, r_z^1$ | $r_x^5, r_y^5, r_z^5$ | Not debunk($z$) | |
| Social media | Not debunk($y$) | $r_x^2, r_y^2, r_z^2$ | $r_x^6, r_y^6, r_z^6$ | Debunk($1-z$) | Leader B |
| Social media | Debunk($1-y$) | $r_x^3, r_y^3, r_z^3$ | $r_x^7, r_y^7, r_z^7$ | Not debunk($z$) | Leader B |
| | Debunk($1-y$) | $r_x^4, r_y^4, r_z^4$ | $r_x^8, r_y^8, r_z^8$ | Debunk($1-z$) | |

The evolution process based on replicated dynamic equation is a basic method for the study of dynamic game theory, where the player's decision-making depends on the value of long-term return, and the game player learns and adjusts his self-decision and finally reaches the ideal state. The statistical description is that if in a round game, the return brought by a decision is greater than the average return of the strategy set, the player will be more likely to take the decision in the next round; and vice versa, the player will be less likely. This process is called the replicated dynamic process, which can be matched well with the results of [39]. In this paper, we apply the method of replicated dynamic equation and the theory that was given in [21] to calculate the expected returns of the strategy sets of the three players in the game. The results are obtained as follows: When the upstream opinion leaders group $A$ choose to block rumors, the expected return is $\pi_x^1 = R_{A1} - C_{A1}$; when the group $A$ choose to drive the rumor spreading, the expected return is given by: $\pi_x^2 = R_{A1} + R_{A2} - C_{A2} - P$. The average expected return of $\pi_x^1$ and $\pi_x^2$ is $\bar{\pi}_x = x\pi_x^1 + (1-x)\pi_x^2$, and then the replicated dynamic equation of the opinion leaders $A$ is given by:

$$F_1(x) = \frac{dx}{dt} = x(\pi_x^1 - \bar{\pi}_x) = x(1-x)(-C_{A1} - R_{A2} + C_{A2} + P). \tag{7}$$

Likewise, the replicated dynamic equations of social medias and opinion leaders $B$ are, respectively,

$$
\begin{aligned}
F_2(y) &= \frac{dy}{dt} = y(\pi_y^1 - \bar{\pi}_y)\\
&= y(1-y)(P_B + E_{G2} - R_{G2} - C_{Gu} + xC_{Gu} + xE_{G1} - xE_{G2}\\
&\quad - xP_B - zP_B + zE_{B2} + xzP_B - xzE_{B2} + xzE_{B1}),
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
F_3(z) &= \frac{dz}{dt} = z(\pi_z^1 - \bar{\pi}_z)\\
&= z(1-z)(\rho E_{B2} - P_B - yR_{B2} - yC_{Bu} + yP_B + y\rho E_{G2}\\
&\quad + xP_B - x\rho E_{B2} + x\rho E_{B1} + xyC_{Bu} - xyP_B - xy\rho E_{G2} + xy\rho E_{G1}).
\end{aligned} \tag{9}
$$

Secondly, the stability analysis of equilibrium points in the tripartite game is conducted by using the method proposed in [13]. Letting Eqns. (7)-(9) be 0, one has

$$\begin{cases} x(1-x)(\pi_x^1 - \pi_x^2) = 0, \\ y(1-y)(\pi_y^1 - \pi_y^2) = 0, \\ z(1-z)(\pi_z^1 - \pi_z^2) = 0, \end{cases} \tag{10}$$

and its Jacobian matrix is given as:

$$J = \begin{pmatrix} (1-2x)A_{11} & x(1-x)A_{12} & x(1-x)A_{13} \\ y(1-y)B_{11} & (1-2y)B_{12} & y(1-y)B_{13} \\ z(1-z)C_{11} & z(1-z)C_{12} & (1-2z)C_{13} \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} A_{11} &= (-C_{A1} - R_{A2} + C_{A2} + P), A_{12} = A_{13} = 0, \\ B_{11} &= (P_B - E_{B2} + E_{B1})z + (C_{Gu} + E_{G1} - E_{G2} - P_B), \\ B_{12} &= (P_B + E_{G2} - R_{G2} - C_{Gu} + xC_{Gu} + xE_{G1} - xE_{G2} - xP_B - zP_B \\ &\quad + zE_{B2} + xzP_B - xzE_{B2} + xzE_{B1}), \\ B_{13} &= (P_B - E_{B2} + E_{B1})x + (E_{B2} - P_B), \\ C_{11} &= (C_{Bu} + \rho E_{G1} - \rho E_{G2} - P_B)y + (P_B - \rho E_{B2} + \rho E_{B1}), \\ C_{12} &= (C_{Bu} + \rho E_{G1} - \rho E_{G2} - P_B)x + (P_B - \rho E_{G2} - R_{G2} - C_{Bu}), \\ C_{13} &= (\rho E_{B2} - P_B - yR_{B2} - yC_{Bu} + yP_B + y\rho E_{G2} + xP_B - x\rho E_{B2} \\ &\quad + x\rho E_{B1} + xyC_{Bu} - xyP_B - xy\rho E_{G2} + xy\rho E_{G1}). \end{aligned}$$

By Eqn. (10), we consider the following cases.

1. If $x(1-x) = 0, y(1-y) = 0, z(1-z) = 0$, then the solution set of equilibrium points only contains 0 or 1, which means that the three game players only choose one of the two strategies. Hence, the game model has eight equilibrium points as followings:

$$H_1(0,0,0), H_2(0,1,0), H_3(0,0,1), H_4(0,1,1), H_5(1,0,0), H_6(1,1,0), H_7(1,0,1), H_8(1,1,1).$$

Below, we analyze the conditions under which each equilibrium point is asymptotically stable. Take point $H_8(1,1,1)$ as an example. Substituting this point into the Jacobian matrix (11) gives its eigenvalues as:

$$\begin{aligned} \lambda_1 &= -C_{A1} - R_{A2} + C_{A2} + P, \\ \lambda_2 &= P_B + E_{G2} - R_{G2} - C_{Gu} + P, \\ \lambda_3 &= P_B + \rho E_{B2} - \rho E_{B1}. \end{aligned}$$

Hence, the equilibrium point $H_8(1,1,1)$ is asymptotically stable if $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$. Similarly, the stability conditions for all the equilibrium points are shown in Table 5.

2. If two of $x(1-x)$, $y(1-y)$ and $z(1-z)$ equal to 0, then the equilibrium point of a game player among the three ones is a non-pure strategic choice, which means that the rumor-control player will choose a feasible strategy with a certain probability. Assume that $x(1-x) = 0$, $y(1-y) = 0$, $C_{13} = 0$, the equilibrium point $(x_0, y_0, z_0)$ and the corresponding Jacobian matrix are given as:

$$J_2 = \begin{pmatrix} (1-2x)A_{11} & 0 & 0 \\ 0 & (1-2y)B_{12} & 0 \\ z(1-z)C_{11} & z(1-z)C_{12} & 0 \end{pmatrix}. \tag{12}$$

Obviously, the matrix has an eigenvalue of 0, which does not satisfy asymptotic stability condition that all the eigenvalues have negative real parts. Thus, when $x(1-x) = 0$, $y(1-y) = 0$, and $C_{13} = 0$, there exists no stable equilibrium point. Similarly, there also exist no stable equilibria

TABLE 5. Stability conditions of equilibrium points in Case 1

| Equilibria | Stability conditions |
|---|---|
| $H_1(0,0,0)$ | $P < C_{A1} + R_{A2} - C_{A2}, C_{Gu} + R_{G2} > E_{G2} + P_B, P_B > \rho E_{B2}$ |
| $H_2(0,1,0)$ | $P < C_{A1} + R_{A2} - C_{A2}, C_{Gu} + R_{G2} < E_{G2} + P_B, \rho E_T < R_{B2} + C_{Bu}$ |
| $H_3(0,0,1)$ | $P < C_{A1} + R_{A2} - C_{A2}, C_{Gu} + R_{G2} > E_T, P_B + R_{B0} < \rho E_{B2}$ |
| $H_4(0,1,1)$ | $P < C_{A1} + R_{A2} - C_{A2}, C_{Gu} + R_{G2} < E_T$ (contrary to reality), $R_{B2} + C_{Bu} < \rho E_T$ (contrary to reality) |
| $H_5(1,0,0)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{G2} > E_{G1}, \rho E_{B1} < 0$ (contrary to reality) |
| $H_6(1,1,0)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{G2} < E_{G1}$ (contrary to reality), $R_{B2} > \rho E_Z$ |
| $H_7(1,0,1)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{G2} > E_Z, \rho E_{B1} > 0$ |
| $H_8(1,1,1)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{G2} < E_Z$ (contrary to reality), $R_{B2} < \rho E_Z$ (contrary to reality) |

under the other two cases of $x(1-x) = 0, z(1-z) = 0, B_{12} = 0$ and $y(1-y) = 0, z(1-z) = 0, A_{11} = 0$.

3. If only one of $x(1-x)$, $y(1-y)$ and $z(1-z)$ equals to 0, there is only one player of the three ones choosing a pure equilibrium strategy. Since $A_{12} = A_{13} = 0$, it is obvious that when $A_{11} = 0$, the corresponding determinant of Jacobian matrix (13) is 0, which does not satisfy the condition of asymptotic stability. So, it suffices to consider the case when $x(1-x) = 0$ and $B_{12} = C_{13} = 0$. Assume that system (10) has a solution $(x_1, y_1, z_1)$, then the corresponding Jacobian matrix is given by:

$$J_3 = \begin{pmatrix} (1-2x_1)A_{11} & 0 & 0 \\ y_1(1-y_1)B_{11} & 0 & y_1(1-y_1)B_{13} \\ z_1(1-z_1)C_{11} & z_1(1-z_1)C_{12} & 0 \end{pmatrix}. \tag{13}$$

The calculated eigenvalues are $\lambda_1 = (1-2x_1)A_{11}$, and $\lambda_{2,3} = \pm\sqrt{y_1(1-y_1)B_{13}z_1(1-z_1)C_{12}}$, which, clearly, can not satisfy the asymptotic stability condition, and then $(x_1, y_1, z_1)$ is not a stable equilibrium point.

4. If none of $x(1-x)$, $y(1-y)$, and $z(1-z)$ is 0, then $A_{11} = B_{12} = C_{13} = 0$. Assume that $(x_2, y_2, z_2)$ is a solution of system (10), which corresponds to the Jacobian matrix as given by:

$$J_4 = \begin{pmatrix} 0 & 0 & 0 \\ y_2(1-y_2)B_{11} & 0 & y_2(1-y_2)B_{13} \\ z_2(1-z_2)C_{11} & z_2(1-z_2)C_{12} & 0 \end{pmatrix}, \tag{14}$$

and its determinant $\det(J_4) = 0$. Thus by the asymptotic stability condition, $(x_2, y_2, z_2)$ is not a stable equilibrium point.

From the above analysis in Cases 1-4, we conclude that the equilibrium point is asymptotically stable if and only if the three players in the game choose the pure strategy set. So in what follows, the stability analysis of the tripartite game in the evolution mechanism of information dissemination is conducted as follows:

1. For the upstream opinion leaders group $A$, which start a rumor or drive its propagation, they are of a leading role in the information dissemination process due to their magnitude and popularity of people following them on social media. Through solving the stable points, we conclude that the choice of evolutionary stability strategy for opinion leaders $A$ only depends on the amount of government penalty. If the penalty amount is greater than the sum of the excess grey income brought by choosing to drive rumors and the reduced operation cost, then the group $A$ choose a blocking strategy to control the rumor propagation.

2. Consider the real situation of rumor debunking. A sound network environment is the overall goal of public opinion construction. The comprehensive returns from the improved public opinion environment is greater than the governance cost of public opinion, which suggests that when the upstream opinion leaders group $A$ chooses to drive the rumor propagation, the

condition that $C_{Gu} + R_{G2} > ET$ holds surely. Meanwhile, for the downstream opinion leaders group $B$, if their reward transferred by the government because of the contribution to debunk rumors is smaller than their operation cost, namely, $P_B < E_{B2}$, then the point $H_3(0, 0, 1)$ is an asymptotically stable equilibrium, which means that the opinion leaders $B$ will hand-off and leave rumor spreading unattended. While the opinion leaders $B$ will participate in rumor control to gain higher returns if and only if the reward from the government exceeds their own operation cost, and thus the evolutionary equilibrium point becomes $H_1(0, 0, 0)$. Consider the responsibility mechanism of official medias and opinion leaders in rumor controlling, when facing malicious information flows and even the evolution of fermented rumors, the official media, as mandatory supervision roles, are unable to hands-off, and must prevent the rumor attacks, while the opinion leaders, including the upstream and downstream ones, choose to drive or block the rumor spreading in accordance with the rational hypothesis of the "Economist" with a short-term outlook, which demonstrates that the opinion leaders' contribution to the rumor control is the icing on the cake instead of timely assistance. The innate differences in responsibility mechanism leads to the threshold effect in cooperation mechanism. In other words, the group $B$ of downstream opinion leaders, who choose to debunk rumor, have a greater initiative in the negotiation of government reward paying and governance cost sharing with the official authorities. Note that when the official authorities impose a small amount of penalty on the upstream opinion leaders group $A$ who provokes or propagates rumors, the cooperation between official media and opinion leaders $B$ will become more difficult due to their different proportions of the cost sharing and the reward income which are both brought by the penalty. On the other side, if the group $B$ choose to debunk rumors, which results in the lack of gimmicks and the decline of popularity, then the operation costs will become high, forcing them to claim higher government rewards, such that $C_{Gu} + R_{G2} < E_T + P_B$. The above argument can explain to some extend why it is difficult for official media and opinion leaders to achieve efficient and accurate cooperation during the evolution of public opinion.

3. Consider the case when the downstream opinion leaders $B$ and the official medias choose not to cooperate to debunk rumors, then the group $B$ cannot get the reward transferred from government, and also does not bear the rumor-control cost, which is similar to the logic of group $B$ without the participation and cooperation, and then has no influence on the final result. Hence, the discussion of this case is omitted in this paper.

4. Consider the case when the upstream opinion leaders $A$ choose to block rumors. Since $\rho E_{B1} > 0$, $R_{G2} < E_Z$, and $R_{B2} < \rho E_Z$, then the rumor-control cost of the downstream rumor debunkers is greater than their comprehensive return brought by improved public opinion, which is clearly contrary to the reality. Hence in this case, there exists a unique stable equilibrium point $H_7(1, 0, 1)$, which indicates that when the official medias choose to debunk rumors, even if the opinion leaders $B$ choose to do nothing and also do not bear the corresponding operation costs, they can still obtain the favorable benefits brought by the improved public opinion, which forms "piggyback" behavior without doing anything, and further aggravates the difficulty of cooperation between official medias and downstream opinion leaders $B$ in guiding and controlling rumors.

4.3. **Collaborative control mechanism based on "penalty-supervision-reward" model.** As analyzed in Section 4.2, the government penalty mechanism can prevent the upstream opinion leaders group $A$ from driving the rumor propagation, but the downstream co-governance between social medias and opinion leaders group $B$ is difficult to develop, which leads to the "piggyback" behavior. If the penalty is relatively small, the group $A$ choose to drive rumors, and the downstream rumor-debunking cooperation of social medias and group $B$ can be reached to a certain extent, but actually the cooperation is impossible to implement due to their heterogeneity of the governance mechanism. It is because the greatest duty of official medias is to purify public opinion and create a kind network environment, while the opinion leaders group $B$ can enjoy the benefits from the improvement of public opinion without any efforts, making the

cooperation between two downstream rumor-control main bodies come to naught. The study based on government-game theory shows that the structure of transfer payment means, such as penalty mechanism and reward mechanism, has been proved to be used to encourage the long-term cooperation of game players. Therefore in this paper, we propose a collaborative mechanism for official medias and opinion leaders in cooperating to control rumor as follows:

1. A certain amount of supervision fee $Q$ will be regularly charged from the opinion leaders group to cover the costs caused by the rumor-debunking process downstream, where the supervision fee will be deferred to the next period if it is not invoked during the current period. Note that the debunking behavior to rumor of official media can help the downstream opinion leaders group $B$ to disseminate positive information, and thus the official media gain additional revenue. Hence, when the official medias choose to debunk rumors, and the opinion leaders group $B$ choose not to debunk, the official media can use the fee $Q$ as its debunking cost; when the group $B$ choose to debunk rumor, they cover the cost with their own funds, where, clearly, the ownership of founds has not been transferred; when the official media and the group $B$ both choose not to debunk rumor, the fee $Q$ is not invoked, and then theoretically satisfies that $Q < \rho E_T$.

2. Consider a general case of rumor control, namely, the social activities and operation process of the downstream rumor-debunking main entities (including the official medias and the opinion leaders $B$) will also have a certain influence on the information flow, and the second influence will result in some operation costs. Moreover, the improper operation in rumor-debunking process will cause the risk of further deterioration of information flow and further diffusion of rumor. This is bound to the social media to make an ongoing effort in the process of debunking rumor, which causes more operation cost. Therefore, we now assume that the cost of social media to block the second outbreak of rumor from the source of information flow is $C_{G0}$. Based on the rationality hypothesis, if the rumor-blocking is conducted by the downstream opinion leaders $B$, then the rumor-blocking cost is $C_{B0}$, and the excess return is $R_{B0}$. It should be mentioned that because the second rumor outbreak caused by the rumor debunking is taken into consideration, the basic return $R_{B1}$ of opinion leaders $B$ set here is somewhat different from that in the original model.

3. If the upstream opinion leaders group $A$ choose to block rumors, and the downstream official medias and opinion leaders group $B$ choose to cooperate to debunk rumors, then the official medias and opinion leaders $B$ share their cost of debunking rumors, and also share the reward from the supervision fee $Q$, the proportion of which is determined by the official media based on the shared cost of cooperation and the differences of responsibilities between the two players in the rumor evolution. The reasonable assumption that the both players share the supervision fee is equivalent to that the opinion leader $B$ will pay an additional amount $q$ to the official media.

4. Consider the case when the opinion leaders $A$ choose to drive rumors, and the official medias and opinion leaders $B$ choose to cooperate to debunk rumors. Since the supervision fee is not enough to cover the contamination cost caused by opinion leaders $A$ driving the rumor evolution, then the two downstream main entities will share the rumor-debunking cost with its amount equal to the penalty imposed by the official institution on upstream opinion leaders $A$. Under this scenario, the supervision fee will not affect the rumor-debunking cost that is finally borne by official medias and opinion leaders $B$.

Based on the above "penalty-supervision-reward" mechanism, the profit and loss parameters of the tripartite parts in the original game model are changed as follows:

$$
\begin{aligned}
&r_x^1 = r_x^2 = r_x^3 = r_x^4 = R_{A1} - C_{A1},\\
&r_x^5 = r_x^6 = r_x^7 = r_x^8 = R_{A1} + R_{A2} - C_{A2} - P,\\
&r_y^1 = r_y^2 = R_{G1}, r_y^3 = R_{G1} + R_{G2} - E_Z + Q, r_y^4 = R_{G1} + R_{G2} - E_{G1} - C_{G0} + q,
\end{aligned}
$$

$$r_y^5 = r_y^6 = R_{G1} - C_{Gu} + P, r_y^7 = R_{G1} + R_{G2} - E_T + P + Q,$$
$$r_y^8 = R_{G1} + R_{G2} - E_{G2} + P_G - C_{G0}, r_z^1 = R_{B1}, r_z^2 = R_{B1} + R_{B2} - \rho E_Z - C_{B0} + R_{B0},$$
$$r_z^3 = R_{B1} + R_{B2} - Q, r_z^4 = R_{B1} + R_{B2} - \rho E_{B1} + R_{B0} - q, r_z^5 = R_{B1} - C_{Bu},$$
$$r_z^6 = R_{B1} + R_{B2} - \rho E_T - C_{B0} + R_{B0}, r_z^7 = R_{B1} + R_{B2} - Q,$$
$$r_z^8 = R_{B1} + R_{B2} - \rho E_{B2} + P_B + R_{B0}.$$

Accordingly, the replicated dynamic equations are given as:

$$\begin{cases} H_1(x) &= x(1-x)(-C_{A1} - R_{A2} + C_{A2} + P), \\ H_2(y) &= y(1-y)(P_B + E_{G2} - R_{G2} - C_{Gu} + C_{G0} + xC_{Gu} + xE_{G1} - xE_{G2} \\ &\quad -xP_B - xq - zP_B + zE_{B2} - zC_{G0} - zQ + xzP_B \\ &\quad -xzE_{B2} + xzE_{B1} + xzq), \\ H_3(z) &= z(1-z)(\rho E_{B2} - R_{B0} - P_B - Q - yR_{B2} - yC_{Bu} + yP_B + yQ \\ &\quad +y\rho E_{G2} + yC_{B0} + xP_B - x\rho E_{B2} + x\rho E_{B1} + xq + xyC_{Bu} - xyP_B \\ &\quad -xy\rho E_{G2} + xy\rho E_{G1} - xyq). \end{cases} \tag{15}$$

Hence, the Jacobian matrix is again used to analyze the asymptotic behaviors of its stable points of (15), and the conditions of asymptotic stability are shown in Table 6.

TABLE 6. Stability conditions for evolution equilibria with collaborative control mechanism.

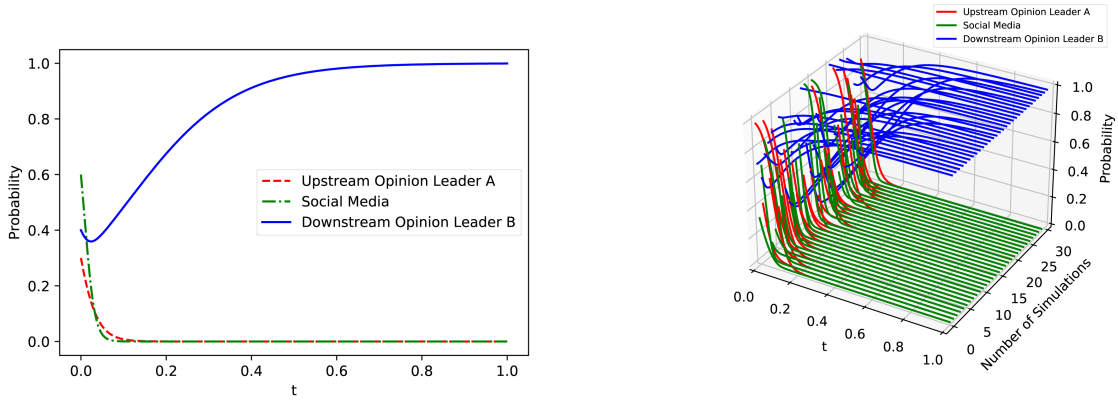
| Equilibria | Stability conditions |
|---|---|
| $H_1(0,0,0)$ | $P < C_{A1} + R_{A2} - C_{A2}, P_B + R_{B0} > \rho E_{B2} - Q, C_{Gu} + R_{G2} > E_{G2} + P_B + C_{G0}$ |
| $H_2(0,1,0)$ | $P < C_{A1} + R_{A2} - C_{A2}, \rho E_T + C_{B0} < R_{B2} + C_{Bu} + R_{B0}, C_{Gu} + R_{G2} < E_{G2} + P_B + C_{G0}$ |
| $H_3(0,0,1)$ | $P < C_{A1} + R_{A2} - C_{A2}, C_{Gu} + R_{G2} + Q > E_T, P_B + R_{B0} < \rho E_{B2} - Q$ |
| $H_4(0,1,1)$ | $P < C_{A1} + R_{A2} - C_{A2}, C_{Gu} + R_{G2} + Q < E_T, \rho E_T + C_{B0} > R_{B2} + C_{Bu} + R_{B0}$ |
| $H_5(1,0,0)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{G2} > E_{G1} + C_{G0} - q, \rho E_{B1} < R_{B0} + Q - q$ |
| $H_6(1,1,0)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{G2} < E_{G1} + C_{G0} - q, R_{B2} + R_{B0} > \rho E_Z + C_{B0}$ |
| $H_7(1,0,1)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{G2} > E_Z - Q, \rho E_{B1} > R_{B0} + Q - q$ |
| $H_8(1,1,1)$ | $P > C_{A1} + R_{A2} - C_{A2}, R_{B2} + R_{B0} < E_Z + C_{B0}$ (Contrary to reality), $R_{G2} < E_Z - Q$ (Contrary to reality) |

According to the stability conditions in Table 6, it can be observed that when the upstream opinion leaders group $A$ choose to drive rumors, the equilibrium point $H_5(1,0,0)$ becomes the new asymptotically stable point only if the conditions that $R_{G2} > E_{G1} + C_{G0} - q$ and $\rho E_{B1} < R_{B0} + Q - q$ hold simultaneously. Then, in the following, we focus on the reasonable setting between $Q$ and $q$ such that $q > E_{G1} + C_{G0} - R_{G2}$ and $Q - q > \rho E_{B1} - R_{B0}$, which implies that there exists a unique asymptotically stable equilibrium point $H_5$ if the opinion leaders $A$ choose to drive rumors. Hence, driven by the the government penalty mechanism, the cooperation of the downstream official medias and opinion leaders group $B$ can be reached. In fact, the most downstream opinion leaders have prepaid the supervision fee because of their location advantages, which can internalize the positive externality generated by the rumor debunking of the upstream official media on its downstream opinion leaders. As a result, the balance between downstream rumor-control players is increased, and meanwhile the promotion effects of the government penalty mechanism and the transfer payment reward mechanism on the long-term cooperation between downstream rumor-debunking players is confirmed again. It should be pointed out that even if the upstream opinion leaders group $A$ choose to block rumor, the minimum amount of government reward that the downstream group $B$ gain for its rumor-debunking work also changes from $\rho E_{B2} - R_{B0}$ to $\rho E_{B2} - R_{B0} - Q$. This, on the one hand, can save the public expenditure of the government, and effectively avoids the "piggyback" behavior of opinion leaders, making it easier for downstream rumor control entities to reach an agreement on cooperation; and on the other hand, can allow the official media, as the main entity of

rumor control, to obtain more compensation through coordination mechanisms to cover their comprehensive costs in response to the second outbreak of rumors.

4.4. **Simulation analysis of tripartite game.** The simulation analysis is implemented by using numerical fitting to explore the evolution process of state strategy selection of three players in different situations in order to verify the feasibility of theoretic results. The simulation environment used in this paper is Python 3.7., and the parameter values are referred to in Tables 9-12 in Appendix.

Firstly, we consider the evolution of game strategy with small penalty in baseline model established in Subsection 4.2. As shown in Fig.6(a), when the penalty amount imposed on the upstream group $A$ of opinion leaders is small, the group $A$ will choose to drive the rumor spreading, while the downstream group $B$ of opinion leaders will choose not to debunk rumor due to their small transfer payment reward. Hence, the strategy set of three main entities is (Driving, Debunking, Non-debunking), which validates the stable equilibrium point $H_3(0,0,1)$ in the baseline model, and gives the worst-case scenario for rumor spreading on social media. In Fig.6(b), the initial strategy values of three main entities are dynamically selected, and 30 groups of random numbers were generated in the interval $(0,1)$ for repeated simulation experiments. The simulated trend is consistent with that of the single experiment, which shows the influence of penalty amount on the strategy choice of upstream opinion leaders.



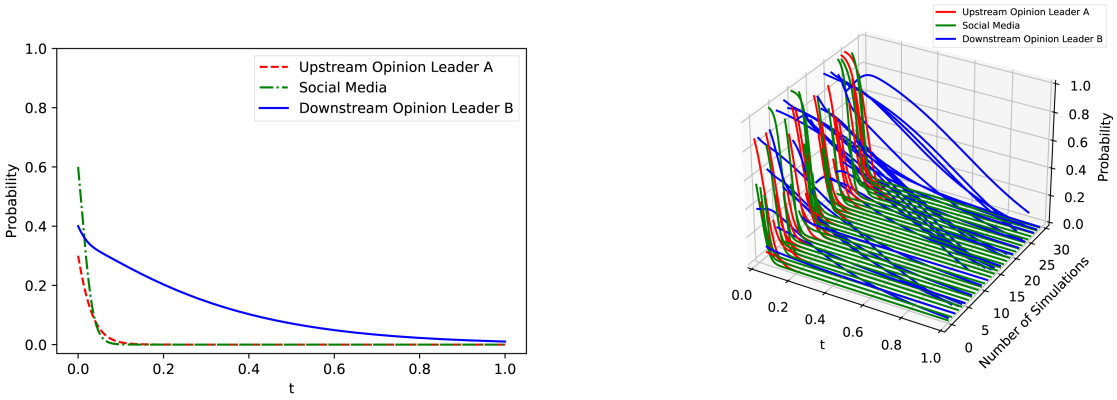

(a) Single experiment simulation

(b) Simulation with repeating 30 times

FIGURE 6. Evolution strategy of the rumor-control players when $P = 60$ and $P_B = 24$.

Secondly, we consider the evolution of game strategy in baseline model when increasing amount of transfer payment reward. The simulated result in Fig.7(a) indicates that the amount of penalty is unchanged, the upstream opinion leaders group $A$ choose to drive the rumor spreading, but with the increase of the proportion of transfer payment reward, the profit of downstream opinion leaders Group $B$ can exceed its comprehensive operation cost, and so chooses the debunking strategy. Therefore, the strategy selection of the three main-players is (Driving, Debunking, Debunking), which corresponds to the stable equilibrium point $H_1(0,0,0)$ of the baseline model, and provides a suboptimal strategy to debunk rumors. Similarly, Fig.7(b) illustrates the simulated curves in 30 repeated fitting experiments, and the simulated trend is consistent with the result of the single experiment.

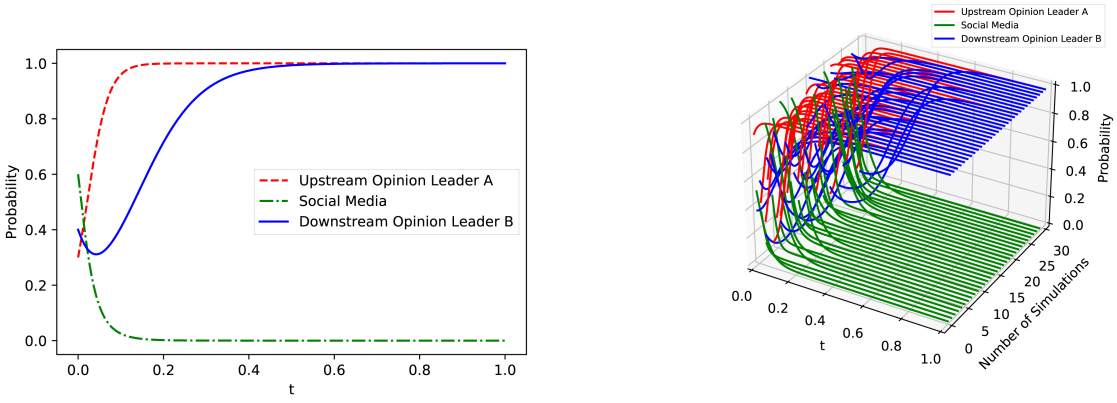
Thirdly, we consider the game strategy evolution in baseline model with increasing penalty amount. From Fig.8(a), it can be seen that as the penalty amount increase based on the value of the baseline model, the upstream opinion leaders group $A$ is wary of the high penalty,




(a) Single experiment simulation (b) Simulation with repeating 30 times

FIGURE 7. Evolution strategy of the rumor-control players when $P = 60$ and $P_B = 40$.

and thus chooses the blocking strategy to stop rumors; while the downstream opinion leaders group $B$, because the proportion of the reward paid to them remains unchanged, chooses not to debunk rumors and waits for gains with doing nothing. So now the strategy choice of the three main-players is (Blocking, Debunking, Non-debunking), namely the equilibrium point here is $H_7(1, 0, 1)$, which demonstrates the objective existence of the "piggyback" effect, and matches well with the theoretic results. And Fig.8(b) gives the numerical simulation of 30 repeated experiments randomly selected for initial strategy, which is in good agreement with that in the single experiment.

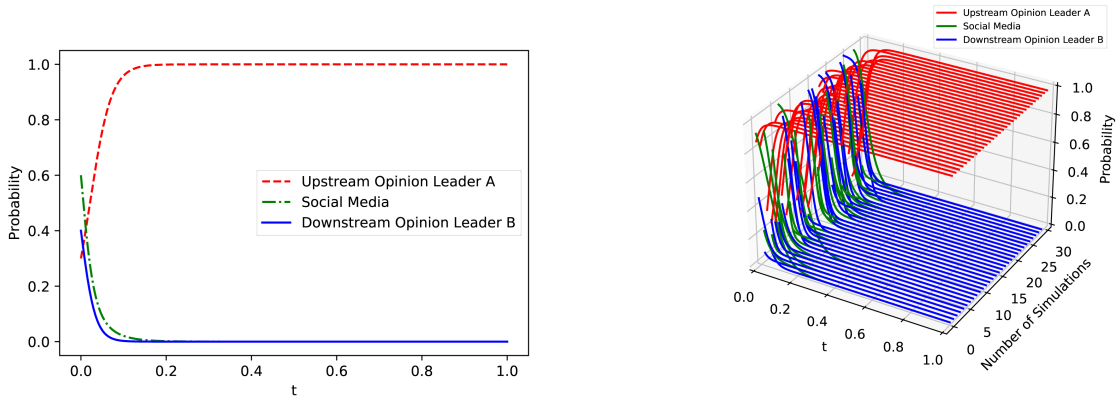



(a) Single experiment simulation (b) Simulation with repeating 30 times

FIGURE 8. Evolution strategy of the rumor-control players when $P = 140$ and $P_B = 60$.

Finally, we consider the evolution of game strategy in the model with "penalty-supervision-reward" mechanism. See Fig.9(a), after setting up the "penalty-regulation-reward" coordinated governance fund, the upstream opinion leaders group $A$ choose to block rumors, the official medias choose to debunk rumors, and the downstream opinion leaders group $B$ also choose to debunk rumors. By reasonable assignment, the tripartite state strategy set is (Blocking,

Debunking, Debunking), and then $H_5(1,0,0)$ is the unique stable equilibrium point. Fig.9(b) shows the simulated results of repeated experiments for 30 times, and the trend of strategy evolution is consistent with that of the single experiment. Obviously, the simulation in Fig.9 validate the theoretic results, and also suggest that the coordination fund for controlling rumor can not only standardize the strategy choice of the upstream opinion leaders $A$, but also eliminate the cooperation threshold of the downstream rumor-control players.




(a) Single experiment simulation

(b) Simulation with repeating 30 times

FIGURE 9. Evolution strategy of the rumor-control players when $P = 140$ and $Q = 30$.

## 5. Conclusions and future research direction

With the ongoing advance of the information age, the speed and magnitude of information transmission is undergoing a rapid upgrading process. The innovation of social network topology and nodes has made the cascading relationships of social media users increasingly complex and intimately connect. In today's world, the frequent occurrence of extreme events and the disruption of social order make public crisis management more important. From the perspective of online public opinion guidance, we analyze the game relationship and interplay mechanism of different main entities in rumor control. The results obtained in this paper consist of the following parts.

1. The SEIDR rumor spreading model based on the dynamic method shows the impact of different attitudes of guiding entities at the macro level on the scale of rumor dissemination. Using the basic reproduction number and considering the differences in the power-responsibility mechanism, action mechanism and feedback time, the active response behavior of official medias is a key factor which affects the emotional transformation of social media users, and the promotion strategy of opinion leaders has a significant positive impact on the rumor scale. Therefore, strengthening the response action of official medias and the supervision of self-medias is a reliable mean to control and guide the online rumor propagation.

2. The single role of opinion leaders group in traditional game theory of public opinion has been broken, and then a tripartite game relationship with dual characteristics between official medias and opinion leaders is established. According to the flow nature of information, the parameters are set rationally and the theoretical analysis is conducted deeply. This study shows that the opinion leaders play an important part in public opinion guidance, and become different roles because of their respective environmental positions and interest demands. Hence, it is

biased to consider opinion leaders to be of hostile relations with official medias. The upstream opinion leaders who choose to drive rumors and take a risk to earn gray income play a hostile game relationship, while for the downstream opinion leaders who play the role in debunking rumors, a reasonable cooperation mechanism with official medias is established to effectively prevent the rumor transmission. However, due to the location advantage in the downstream of information flow and the mandatory power-responsibility mechanism of official medias, the positive guidance of downstream opinion leaders is no longer an unnecessary task, which leads to a threshold aggravating the difficulty of cooperation between both.

3. A collaborative governance model with the government penalty mechanism and the transfer payment reward mechanism is developed. The theoretical results show that when the penalty, supervision cost, and transfer payment reward are reasonably set up, the action mechanism is significantly effective. On the one hand, since the purpose of setting up penalty is for the downstream opinion leaders to prepay for their operations, the collaborative governance method can effectively alleviates the inequality issues caused by location, interest and responsibility to a certain extent, even if the two rumor-control main entities downstream can not achieve cooperation. On the other hand, the specific amount of penalty can be set to achieve a relatively ideal level, so as to ensure the effectiveness of the penalty coordination mechanism. This indicates that the threshold effect of cooperation is significantly reduced, which promotes the cooperation of downstream main entities of rumor control, reduces the cost required by the government to purify the network public opinion space, and then propose a collaborative control mechanism based on "penalty-supervision-reward" model between official medias and opinion leaders.

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## Appendix.

## 1. Proofs of theorems

**A.1. Proof of Theorem 1.** The equilibrium points are the solutions of the following system:

$$\begin{cases} S'(t) = 0, \\ E'(t) = 0, \\ I'(t) = 0, \\ D'(t) = 0, \\ R'(t) = 0. \end{cases}$$

From $I'(t) = 0$, we have

$$E = \frac{\zeta + \xi + \gamma + \epsilon}{\alpha} I, \tag{A.16}$$

which, along with $D'(t) = 0$, gives

$$D = \frac{\beta(\zeta + \xi + \epsilon) + \alpha\xi}{\alpha(\theta + \epsilon)} I. \tag{A.17}$$

Substituting (A.16) and (A.17) into $E'(t) = 0$ leads to

$$\left( \mu\omega(1-\phi)S + \frac{\kappa(1-\omega)\phi(\beta(\zeta+\xi+\epsilon)+\alpha\xi)}{\alpha(\theta+\epsilon)} S - \frac{(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)}{\alpha} \right) I = 0.$$

Then, the equilibrium occurs at:

$$I = 0, \quad \text{or} \quad I > 0 \ \text{ and } \ S = \frac{(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)(\theta+\epsilon)}{\alpha(\theta+\epsilon)\mu\omega(1-\phi) + \kappa(1-\omega)\phi(\beta(\zeta+\xi+\epsilon)+\alpha\xi)}. \tag{A.18}$$

For $I = 0$, it is easy to see that the system (1) always has a unique rumor-free equilibrium point of the form (3). For $I > 0$, we substitute the expression of $S$ in (A.18) into $S'(t) = 0$ to get

$$\Lambda - \epsilon S - \left( \mu\omega(1-\phi)S + \frac{\kappa(1-\omega)\phi(\beta(\zeta+\xi+\epsilon)+\alpha\xi)}{\alpha(\theta+\epsilon)} S - \epsilon \right) I = 0.$$

Recalling the expression (4) of the basic reproduction number $R_0$, we derive that, if $R_0 \leq 1$, there is no positive equilibrium, and the rumor-free equilibrium is the only equilibrium in $\Omega$; if $R_0 > 1$, there exists a unique positive equilibrium $P^* = (S^*, E^*, I^*, D^*, R^*)$ of the form (5), called rumor-existence equilibrium, in $\text{int}(\Omega)$. □

**A.2. Proof of Theorem 2.** According to the system (1) and Eqn. (3), the Jacobian matrix at the rumor-free equilibrium $P_0$ is given by:

$$J(P_0)=\begin{pmatrix} -\epsilon & 0 & \frac{-\mu\omega(1-\phi)\Lambda}{\epsilon} & \frac{-k(1-\omega)\phi\Lambda}{\epsilon} & 0 \\ 0 & -(\alpha+\beta+\gamma+\epsilon) & \frac{\mu\omega(1-\phi)\Lambda}{\epsilon} & \frac{k(1-\omega)\phi\Lambda}{\epsilon} & 0 \\ 0 & \alpha & -(\zeta+\xi+\epsilon) & 0 & 0 \\ 0 & \beta & \xi & -(\theta+\epsilon) & 0 \\ 0 & \gamma & \zeta & \theta & -\epsilon \end{pmatrix}.$$

By calculation, the Jacobian matrix $J(P_0)$ has three negative eigenvalues:

$$\lambda_1=\lambda_2=-\epsilon<0 \quad \text{and} \quad \lambda_3=-(\theta+\epsilon)<0;$$

the other eigenvalues $\lambda_4$ and $\lambda_5$ are determined by the following matrix:

$$\Gamma=\begin{pmatrix} -(\alpha+\beta+\gamma+\epsilon) & \frac{\mu\omega(1-\phi)\Lambda}{\epsilon} \\ \alpha & -(\zeta+\xi+\epsilon) \end{pmatrix},$$

which leads to

$$\begin{aligned} tr(\Gamma)&=-\alpha-\beta-\gamma-\zeta-\xi-2\epsilon<0, \\ det(\Gamma)&=(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)-\frac{\mu\omega(1-\phi)\Lambda\alpha}{\epsilon}. \end{aligned}$$

If $R_0<1$, then $\Lambda\mu\omega(1-\phi)\alpha(\theta+\epsilon)+\Lambda k(1-\omega)\phi\beta(\zeta+\xi+\epsilon)+\alpha\xi\Lambda<\epsilon(\alpha+\beta+\gamma+\epsilon)(\theta+\epsilon)(\zeta+\xi+\epsilon)$, which implies that

$$(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)>\frac{\Lambda\mu\omega(1-\phi)\alpha}{\epsilon}+\frac{\Lambda k(1-\omega)\phi\beta(\xi+\zeta+\epsilon+)\alpha\xi\Lambda}{\epsilon(\theta+\epsilon)}.$$

From the above derivations, we obtain that when $R_0<1$, then $det(\Gamma)>0$, and further $\lambda_4<0$, $\lambda_5<0$. Finally by the stability theory [36], when $R_0\leq 1$, the rumor-free equilibrium $P_0$ is locally asymptotically stable, which completes the proof. □

**A.3. Proof of Theorem 3.** Consider the Lyapunov function:

$$\Psi(E,I)=E+\frac{\alpha+\beta+\gamma+\epsilon}{\alpha}I. \tag{A.19}$$

Then, the derivative of $L(E,I)$ with respective to $t$ gives

$$\begin{aligned} \frac{d\Psi(E,I)}{dt}&=\mu\omega(1-\phi)SI+k(1-\omega)\phi SD-(\alpha+\beta+\gamma+\epsilon)E \\ &\quad+\frac{\alpha+\beta+\gamma+\epsilon}{\alpha}+(\alpha E-(\zeta+\xi+\epsilon)I) \\ &=(\mu\omega(1-\phi)I+k(1-\omega)\phi D)S-\frac{(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)}{\alpha}. \end{aligned} \tag{A.20}$$

By Eqn. (5), it follows that

$$\begin{aligned} \frac{d\Psi(E,I)}{dt}&=\mu\omega(1-\phi)SI+k(1-\omega)\phi SI\frac{\beta(\alpha+\beta+\epsilon)+\xi}{\theta+\epsilon}-\frac{(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)}{\alpha}I \\ &=\left(\mu\omega(1-\phi)S+k(1-\omega)\phi S\frac{\beta(\alpha+\beta+\epsilon)+\xi}{\theta+\epsilon}-\frac{(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)}{\alpha}\right)I \\ &=\left(\frac{R_0\epsilon S}{\Lambda}-1\right)\frac{(\alpha+\beta+\gamma+\epsilon)(\zeta+\xi+\epsilon)}{\alpha}I, \end{aligned}$$

which, along with (2), yields that when $R_0\leq 1$, then $\frac{d\Psi(E,I)}{dt}\leq 0$. Furthermore, $\frac{d\Psi(E,I)}{dt}=0$ if and only if $I=0$. Hence, when $R_0\leq 1$, the largest compact invariant set in $\{(S,E,I,D,R)\in\Omega|\frac{d\Psi}{dt}=0\}$ is the singleton set containing $P_0$. Applying the LaSalle's invariance principle [23], the rumor-free equilibrium $P_0$ is globally asymptotically stable in $\Omega$ when $R_0\leq 1$. □

**A.4. Proof of Theorem 5.** According to $P^* = (S^*, E^*, I^*, D^*, R^*)$ with its coordinates satisfying (5), the Jacobian matrix of subsystem (6) at the rumor-existence equilibrium $P^*$ is given by:

$$H = \begin{pmatrix} -\mu\omega(1-\phi)I - k(1-\omega)\phi D - \epsilon & 0 & -\mu\omega(1-\phi)S & -k(1-\omega)\phi S \\ \mu\omega(1-\phi)I + k(1-\omega)\phi D & -(\alpha+\beta+\gamma+\epsilon) & \mu\omega(1-\phi)S & k(1-\omega)\phi S \\ 0 & \alpha & -(\xi+\zeta+\epsilon) & 0 \\ 0 & \beta & \xi & -(\theta+\epsilon) \end{pmatrix},$$

which implies that

$$\begin{aligned} tr(H) &= -(\mu\omega(1-\phi)I + k(1-\omega)\phi D + \epsilon) - (\alpha+\beta+\gamma+\epsilon) - (\zeta+\xi+\epsilon-\theta+\epsilon) < 0, \\ det(H) &= (\mu\omega(1-\phi)I + k(1-\omega)\phi D + \epsilon)((\alpha+\beta+\gamma+\epsilon)(\alpha+\xi+\epsilon)(\theta+\epsilon) \\ &\quad - \mu\omega(1-\phi)S\alpha(\theta+\epsilon) - k(1-\omega)\phi S\alpha\xi - k(1-\omega)\phi S\beta(\zeta+\xi+\epsilon)) \\ &\quad + \mu\omega(1-\phi)S\alpha(\theta+\epsilon)(\mu\omega(1-\phi)I + k(1-\omega)\phi D) \\ &\quad + k(1-\omega)\phi S\beta(\mu\omega(1-\phi)I + k(1-\omega)\phi D)(\alpha+\xi+\epsilon). \end{aligned}$$

By Eqn. (5), $(\alpha+\beta+\gamma+\epsilon)(\theta+\epsilon)(\zeta+\xi+\epsilon) - \mu\omega(1-\phi)\alpha S(\theta+\epsilon) - Sk(1-\omega)\phi\beta(\xi+\zeta+\epsilon) - \alpha k(1-\omega)\phi\xi S = 0$. Then, it follows that

$$\begin{aligned} det(H) &= \mu\omega(1-\phi)\alpha S(\theta+\epsilon)(\mu\omega(1-\phi)I + k(1-\omega)\phi D) \\ &\quad + k(1-\omega)\phi S\alpha\xi(\mu\omega(1-\phi)I + k(1-\omega)\phi D) \\ &\quad + k(1-\omega)\phi S\beta(\mu\omega(1-\phi)I + k(1-\omega)\phi D)(\alpha+\xi+\epsilon) > 0, \end{aligned}$$

which shows that all the eigenvalues of Jacobian matrix $H$ have negative real parts. Hence by the stability theory [36], the rumor-existence equilibrium $P^*$ is locally asymptotically stable with respect to $\Omega$ when $R_0 > 1$. □

**A.5. Proof of Theorem 6.** We now prove Theorem 6 by using the Li-Muldowney geometric approach, where we omit the detailed introduction of this approach and refer readers to [29].

To simplify the calculation, we set

$$\begin{aligned} &\mu_1 = \frac{\mu\omega(1-\phi)}{\epsilon}, k_1 = \frac{k(1-\omega)\phi}{\epsilon}, \alpha_1 = \frac{\alpha}{\epsilon}, \beta_1 = \frac{\beta}{\epsilon}, \gamma_1 = \frac{\gamma}{\epsilon}, \zeta_1 = \frac{\zeta}{\epsilon}, g = \epsilon t, \\ &\xi_1 = \frac{\xi}{\epsilon}, \theta_1 = \frac{\theta}{\epsilon}, S_1 = \frac{S}{\epsilon}, E_1 = \frac{E}{\epsilon}, I_1 = \frac{I}{\epsilon}, D_1 = \frac{D}{\epsilon}, R_1 = \frac{R}{\epsilon}, \tau = \frac{\Lambda}{\epsilon}. \end{aligned}$$

Thus, the subsystem (6) is equivalent to

$$\begin{cases} \frac{dS_1(t)}{dg} = 1 - \mu_1 S_1 I_1 \tau - k_1 S_1 D_1 \tau - S_1 \\ \frac{dE_1(t)}{dg} = \mu_1 S_1 I_1 \tau + k_1 S_1 D_1 \tau - (\alpha_1 + \beta_1 + \gamma_1 + 1) E_1 \\ \frac{dI_1(t)}{dg} = \alpha_1 E_1 - (\zeta_1 + \xi_1 + 1) I_1 \\ \frac{dD_1(t)}{dg} = \beta_1 E_1 + \xi_1 R_1 - (\theta_1 + 1) D_1. \end{cases} \tag{A.21}$$

Construct the following diagonal matrix as $P = \mathrm{diag}(I_1, E_1, E_1, E_1, E_1, E_1)$, which is differentially continuous and non-singular in $\Omega^0$. Then, we have

$$P_f P^{-1} = \mathrm{diag}(\frac{R_1'}{R_1}, \frac{E_1'}{E_1}, \frac{E_1'}{E_1}, \frac{E_1'}{E_1}, \frac{E_1'}{E_1}, \frac{E_1'}{E_1}),$$

where $f$ is the vector domain of (A.21). After a direct calculation, the Jacobian matrix of subsystem (A.21) is given by:

$$J = \begin{pmatrix} -(\mu_1 I_1 \tau + k_1 D_1 \tau + 1) & 0 & -\mu_1 S_1 \tau & -k_1 S_1 \tau \\ \mu_1 I_1 \tau + k_1 D_1 \tau & -(\alpha_1 + \beta_1 + \gamma_1 + 1) & \mu_1 S_1 \tau & \mu_1 S_1 \tau \\ 0 & \alpha_1 & -(\xi_1 + \zeta_1 + 1) & 0 \\ 0 & \beta_1 & \xi_1 & -(\theta_1 + 1) \end{pmatrix},$$

and its second additive compound matrix is

$$J^{[2]} = \begin{pmatrix} \omega_1 & \mu_1 S_1 \tau & k_1 S_1 \tau & \mu_1 S_1 \tau & k_1 S_1 \tau & 0 \\ \alpha_1 & \omega_2 & 0 & 0 & 0 & k_1 S_1 \tau \\ \beta_1 & \xi_1 & \omega_3 & 0 & 0 & -\mu_1 S_1 \tau \\ 0 & \omega_7 & 0 & \omega_5 & 0 & -k_1 S_1 \tau \\ 0 & 0 & \omega_7 & \xi_1 & \omega_6 & \mu_1 S_1 \tau \\ 0 & 0 & 0 & -\beta_1 & \alpha_1 & \omega_4 \end{pmatrix},$$

where

$$\begin{aligned} \omega_1 &= -\mu_1 I_1 \tau - k_1 D_1 \tau - \alpha_1 - \beta_1 - \gamma_1 - 2, \\ \omega_2 &= -\mu_1 I_1 \tau - k_1 D_1 \tau - \zeta_1 - \xi_1 - 2, \\ \omega_3 &= -\mu_1 I_1 \tau - k_1 D_1 \tau - \theta_1 - 2, \\ \omega_4 &= -\zeta_1 - \xi_1 - \theta_1 - 2, \\ \omega_5 &= -\alpha_1 - \beta_1 - \gamma_1 - \zeta_1 - \xi_1 - 2, \\ \omega_6 &= -\alpha_1 - \beta_1 - \gamma_1 - \theta_1 - 2, \\ \omega_7 &= \mu_1 I_1 \tau + k_1 D_1 \tau. \end{aligned}$$

Set $B = A_f A^{-1} + A J^{[2]} A^{-1}$, which can be written in matrix form:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

with

$$\begin{aligned} B_{11} &= \frac{I_1'}{I_1} - \mu_1 I_1 \tau - k_1 D_1 \tau - \alpha_1 - \beta_1 - \gamma_1 - 2, \\ B_{12} &= \Big( \frac{\mu_1 S_1 \tau I_1}{E_1} \quad \frac{k_1 S_1 \tau I_1}{E_1} \quad \frac{\mu_1 S_1 \tau I_1}{E_1} \quad \frac{k_1 S_1 \tau I_1}{E_1} \quad 0 \Big), \\ B_{21} &= \Big( \frac{\alpha_1 E_1}{I_1} \quad \frac{\beta_1 E_1}{I_1} \quad 0 \quad 0 \quad 0 \Big)^T. \end{aligned}$$

$$B_{22} = \begin{pmatrix} \frac{E_1'}{E_1} + \omega_2 & 0 & 0 & 0 & k_1 S_1 \tau \\ \xi_1 & \frac{E_1'}{E_1} + \omega_3 & 0 & 0 & -\mu_1 S_1 \tau \\ \omega_7 & 0 & \frac{E_1'}{E_1} + \omega_5 & 0 & -k_1 S_1 \tau \\ 0 & \omega_7 & \xi_1 & \frac{E_1'}{E_1} + \omega_6 & \mu_1 S_1 \tau \\ 0 & 0 & -\beta_1 & \alpha_1 & \frac{E_1'}{E_1} + \omega_4 \end{pmatrix}.$$

Let $Y$ be a vector in $\mathbb{R}^5$ with its norm defined by:

$$|Y| = \max\{|y_1| + |y_2|, |y_3| + |y_4| + |y_5|\},$$

where $y_i$ is the $i$-th component of $Y$, $i = 1, 2, ..., 5$. Let $\sigma(B)$ be the *Lozinskiĭ* measure with respect to this norm, then

$$\sigma(B) \leq \sup\{\omega_1, \omega_2\},$$

where $\omega_1 = \sigma_1(B_{11}) + |B_{12}|$, $\omega_2 = |B_{21}| + \sigma_1(B_{22})$, $|B_{12}|$, $|B_{21}|$ are matrix norms with respect to the $l_1$ vector norm, and $\sigma_1$ is the *Lozinskiĭ* measure with respect to $l_1$ norm. As a result,

$$\begin{aligned}
\sigma_1(B_{11}) &= \frac{I_1'}{I_1} - \mu_1 I_1 \tau - k_1 D_1 \tau - \alpha_1 - \beta_1 - \gamma_1 - 2, \\
|B_{12}| &= \frac{\mu_1 S_1 \tau I_1}{E_1} + \frac{k_1 S_1 \tau I_1}{E_1}, \\
|B_{21}| &= \frac{\alpha_1 E_1}{I_1} + \frac{\beta_1 E_1}{I_1}, \\
\sigma_1(B_{22}) &= \frac{E_1'}{E_1} - \frac{\beta_1 E_1}{I_1} - 2 - min(\theta_1, \xi_1).
\end{aligned} \tag{A.22}$$

Therefore,

$$\begin{aligned}
\omega_1 &= \sigma_1 B11 + |B_{12}| \\
&= \frac{I_1'}{I_1} - \mu_1 I_1 \tau - k_1 D_1 \tau - \alpha_1 - \beta_1 - \gamma_1 - 2 + \frac{\mu_1 S_1 \tau I_1}{E_1} + \frac{k_1 S_1 \tau I_1}{E_1} \\
&= \frac{I_1'}{I_1} + \frac{E_1'}{E_1} - 1 - \mu_1 I_1 \tau - k_1 D_1 \tau. \\
\omega_2 &= |B_{21}| + \sigma_1(B_{22}) = \frac{\alpha_1 E_1}{I_1} + \frac{\beta_1 E_1}{I_1} + \frac{E_1'}{E_1} - \frac{\beta_1 E_1}{I_1} - 2 - min(\theta_1, \xi_1) \\
&= \frac{I_1'}{I_1} + \frac{E_1'}{E_1} - 1 + \xi_1 + \zeta_1 - min(\theta_1, \zeta_1) \leq \frac{I_1'}{I_1} + \frac{E_1'}{E_1} - 1 + \xi_1,
\end{aligned} \tag{A.23}$$

which gives that $\sigma(B) \leq \sup\{\omega_1, \omega_2\} = \frac{I_1'}{I_1} + \frac{E_1'}{E_1} - 1 + \xi_1 + \zeta_1 - \min(\theta_1, \zeta_1)$.

Let $\Upsilon = 1 - \xi_1 - \zeta_1 + \min(\theta_1, \zeta_1) \geq 0$, then $\sigma(B) \leq \frac{I_1'}{I_1} + \frac{E_1'}{E_1} - \Upsilon$. For any root $(S_1, E_1, R_1, I_1)$ (e.g. $\Big(S_1(0), E_1(0), R_1(0), I_1(0) \in \Theta\Big)$), the absorbing set is obtained as:

$$\frac{1}{t}\int_0^1 \sigma(B)ds \leq \frac{1}{t}\int_0^1 \Big(\frac{I_1'}{I_1} + \frac{E_1'}{E_1} - \Upsilon\Big)ds = \frac{1}{t} ln\frac{I_1(t)}{I_1(0)} + \frac{1}{t} ln\frac{E_1(t)}{E_1(0)} - \Upsilon,$$

which leads to $\lim\limits_{t\to\infty} \sup_{[S,E,I,D\in\Theta]} \frac{1}{t}\int_0^1 \sigma(B)ds \leq -\frac{\Upsilon}{2} < 0$. Hence by Theorem 3.5 of [29], the positive equilibrium $(S^*, E^*, I^*, D^*)$ of system (A.21) is globally asymptotically stable, and further $(S, E, I, D)$ of the subsystem (6) is still globally asymptotically stable.

Subsequently, we consider $R'(t) = \zeta I + \theta D + \gamma E - \epsilon R$ with its limit as $R'(t) = \zeta I^* + \theta D^* + \gamma E^* - \epsilon R$, then we have

$$\begin{aligned}
R(t) &= e^{-\epsilon t}\Big(I(0) + \int_0^t [\zeta I^* + \theta D^* + \gamma E^*] e^{\epsilon s} ds\Big), \\
\lim_{t\to\infty} R(t) &= R^*.
\end{aligned}$$

Consequently, the rumor-existence equilibrium $(S^*, E^*, I^*, D^*)$ of the subsystem (6) is globally asymptotically stable in int$(\Omega)$ when $R_0 > 1$, which completes the proof. □

## 2. Parameter setting in numerical simulation

The simulation environment used in this paper is Python 3.7.

TABLE 7. Parameter setting when $R_0 < 1$.

| Parameters | Values | Parameters | Values |
|---|---|---|---|
| $I(0)$ | 1 | $S(0)$ | 9999 |
| $E(0)$ | 0 | $D(0)$ | 0 |
| $R(0)$ | 0 | $\Lambda$ | 0.01 |
| $\epsilon$ | 0.0006 | $\mu$ | 0.00008 |
| $\omega$ | 0.5 | $\phi$ | 0.5 |
| $k$ | 0.00008 | $\alpha$ | 0.0002 |
| $\beta$ | 0.0005 | $\theta$ | 0.002 |
| $\zeta$ | 0.0001 | $\gamma$ | 0.0001 |
| $\xi$ | 0.0001 | $t$ | 4000 |
| $R_0 = 0.0174 < 1$ | | | |

TABLE 8. Parameter setting when $R_0 > 1$.

| Parameters | Values | Parameters | Values |
|---|---|---|---|
| $I(0)$ | 1 | $S(0)$ | 9999 |
| $E(0)$ | 0 | $D(0)$ | 0 |
| $R(0)$ | 0 | $\Lambda$ | 10 |
| $\epsilon$ | 0.0006 | $\mu$ | 0.00008 |
| $\omega$ | 0.5 | $\phi$ | 0.5 |
| $k$ | 0.00008 | $\alpha$ | 0.0005 |
| $\beta$ | 0.0005 | $\theta$ | 0.0002 |
| $\zeta$ | 0.0005 | $\gamma$ | 0.0005 |
| $\xi$ | 0.00005 | $t$ | 4000 |
| Note: $R_0 = 7.535 > 1$ | | | |

TABLE 9. Parameter setting of evolutionary game baseline model.

| Parameters | Values | Parameters | Values | Parameters | Values |
|---|---|---|---|---|---|
| $(x, y, z)$ | $(0.3, 0.6, 0.4)$ | $C_{A1}$ | 100 | $R_{A2}$ | 80 |
| $C_{A2}$ | 80 | $P$ | 60 | $P_B$ | 24 |
| $E_{G2}$ | 60 | $R_{G2}$ | 60 | $C_{Gu}$ | 120 |
| $E_{G1}$ | 25 | $E_{B2}$ | 40 | $E_{B1}$ | 15 |
| $\rho$ | 0.8 | $R_{B2}$ | 40 | $C_{Bu}$ | 80 |

TABLE 10. Parameter setting when increasing $P_B$.

| Parameters | Values | Parameters | Values | Parameters | Values |
|---|---|---|---|---|---|
| $(x, y, z)$ | $(0.3, 0.6, 0.4)$ | $C_{A1}$ | 100 | $R_{A2}$ | 80 |
| $C_{A2}$ | 80 | $P$ | 60 | $P_B$ | 40 |
| $E_{G2}$ | 60 | $R_{G2}$ | 60 | $C_{Gu}$ | 120 |
| $E_{G1}$ | 25 | $E_{B2}$ | 40 | $E_{B1}$ | 15 |
| $\rho$ | 0.8 | $R_{B2}$ | 40 | $C_{Bu}$ | 80 |

TABLE 11. Parameter setting when increasing $P$.

| Parameters | Values | Parameters | Values | Parameters | Values |
|---|---|---|---|---|---|
| $(x, y, z)$ | $(0.3, 0.6, 0.4)$ | $C_{A1}$ | 100 | $R_{A2}$ | 80 |
| $C_{A2}$ | 80 | $P$ | 140 | $P_B$ | 80 |
| $E_{G2}$ | 60 | $R_{G2}$ | 60 | $C_{Gu}$ | 120 |
| $E_{G1}$ | 25 | $E_{B2}$ | 40 | $E_{B1}$ | 15 |
| $\rho$ | 0.8 | $R_{B2}$ | 40 | $C_{Bu}$ | 80 |

TABLE 12. Parameter setting with coordinated debunking funds.

| Parameters | Values | Parameters | Values | Parameters | Values |
|---|---|---|---|---|---|
| $(x, y, z)$ | $(0.3, 0.6, 0.4)$ | $C_{A1}$ | 100 | $R_{A2}$ | 80 |
| $C_{A2}$ | 80 | $P$ | 140 | $P_B$ | 60 |
| $E_{G2}$ | 60 | $R_{G2}$ | 60 | $C_{Gu}$ | 120 |
| $E_{G1}$ | 25 | $E_{B2}$ | 40 | $E_{B1}$ | 15 |
| $\rho$ | 0.8 | $R_{B2}$ | 40 | $C_{Bu}$ | 80 |
| $C_{G0}$ | 20 | $q$ | 10 | $Q$ | 30 |
| $R_{B0}$ | 50 | $C_{B0}$ | 30 | | |



**Qingwu Gao** - received his Ph.D. degree in Mathematics from Soochow University, China, in 2010. He is now a Professor at School of Mathematics, Nanjing Audit University, Nanjing, China. His research interests include modelling, simulation, dynamics, stochastics, game and control.



**Jia-Nan Lin** - is a master student of Statistics at School of Statistics and Data Science, Nanjing Audit University, Nanjing, China. His research interests include data mining and analysis, game optimization, and statistical analysis.



**Qian Zhang** - is a master student of Statistics at School of Statistics and Data Science, Nanjing Audit University, Nanjing, China. Her research interests include online finance, data crawling, and game control.



**Yuhua Xu** - received his Ph.D. degree in Control Theory and Control Engineering from Donghua University, China, in 2011. He is currently a Professor at the School of Statistics and Data Science, Nanjing Audit University, Nanjing, China. His current research interests include complex networks, nonlinear dynamics, nonlinear finance systems, and chaos control.