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# VARIABLE TIME FILTER APPROACH FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS BASED ON A SCALAR AUXILIARY VARIABLE

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Abstract. In this work, based on a scalar auxiliary variable in exponential function, an unconditionally stable variable time filter method is designed for the incompressible Navier-Stokes equations. This method combines the backward Euler scheme for temporal discretization with variable time step and a variable time filter. Without extra complexity, this method improves the solution of the backward Euler by adding several lines to the original code, and increases the time accuracy from the first order to the second order. Moreover, we prove and obtain the unconditional stability and second-order convergence of the proposed method. Finally, numerical tests illustrate the performance of the considered approach.

Keywords: Variable Time Step, Navier-Stokes Equations, Scalar Auxiliary Variable, Unconditional Stability, Variable Time Filter, Regularity, Incompressible Equations.

AMS Subject Classification: 65M15, 65M60.

## 1. Introduction

For numerical simulation and computation, efficiency and accuracy are the core parts of numerical method. In the field of computational fluid dynamics, time accuracy is critical to obtaining physically relevant solutions (see e.g. [2,22,28,35] and reference therein). For unsteady partial differential equations, the temporal adaptive method can improve the effectiveness of the numerical approach without losing accuracy. The key of this method is the variable time step. In other words, a smaller time step is used when the system dissipates fast and a larger time step is used when the system dissipates relatively slowly [27]. The time filter technique is firstly proposed in [31] and analyzed in [4], called the Robert-Asselin filter. Recently, for unsteady multi-physical coupling models, time filter was applied [9, 14, 25, 29], which can improve the accuracy of solutions without significantly increasing the complexity of the system. From a practical point of view, because the time filter is added as a separate post-processing, it is easy to combine existing backward Euler code to improve the accuracy of the backward Euler scheme from the first order to the second order [13].

In [10], the authors applied the time filter proposed in [13] for the Navier-Stokes equations and conducted theoretical results concerning stability and second-order time accuracy. However, to the best of authors' knowledge, there is little relevant theoretical analysis provided for the Navier-Stokes equations with variable time filter. Besides, we notice that the stability and convergence of incompressible Navier-Stokes equations with variable time step were proven in [11], but the results obtained are conditional, which is dependent on temporal and spatial step sizes. The scalar auxiliary variable technique is an effective method to design unconditional energy stable scheme, where the definition of energy involves a scalar auxiliary variable [33]. Subsequently, for the Navier-Stokes equations, by introducing the scalar auxiliary variable, a fully discrete

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implicit/explicit numerical scheme with constant time step is constructed [36], and two kinds of adaptive time step algorithms introduced in [8].

The goal of this paper is to extend the idea of the aforementioned researches in [33] and [10], and construct an unconditional stable method for incompressible Navier-Stokes equations, which is a combination of the variable time filter and exponential scalar auxiliary variable to achieve second-order temporal accuracy. In addition, we theoretically analyze the unconditional stability and second-order convergence of the proposed approach. The contents of this paper are listed as follows. In Section 2, we introduce some basic results and function spaces. In Section 3, based on an exponential scalar auxiliary variable, we design a variable time filter approach for the incompressible Navier-Stokes equations and discuss the unconditional stability results. Besides, error estimates of numerical solution are given. Some numerical experiments are carried out to illustrate the theoretical results in Section 4. We draw a conclusion in final section.

## 2. Preliminary setting

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. We consider the time-dependent incompressible Navier-Stokes equations as followings:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega\times J,\\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega\times J,\\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega\times J,\\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega, \end{aligned} \tag{1}$$

for $(\mathbf{x},t)\in\Omega\times J$ and $J=(0,T]$ with a fixed $T\in(0,\infty)$. Here, $\mathbf{u}(\mathbf{x},t):\Omega\times J\to\mathbb{R}^2$ and $p(\mathbf{x},t):\Omega\times J\to\mathbb{R}$ represent the unknown fluid velocity and fluid pressure in $\Omega$, respectively. Besides, the body force $\mathbf{f}(\mathbf{x},t):\Omega\times J\to\mathbb{R}^2$ is known on the flow, and $\nu>0$ is the kinematic viscosity of the fluid.

We introduce the usual $L^2(\Omega)$ norm and its inner product by $\|\cdot\|_0$ and $(\cdot,\cdot)$, respectively. The $H^m(\Omega)$ is used to represent the Sobolev space $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ denotes the norm in $H^m(\Omega)$. In particular, $L^p(\Omega)$ norm and the Sobolev space $W^{m,p}(\Omega)$ norm are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$ for $m\in\mathbb{N}^+$, $1\le p<\infty$. Additionally, for $\mathbf{X}$ being a normed function space in $\Omega$, $L^p(0,T;\mathbf{X})$ is the space of all functions defined on $[0,T]\times\Omega$ for which the norm is given by:

$$\|\mathbf{u}\|_{L^p(0,T;\mathbf{X})} = \left(\int_0^T \|\mathbf{u}\|_{\mathbf{X}}^p \mathrm{d}t\right)^{\frac{1}{p}},\ p\in[1,\infty)$$

is finite. For $p=\infty$, the usual modification is used in the definition of this space,

$$\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{X})} = ess\sup_{0\le t\le T}\|\mathbf{u}\|_{\mathbf{X}}.$$

For the mathematical setting of (1), we introduce the following function spaces:

$$\mathbf{X} := \{\mathbf{v}\in H^1(\Omega)^2:\ \mathbf{v}=0 \text{ on } \partial\Omega\},\quad \mathbf{V} := \{\mathbf{v}\in\mathbf{X}:\ \nabla\cdot\mathbf{v}=0\},$$
$$\mathbf{Y} := L^2(\Omega)^2,\quad \mathbf{H} := \{\mathbf{v}\in\mathbf{Y}:\ \nabla\cdot\mathbf{v}=0,\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}=0\},\quad M := \{q\in L^2(\Omega):(q,1)=0\}.$$

Besides, the dual space of $\mathbf{X}$ is denoted by $\mathbf{X}'$, and the norm of its element $\mathbf{f}$ is defined by:

$$\|\mathbf{f}\|_{-1} = \sup_{0\neq\mathbf{v}\in\mathbf{X}}\frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|_0},\qquad \forall\ \mathbf{f}\in\mathbf{X}'.$$

We refer readers to [20, 30] for details on these spaces.

Next, we define the Stokes operator $A$ by:

$$A\mathbf{u} = -P\Delta\mathbf{u}, \quad \forall\, \mathbf{u} \in D(A) := H^2(\Omega)^2 \cap \mathbf{V},$$

where $P$ is $L^2$-orthogonal projector of $\mathbf{Y}$ onto $\mathbf{H}$.

Moreover, an assumption about the domain $\Omega$ is needed (see [24, 36]).

**Assumption A1.** Assume that $\Omega$ is smooth so that the unique solution $(\mathbf{v}, q) \in \mathbf{X} \times M$ of the steady Stokes problem

$$-\nu\Delta\mathbf{v} + \nabla q = \mathbf{g}, \ \ \nabla \cdot \mathbf{v} = 0, \ \text{in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0,$$

exists and satisfies

$$\|\mathbf{v}\|_2 + \|q\|_1 \le C_0\|\mathbf{g}\|_0,$$

where $C_0 > 0$ is a generic constant depending on the data $\nu$ and $\Omega$.

In fact, the existence of this solution pair can be obtained by the Lax-Milgram theorem. Note that the bilinear form $\nu(\nabla\cdot, \nabla\cdot)$ is symmetric, continuous and coercive, and the bilinear $(\cdot, \mathrm{div}\cdot)$ is continuous and satisfies the Babuska-Brezzi condition. Then for all $\mathbf{g} \in \mathbf{X}'$, according to the Lax-Milgram theorem, the solution pair exists. Here and after, we denote $C$ (with or without a subscript or superscript) a general positive constant depending at most on the values of $\Omega, T, \nu, \mathbf{u}_0, \mathbf{f}$, and may represent different values at its different circumstances. We remark that the validity of the assumption is known (see [3, 20, 30]) if $\partial\Omega$ is of $C^2$ or if $\Omega$ is a convex polygon.

Furthermore, it is well known that (see [1, 17, 36])

$$\|\mathbf{v}\|_{L^\infty}^2 \le C\|\mathbf{v}\|_0\|A\mathbf{v}\|_0, \quad \|\mathbf{v}\|_2 \le C\|A\mathbf{v}\|_0, \ \ \forall\, \mathbf{v} \in D(A), \tag{2}$$

and

$$\|\mathbf{v}\|_{L^4}^2 \le C\|\mathbf{v}\|_0\|\nabla\mathbf{v}\|_0, \qquad \forall\, \mathbf{v} \in \mathbf{X}. \tag{3}$$

The following Poincaré inequalities hold

$$\|\mathbf{v}\|_0^2 \le C_p\|\mathbf{v}\|_1^2, \ \ \forall\, \mathbf{v} \in \mathbf{X}, \quad \|\mathbf{v}\|_1^2 \le C_p\|\mathbf{v}\|_2^2, \ \ \forall\, \mathbf{v} \in H^2(\Omega)^2,$$

where $C_p$ is a positive constant depending only on $\Omega$.

Now, we give the assumption on data regularity.

**Assumption A2.** For the positive constant $C_1$ depending on $\Omega$, assume that the initial data $\mathbf{u}_0(\mathbf{x})$ and the body force $\mathbf{f}$ satisfy

$$\mathbf{u}_0 \in D(A), \quad \mathbf{f} \in L^\infty(0, T; H^1(\Omega)^2), \quad \|A\mathbf{u}_0\|_0 + \|\mathbf{f}\|_1 \le C_1.$$

Next, we define the trilinear form $b_1(\cdot,\cdot,\cdot)$ on $\mathbf{X} \times \mathbf{X} \times \mathbf{X}$ by:

$$b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}).$$

Denote the skew-symmetric trilinear form $b(\cdot,\cdot,\cdot)$ as:

$$\begin{aligned} b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) + 0.5((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}) = 0.5b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) - 0.5b_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) \\ &= -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}. \end{aligned}$$

Based on the above definitions of the function spaces, the corresponding variational formulation of the problem (1) is given as follows: for all $(\mathbf{v}, q) \in \mathbf{X} \times M$, find $(\mathbf{u}, p) \in \mathbf{X} \times M$ for all $t \in (0, T]$ such that

$$\begin{cases} (\mathbf{u}_t, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{X}, \\ (\nabla \cdot \mathbf{u}, q) = 0, & \forall\, q \in M. \end{cases} \tag{4}$$

The following results are recalled which are given in [15, 20, 21].

**Theorem 1.** *Under Assumption A1 and A2, the problem (4) admits a unique solution* $(\mathbf{u}, p)$ *satisfying the following estimates*

$$\|\nabla \mathbf{u}\|_0 + \|\mathbf{u}_t\|_0 + \|A\mathbf{u}\|_0 + \|p\|_1 + \left(\int_0^t (\|\mathbf{u}_{tt}\|_0^2 + \|A\mathbf{u}_t\|_0^2 + \|p_t\|_1^2)\mathrm{d}s\right)^{1/2} \leq C_2.$$

*for all* $t \in (0, T]$

From now on, let $h$ be a real positive parameter. The finite element subspace $(\mathbf{X}_h, M_h)$ of $(\mathbf{X}, M)$ is characterized by $\mathcal{T}_h = \mathcal{T}_h(\Omega)$, a partitioning of $\overline{\Omega}$ into triangles $K$, assumed to be uniformly regular as $h \to 0$. For further details, we can refer to [6, 30, 34].

Define the subspace $\mathbf{V}_h$ of $\mathbf{X}_h$ given by:

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h : (\nabla \cdot \mathbf{v}_h, q_h) = 0, \ \forall \ q_h \in M_h\}.$$

Let $P_h : \mathbf{Y} \to \mathbf{V}_h$ denote the $L^2$-orthogonal projection, which is defined by:

$$(P_h\mathbf{u}, \mathbf{v}_h) = (\mathbf{u}, \mathbf{v}_h), \ \forall \ \mathbf{u} \in \mathbf{Y}, \ \ \mathbf{v}_h \in \mathbf{V}_h.$$

With above statements, as [19, 20], a discrete analogue $A_h = -P_h\Delta_h$ of the Stokes operator $A$ is defined by $(-\Delta_h\mathbf{u}_h, \mathbf{v}_h) = (\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)$ for all $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h$. Denote discrete Sobolev norm by $\|\mathbf{v}_h\|_r = \|A_h^{r/2}\mathbf{v}_h\|_0$ for any order $r \in \mathbb{R}$.

Next, let $(\mathbf{X}_h, M_h)$ be finite element spaces consisting piecewise polynomials of degrees $j$ and $l$ on $\pi^h \in K$, respectively. Moreover, assume that they satisfy the following approximation properties [34], for $1 \leq j, l$ :

$$\inf_{\mathbf{v}_h \in \mathbf{X}_h} \{\|\mathbf{u} - \mathbf{v}_h\|_0 + h\|\nabla(\mathbf{u} - \mathbf{v}_h)\|\} \leq Ch^{j+1}\|\mathbf{u}\|_{j+1},$$
$$\inf_{q_h \in M_h} \|p - q_h\|_0 \leq Ch^l\|p\|_l,$$

for all $\mathbf{u} \in \mathbf{X} \cap H^{j+1}(\Omega)^2$ and $p \in M \cap H^l(\Omega)$. Furthermore, we assume the finite element space pair for which the discrete inf-sup condition is satisfied

$$\inf_{q_h \in M_h} \sup_{\mathbf{v}_h \neq \mathbf{0}, \ \mathbf{v}_h \in \mathbf{X}_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|q_h\|_0 \|\nabla \mathbf{v}_h\|_0} \geq \beta > 0,$$

where $\beta$ is independent of $h$.

The following important estimates of the trilinear form $b(\cdot, \cdot, \cdot)$ can be found in [16, 18, 19, 30]:

$$\begin{aligned} |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{C}{2}(&\|A_h\mathbf{v}\|_0^{1/2}\|\mathbf{v}\|_1^{1/2}\|\mathbf{u}\|_0^{1/2}\|\mathbf{u}\|_1^{1/2} \\ &+ \|A_h\mathbf{v}\|_0^{1/2}\|\mathbf{v}\|_0^{1/2}\|\mathbf{u}\|_1)\|\mathbf{w}\|_0, \ \ \forall \ \mathbf{u}, \mathbf{v} \in \mathbf{V}_h, \ \mathbf{w} \in \mathbf{X}_h, \end{aligned} \tag{5}$$

$$\begin{aligned} |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{C}{3}(&\|\mathbf{u}\|_0^{1/2}\|A_h\mathbf{v}\|_0^{1/2} + \|\mathbf{v}\|_0^{1/2}\|A_h\mathbf{u}\|_0^{1/2} \\ &+ \|\mathbf{v}\|_1^{1/2}\|\mathbf{u}\|_1^{1/2})\|A_h\mathbf{v}\|_0^{1/2}\|A_h\mathbf{u}\|_0^{1/2}\|\mathbf{w}\|_{-1}, \ \ \forall \ \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}_h. \end{aligned} \tag{6}$$

In the last of this section, we recall the following lemma.

**Lemma 1.** *[12] Consider a two-step A-stable scheme. If* $\delta = \beta_2 - \frac{\alpha_2}{2} > 0$, *then the coefficients* $\alpha_i$ *and* $\beta_i$ $(i = 0, 1, 2)$ *satisfy the following relation:*

$$\begin{aligned} 2\left(\sum_{i=0}^{2}\alpha_i\xi_i\right)\left(\sum_{i=0}^{2}\beta_i\xi_i\right) \geq &(\alpha_2^2 + \delta)\xi_2^2 - (2\alpha_2 - 1)\xi_1^2 - ((\alpha_2 - 1)^2 + \delta)\xi_0^2 \\ &- 2(\alpha_2(\alpha_2 - 1) + \delta)(\xi_2\xi_1 - \xi_1\xi_0), \quad \forall \ \xi_0, \xi_1, \xi_2 \in \mathbb{R}. \end{aligned}$$

## 3. Variable time filter approach based on a scalar auxiliary variable

In this section, we construct a variable time filter approach based on a scalar auxiliary variable in exponential function for solving the Navier-Stokes equations . Then, we consider stability and error estimate of the presented method.

3.1. **Variable time filter approach.** For the Navier-Stokes equations, the stability of first-order time accuracy method with variable time step will result in a CFL-type condition [11]. In fact, the unconditional stability is crucial for the numerical method. Hence, in order to construct an unconditionally stable numerical method, we will introduce an exponential scalar auxiliary variable.

As the work in [23] and reference therein, let a scalar auxiliary variable $Q$ in exponential function be given as $Q(t) = \exp(-\frac{t}{T})$. Then, rewrite the governing system (1) as the following equivalent form:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + Q(t)\exp\left(\dfrac{t}{T}\right)(\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = -\dfrac{1}{T}Q(t) + \exp\left(\dfrac{t}{T}\right) b(\mathbf{u},\mathbf{u},\mathbf{u}). \end{cases} \tag{7}$$

The corresponding variational formulation of the problem (7) reads as: find $(\mathbf{u}, p) \in \mathbf{X}\times M$ for all $t \in (0, T]$ such that

$$\begin{cases} (\mathbf{u}_t, \mathbf{v}) + Q(t)\exp\left(\dfrac{t}{T}\right) b(\mathbf{u},\mathbf{u},\mathbf{v}) + \nu(\nabla\mathbf{u},\nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}), \\ (\nabla\cdot\mathbf{u}, q) = 0, \qquad \forall(\mathbf{v}, q) \in \mathbf{X}\times M \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = -\dfrac{1}{T}Q(t) + \exp\left(\dfrac{t}{T}\right) b(\mathbf{u},\mathbf{u},\mathbf{u}). \end{cases} \tag{8}$$

Let $\{t_n\}_{n=0}^N$ with $N > 0$ be a non-uniform partition of $[0, T]$ with variable time step $\Delta t_n = t_{n+1} - t_n$ and $\Delta t_{\max} = \max\limits_{1\le n\le N}\{\Delta t_n\}$. Denote variable time step ratio $\omega_n = \frac{\Delta t_n}{\Delta t_{n-1}}$ with $\omega_{\min} = \min\limits_{1\le n\le N}\{\omega_n\}$, $\omega_{\max} = \max\limits_{1\le n\le N}\{\omega_n\}$ and $\omega_0 = 1$. We assume that $\{\omega_n\}_{n=1}^N$ is a uniformly bounded sequence. Besides, we denote $E(\mathbf{u}_h^n) := (1+\omega_n)\mathbf{u}_h^n - \omega_n\mathbf{u}_h^{n-1}$ as a second-order extrapolation of $\mathbf{u}_h^{n+1}$. Especially, $E(\mathbf{u}_h^0) := \mathbf{u}_h^0$.

Based on above denotation, the variable time filter method for (8) is given as follows:

**Step I**: Given $\tilde{\mathbf{u}}_h^n, \tilde{\mathbf{u}}_h^{n-1}, \mathbf{u}_h^n \in \mathbf{X}_h$ and $Q^{n-1}, Q^n \in \mathbb{R}$, find $(\tilde{\mathbf{u}}_h^{n+1}, \tilde{p}_h^{n+1}) \in \mathbf{X}_h \times M_h$ satisfying

$$\begin{aligned} &\left(\frac{\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n}{\Delta t_n}, \mathbf{v}_h\right) + \nu(\nabla\tilde{\mathbf{u}}_h^{n+1}, \nabla\mathbf{v}_h) + Q^{n+1}\exp(\frac{t_{n+1}}{T}) b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \mathbf{v}_h) \\ &\quad - (\tilde{p}_h^{n+1}, \nabla\cdot\mathbf{v}_h) = (\mathbf{f}(t_{n+1}), \mathbf{v}_h), \\ &(\nabla\cdot\tilde{\mathbf{u}}_h^{n+1}, q_h) = 0, \\ &\frac{\frac{1+2\omega_n}{1+\omega_n}Q^{n+1} - (1+\omega_n)Q^n + \frac{\omega_n^2}{1+\omega_n}Q^{n-1}}{\Delta t_n} = -\frac{1}{T}Q^{n+1} + \exp(\frac{t_{n+1}}{T}) b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \tilde{\mathbf{u}}_h^{n+1}). \end{aligned} \tag{9}$$

**Step II**: Based on $\tilde{\mathbf{u}}_h^{n+1}, \tilde{p}_h^{n+1}$ from **Step I**, calculate $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ by:

$$\begin{aligned} \mathbf{u}_h^{n+1} &= \tilde{\mathbf{u}}_h^{n+1} - \frac{\tau}{2}\left(\frac{2}{1+\omega_n}\tilde{\mathbf{u}}_h^{n+1} - 2\mathbf{u}_h^n + \frac{2\omega_n}{1+\omega_n}\mathbf{u}_h^{n-1}\right), \\ p_h^{n+1} &= \tilde{p}_h^{n+1} - \frac{\tau}{2}\left(\frac{2}{1+\omega_n}\tilde{p}_h^{n+1} - 2p_h^n + \frac{2\omega_n}{1+\omega_n}p_h^{n-1}\right), \end{aligned} \tag{10}$$

where $\tau$ is a parameter to be determined.

*Remark* 1. As [13], we know that the above method is $A$-stable for $-\frac{1+\omega_n}{1+2\omega_n} \le \tau \le \min\{\frac{1+\omega_n}{3\omega_n}, 1+\omega_n\}$, where $\tau$ is chosen as $\tau = \frac{\omega_n(1+\omega_n)}{(1+2\omega_n)}$ for the second order accuracy. Besides, for sake of simplicity, we rewrite (10) as:

$$\tilde{s}^{n+1} = \frac{1+2\omega_n}{1+\omega_n} s^{n+1} - \omega_n s^n + \frac{\omega_n^2}{1+\omega_n} s^{n-1}, \qquad \text{for} \quad s = \mathbf{u}_h, p_h. \tag{11}$$

Note that $\tilde{\mathbf{u}}_h^{n+1}$ and $\tilde{p}_h^{n+1}$ are coupled with $Q^{n+1}$ in (9). If the solving system is decoupled, then it will get well computational efficiency. Hence, we now describe how to solve (9) under a decoupled idea.

Firstly, we denote $S^{n+1} = \exp(\frac{t_{n+1}}{T})Q^{n+1}$ and set

$$\tilde{\mathbf{u}}_h^{n+1} = \hat{\mathbf{u}}_h^{n+1} + S^{n+1}\bar{\mathbf{u}}_h^{n+1}, \qquad \tilde{p}_h^{n+1} = \hat{p}_h^{n+1} + S^{n+1}\bar{p}_h^{n+1}. \tag{12}$$

Applying (12) to (9), we get the following two equations: given $\mathbf{u}_h^n \in \mathbf{X}_h$, find $(\hat{\mathbf{u}}_h^{n+1}, \hat{p}_h^{n+1}) \in \mathbf{X}_h \times M_h$ satisfying for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h$,

$$\left(\frac{\hat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n}{\Delta t_n}, \mathbf{v}_h\right) + \nu(\nabla\hat{\mathbf{u}}_h^{n+1}, \nabla\mathbf{v}_h) - (\hat{p}_h^{n+1}, \nabla\cdot\mathbf{v}_h) + (\nabla\cdot\hat{\mathbf{u}}_h^{n+1}, q_h) = (\mathbf{f}(t_{n+1}), \mathbf{v}_h), \tag{13}$$

and given $\tilde{\mathbf{u}}_h^{n-1}, \tilde{\mathbf{u}}_h^n \in \mathbf{X}_h$, find $(\bar{\mathbf{u}}_h^{n+1}, \bar{p}_h^{n+1}) \in \mathbf{X}_h \times M_h$ satisfying

$$(\frac{\bar{\mathbf{u}}_h^{n+1}}{\Delta t_n}, \mathbf{v}_h) + \nu(\nabla\bar{\mathbf{u}}_h^{n+1}, \nabla\mathbf{v}_h) + b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \mathbf{v}_h) - (\bar{p}_h^{n+1}, \nabla\cdot\mathbf{v}_h) + (\nabla\cdot\bar{\mathbf{u}}_h^{n+1}, q_h) = 0. \tag{14}$$

Secondly, the second equation in (9) can be rewritten as:

$$\begin{aligned} &\left(\frac{\Delta t_n + \frac{1+2\omega_n}{1+\omega_n}T}{T\Delta t_n} - \exp(\frac{2t_{n+1}}{T})b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \bar{\mathbf{u}}_h^{n+1})\right) S^{n+1} \\ &= \exp(\frac{t_{n+1}}{T})\left(\frac{1+\omega_n}{\Delta t_n}Q^n - \frac{\omega_n^2}{\Delta t_n(1+\omega_n)}Q^{n-1}\right) + \exp(\frac{2t_{n+1}}{T})b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \hat{\mathbf{u}}_h^{n+1}). \end{aligned} \tag{15}$$

Thus, based on $\hat{\mathbf{u}}_h^{n+1}$ and $\bar{\mathbf{u}}_h^{n+1}$ obtained by (13) and (14), we get the value of $S^{n+1}$ by solving (15).

Finally, according the $(\hat{\mathbf{u}}_h^{n+1}, \hat{p}_h^{n+1})$, $(\bar{\mathbf{u}}_h^{n+1}, \bar{p}_h^{n+1})$ and $S^{n+1}$ obtained from (13), (14) and (15), respectively, we achieve $\tilde{\mathbf{u}}_h^{n+1}$ and $\tilde{p}_h^{n+1}$ from (12) that we can get $\mathbf{u}_h^{n+1}$ and $p_h^{n+1}$ from (10).

Note that the solvability of (15) can be proved. In fact, setting $(\mathbf{v}_h, q_h) = (\bar{\mathbf{u}}_h^{n+1}, \bar{p}_h^{n+1})$ in (14), we obtain

$$\frac{1}{\Delta t_n}\|\bar{\mathbf{u}}_h^{n+1}\|_0^2 + \nu\|\nabla\bar{\mathbf{u}}_h^{n+1}\|_0^2 + b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \bar{\mathbf{u}}_h^{n+1}) = 0,$$

which implies the coefficient of $S^{n+1}$ is non-zero.

3.2. **Unconditional stability.** In this subsection, we will prove the unconditional stability of the variable time filter method (9) and (10). Before considering the unconditional stability, we show the following equality for the approximation of the scalar auxiliary variable $Q^{n+1}$.

$$Q^{n+1}\left(\frac{1+2\omega_n}{1+\omega_n}Q^{n+1} - (1+\omega_n)Q^n + \frac{\omega_n^2}{1+\omega_n}Q^{n-1}\right) = \frac{1}{2}\left(|\tilde{Q}^{n+1}|^2 - |\tilde{Q}^n|^2\right) + \frac{1}{2}|\tilde{D}^{n+1}|^2, \tag{16}$$

where

$$|\tilde{Q}^n|^2 = \left|\frac{\sqrt{r_2} - \sqrt{r_2 - 4r_1}}{2}Q^n\right|^2 + \left|\sqrt{r_2}Q^n - \frac{\sqrt{r_2} - \sqrt{r_2 - 4r_1}}{2}Q^{n-1}\right|^2, \quad r_1 = \frac{\omega_n^2}{r_2}, \quad r_2 = 1+\omega_n,$$

$$\tilde{D}^{n+1} = \frac{2r_1}{\sqrt{r_2} - \sqrt{r_2 - 4r_1}}Q^{n+1} - \sqrt{r_2}Q^n + \frac{\sqrt{r_2} - \sqrt{r_2 - 4r_1}}{2}Q^{n-1}.$$

**Theorem 2.** *Under the assumptions of Theorem 1, we assume that the initial values* $\mathbf{u}_h^1$, $\mathbf{u}_h^0$, $Q^1$ *and* $Q^0$ *satisfy*

$$\|\mathbf{u}_h^0\|_0^2 + \|\mathbf{u}_h^1\|_0^2 + |Q^1|^2 + |Q^0|^2 \leq C.$$

*Then, for any* $N \geq 2$, *there exists a constant* $M_0 > 0$ *which is independent on* $h$ *and* $\Delta t_n$ *such that the approximate solution to (9)-(10) for the Navier-Stokes equations satisfies*

$$\|\mathbf{u}_h^N\|_0^2 + \frac{1}{2}\sum_{n=1}^{N-1}\Delta t_n \nu \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{1}{2}|\tilde{Q}^N|^2 + \frac{1}{T}\sum_{n=1}^{N-1}\Delta t_n |Q^{n+1}|^2 \leq M_0.$$

*Proof.* Taking $\mathbf{v}_h = \tilde{\mathbf{u}}_h^{n+1}\Delta t_n$ and $q_h = \tilde{p}_h^{n+1}\Delta t_n$ in the first equation of (9), we obtain

$$\begin{aligned} &H_1(\tilde{\mathbf{u}}_h^{n+1}) + \nu\Delta t_n \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \Delta t_n \exp(\frac{t_{n+1}}{T})Q^{n+1} b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \tilde{\mathbf{u}}_h^{n+1}) \\ &= \Delta t_n (\mathbf{f}(t_{n+1}), \tilde{\mathbf{u}}_h^{n+1}), \end{aligned} \tag{17}$$

where $H_1(\tilde{\mathbf{u}}_h^{n+1}) := \left(\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}\right)$.

Multiplying both sides of the second equation in (9) by $\Delta t_n Q^{n+1}$ yields

$$\begin{aligned} &Q^{n+1}\left(\frac{1+2\omega_n}{1+\omega_n}Q^{n+1} - (1+\omega_n)Q^n + \frac{\omega_n^2}{1+\omega_n}Q^{n-1}\right) \\ &= -\frac{1}{T}\Delta t_n |Q^{n+1}|^2 + \Delta t_n \exp(\frac{t_{n+1}}{T})Q^{n+1} b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), \tilde{\mathbf{u}}_h^{n+1}). \end{aligned} \tag{18}$$

Moreover, combining (17) and (18) with (16), and applying the Cauchy-Schwarz-Young inequality, we arrive at

$$\begin{aligned} &H_1(\tilde{\mathbf{u}}_h^{n+1}) + \frac{1}{2}\nu\Delta t_n \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{1}{2}(|\tilde{Q}^{n+1}|^2 - |\tilde{Q}^n|^2) + \frac{1}{2}|\tilde{D}^{n+1}|^2 + \frac{1}{T}\Delta t_n |Q^{n+1}|^2 \\ &\leq \frac{1}{2\nu}\Delta t_n \|\mathbf{f}(t_{n+1})\|_0^2. \end{aligned} \tag{19}$$

Next, using Lemma 1 with $\alpha_0 = \beta_0 = \frac{\omega_n^2}{1+\omega_n}$, $\alpha_1 = -(1+\omega_n)$, $\beta_1 = -\omega_n$, $\alpha_2 = \beta_2 = \frac{1+2\omega_n}{1+\omega_n}$, $\xi_0 = \|\mathbf{u}_h^{n-1}\|_0$, $\xi_1 = \|\mathbf{u}_h^n\|_0$, $\xi_2 = \|\mathbf{u}_h^{n+1}\|_0$ and $\delta = \frac{1+2\omega_n}{2(1+\omega_n)}$, and utilizing (11) for $H_1(\tilde{\mathbf{u}}_h^{n+1})$, we have

$$\begin{aligned} H_1(\tilde{\mathbf{u}}_h^{n+1}) \geq\ & A_1(\omega_n)\left(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2\right) + A_2(\omega_n)\left(\|\mathbf{u}_h^n\|_0^2 - \|\mathbf{u}_h^{n-1}\|_0^2\right) \\ & - A_3(\omega_n)\left(\|\mathbf{u}_h^{n+1}\|_0\|\mathbf{u}_h^n\|_0 - \|\mathbf{u}_h^n\|_0\|\mathbf{u}_h^{n-1}\|_0\right), \end{aligned} \tag{20}$$

where $A_1(\omega_n) := \frac{\alpha_2^2+\delta}{2} = \frac{10\omega_n^2+11\omega_n+3}{4(1+\omega_n)^2}$, $A_2(\omega_n) := \frac{(\alpha_2-1)^2+\delta}{2} = \frac{4\omega_n^2+3\omega_n+1}{4(1+\omega_n)^2}$ and $A_3(\omega_n) := \alpha_2(\alpha_2 - 1) + \delta = \frac{6\omega_n^2+5\omega_n+1}{2(1+\omega_n)^2}$. According the equality $ab - bc = \frac{1}{2}\left[a^2 - b^2 - (a-b)^2 + b^2 - c^2 + (b-c)^2\right]$, we gain

$$\begin{aligned} \|\mathbf{u}_h^{n+1}\|_0\|\mathbf{u}_h^n\|_0 - \|\mathbf{u}_h^n\|_0\|\mathbf{u}_h^{n-1}\|_0 = \frac{1}{2}\big[&\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2 \\ &-(\|\mathbf{u}_h^{n+1}\|_0 - \|\mathbf{u}_h^n\|_0)^2 + \|\mathbf{u}_h^n\|_0^2 - \|\mathbf{u}_h^{n-1}\|_0^2 + (\|\mathbf{u}_h^n\|_0 - \|\mathbf{u}_h^{n-1}\|_0)^2\big]. \end{aligned} \tag{21}$$

The combination of (20) and (21) implies that

$$\begin{aligned} H_1(\tilde{\mathbf{u}}_h^{n+1}) \geq\ & \left[A_1(\omega_n) - \frac{1}{2}A_3(\omega_n)\right]\left(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2\right) + \left[A_2(\omega_n) - \frac{1}{2}A_3(\omega_n)\right]\left(\|\mathbf{u}_h^n\|_0^2 - \|\mathbf{u}_h^{n-1}\|_0^2\right) \\ & + \frac{1}{2}A_3(\omega_n)\left[\left(\|\mathbf{u}_h^{n+1}\|_0 - \|\mathbf{u}_h^n\|_0\right)^2 - \left(\|\mathbf{u}_h^n\|_0 - \|\mathbf{u}_h^{n-1}\|_0\right)^2\right]. \end{aligned}$$

Obviously, $A_1(\omega_n) - \frac{1}{2}A_3(\omega_n) > 0$ and $A_2(\omega_n) - \frac{1}{2}A_3(\omega_n) < 0$. Hence, there exists a variable time step ratio $\tilde{\omega} > 0$ such that for some positive constants $C^1$, $C^2$, $C^3$ the following inequalities

hold:

$$\left[A_1(\omega_n)-\frac{A_3(\omega_n)}{2}\right]\left(\|\mathbf{u}_h^{n+1}\|_0^2-\|\mathbf{u}_h^n\|_0^2\right)\geq C^1\left[A_1(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}\right]\left(\|\mathbf{u}_h^{n+1}\|_0^2-\|\mathbf{u}_h^n\|_0^2\right), \tag{22}$$

$$\left[A_2(\omega_n)-\frac{A_3(\omega_n)}{2}\right]\left(\|\mathbf{u}_h^{n}\|_0^2-\|\mathbf{u}_h^{n-1}\|_0^2\right)\geq C^2\left[A_2(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}\right]\left(\|\mathbf{u}_h^{n}\|_0^2-\|\mathbf{u}_h^{n-1}\|_0^2\right), \tag{23}$$

and

$$\begin{aligned}&\frac{A_3(\omega_n)}{2}\left[\left(\|\mathbf{u}_h^{n+1}\|_0-\|\mathbf{u}_h^n\|_0\right)^2-\left(\|\mathbf{u}_h^n\|_0-\|\mathbf{u}_h^{n-1}\|_0\right)^2\right]\\&\geq C^3\frac{A_3(\tilde{\omega})}{2}\left[\left(\|\mathbf{u}_h^{n+1}\|_0-\|\mathbf{u}_h^n\|_0\right)^2-\left(\|\mathbf{u}_h^n\|_0-\|\mathbf{u}_h^{n-1}\|_0\right)^2\right].\end{aligned} \tag{24}$$

Further, we can find the $\tilde{\omega}$ such that $C^1\geq C^2$. In fact, according to (22) and (23), we have the following four cases:

**Case I.** If $\|\mathbf{u}_h^{n+1}\|_0^2-\|\mathbf{u}_h^n\|_0^2>0$ and $\|\mathbf{u}_h^n\|_0^2-\|\mathbf{u}_h^{n-1}\|_0^2>0$, then we have $C^1\leq\frac{A_1(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_1(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$ and $C^2\geq\frac{A_2(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_2(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$. In order to ensure $C^1\geq C^2$, we only require the following inequality to hold

$$\frac{A_1(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_1(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}\geq C^1\geq C^2\geq\frac{A_2(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_2(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}>0.$$

Due to the definitions of $A_1(\omega_n), A_2(\omega_n)$ and $A_3(\omega_n)$, the above inequality can be reduced to $(\omega_n+1)\tilde{\omega}^2+(1-\omega_n^2)\tilde{\omega}-(\omega_n^2+\omega_n)\geq 0$. It is evident that quadratic equation $(\omega_n+1)x^2+(1-\omega_n^2)x-(\omega_n^2+\omega_n)=0$ has a unique positive root $x_1=\omega_n$. Therefore, when $\tilde{\omega}\geq\omega_n$, the above inequality holds.

**Case II.** If $\|\mathbf{u}_h^{n+1}\|_0^2-\|\mathbf{u}_h^n\|_0^2>0$ and $\|\mathbf{u}_h^n\|_0^2-\|\mathbf{u}_h^{n-1}\|_0^2<0$, then we have $C^1\leq\frac{A_1(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_1(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$ and $C^2\leq\frac{A_2(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_2(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$. In order to ensure $C^1\geq C^2$, it requires $\tilde{\omega}\geq\omega_n$.

**Case III.** If $\|\mathbf{u}_h^{n+1}\|_0^2-\|\mathbf{u}_h^n\|_0^2<0$ and $\|\mathbf{u}_h^n\|_0^2-\|\mathbf{u}_h^{n-1}\|_0^2>0$, then we have $C^1\geq\frac{A_1(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_1(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$ and $C^2\geq\frac{A_2(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_2(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$. In order to ensure $C^1\geq C^2$, we need $\tilde{\omega}\geq\omega_n$.

**Case IV.** If $\|\mathbf{u}_h^{n+1}\|_0^2-\|\mathbf{u}_h^n\|_0^2<0$ and $\|\mathbf{u}_h^n\|_0^2-\|\mathbf{u}_h^{n-1}\|_0^2<0$, then we have $C^1\geq\frac{A_1(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_1(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$ and $C^2\leq\frac{A_2(\omega_n)-\frac{1}{2}A_3(\omega_n)}{A_2(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})}$. In order to ensure $C^1\geq C^2$, we require $\tilde{\omega}\geq\omega_n$.

Then, applying (22)-(24) to (19) and summing from $n=1$ to $N-1$, we obtain

$$\begin{aligned}&C^1\left[A_1(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})\right]\left(\|\mathbf{u}_h^N\|_0^2-\|\mathbf{u}_h^1\|_0^2\right)+C^2\left[A_2(\tilde{\omega})-\frac{1}{2}A_3(\tilde{\omega})\right]\left(\|\mathbf{u}_h^{N-1}\|_0^2-\|\mathbf{u}_h^0\|_0^2\right)\\&+C^3\frac{1}{2}A_3(\tilde{\omega})\left[\left(\|\mathbf{u}_h^N\|_0-\|\mathbf{u}_h^{N-1}\|_0\right)^2-\left(\|\mathbf{u}_h^1\|_0-\|\mathbf{u}_h^0\|_0\right)^2\right]+\frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|\nabla\tilde{\mathbf{u}}_h^{n+1}\|_0^2+\frac{1}{2}|\tilde{Q}^N|^2\\&+\frac{1}{2}\sum_{n=1}^{N-1}|\tilde{D}^{n+1}|^2+\frac{1}{T}\sum_{n=1}^{N-1}\Delta t_n|Q^{n+1}|^2\leq\frac{1}{2\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2+\frac{1}{2}|\tilde{Q}^1|^2.\end{aligned} \tag{25}$$

From the definitions of $A_1(\omega_n), A_2(\omega_n)$ and $A_3(\omega_n)$, we have

$$A_1(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}=\frac{4\tilde{\omega}^2+6\tilde{\omega}+2}{4(1+\tilde{\omega})^2},\quad -\left(A_2(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}\right)=\frac{2\tilde{\omega}^2+2\tilde{\omega}}{4(1+\tilde{\omega})^2},$$

which implies

$$A_1(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}>-\left(A_2(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}\right).$$

Hence, there exists a constant $\alpha > 0$ such that $\alpha := \frac{\tilde{\omega}}{2\tilde{\omega}+1} < 1$ and

$$-\left(A_2(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right) = \alpha\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right). \tag{26}$$

Finally, applying (26) to (25) leads to

$$\begin{aligned}
&C^1\left[A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right]\|\mathbf{u}_h^N\|_0^2 - C^2\left[A_2(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right]\|\mathbf{u}_h^0\|_0^2 + C^3\frac{A_3(\tilde{\omega})}{2}\left(\|\mathbf{u}_h^N\|_0 - \|\mathbf{u}_h^{N-1}\|_0\right)^2 \\
&\quad + \frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|\nabla\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{1}{2}|\tilde{Q}^N|^2 + \frac{1}{2}\sum_{n=1}^{N-1}|\tilde{D}^{n+1}|^2 + \frac{1}{T}\sum_{n=1}^{N-1}\Delta t_n|Q^{n+1}|^2 \\
&\le \frac{1}{2\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2 + \frac{1}{2}|\tilde{Q}^1|^2 + C^1\left[A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right]\|\mathbf{u}_h^1\|_0^2 + \alpha C^1\left[A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right] \\
&\quad \times \|\mathbf{u}_h^{N-1}\|_0^2 + C^3\frac{A_3(\tilde{\omega})}{2}\left(\|\mathbf{u}_h^1\|_0 - \|\mathbf{u}_h^0\|_0\right)^2 =: G_1 + \alpha C^1\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right)\|\mathbf{u}_h^{N-1}\|_0^2 \\
&\le G_1 + \alpha G_1 + \alpha^2 G_1 + \cdots + \alpha^{N-2}G_1 + \alpha^{N-1}C^1\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right)\|\mathbf{u}_h^1\|_0^2 \\
&\le \frac{G_1}{1-\alpha} + \alpha^{N-1}C^1\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right)\|\mathbf{u}_h^1\|_0^2,
\end{aligned}$$

where $G_1 = \frac{1}{2\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2 + \frac{1}{2}|\tilde{Q}^1|^2 + C^1\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right)\|\mathbf{u}_h^1\|_0^2 + C^3\frac{A_3(\tilde{\omega})}{2}\left(\|\mathbf{u}_h^1\|_0 - \|\mathbf{u}_h^0\|_0\right)^2$. Then, we get the desired result. □

**Theorem 3.** *Under assumptions of Theorem 2, if $\|A_h\tilde{\mathbf{u}}_h^1\|_0^2 < C$, then there exists a constant $M_1 > 0$ which is independent on $h$ and $\Delta t_n$ such that the approximate solution to (9)-(10) satisfies*

$$\|\nabla\mathbf{u}_h^N\|_0^2 + \frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 \le M_1, \quad \forall N \ge 2.$$

*Proof.* Choosing $\mathbf{v}_h = -A_h\tilde{\mathbf{u}}_h^{n+1}\Delta t_n \in \mathbf{V}_h$ in the first equation of (9), we obtain

$$\begin{aligned}
H_2(\nabla\tilde{\mathbf{u}}_h^{n+1}) + \nu\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 =& \Delta t_n Q^{n+1}\exp\left(\frac{t_{n+1}}{T}\right)b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), A_h\tilde{\mathbf{u}}_h^{n+1}) \\
&- \Delta t_n(\mathbf{f}(t_{n+1}), A_h\tilde{\mathbf{u}}_h^{n+1}),
\end{aligned} \tag{27}$$

where $H_2(\nabla\tilde{\mathbf{u}}_h^{n+1}) := \left(\nabla(\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n), \nabla\tilde{\mathbf{u}}_h^{n+1}\right)$.

For the first term in the right-hand side of (27), using (5), Theorem 2 and the Young inequality, we have

$$\begin{aligned}
&\Delta t_n Q^{n+1}\exp\left(\frac{t_{n+1}}{T}\right)b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), A_h\tilde{\mathbf{u}}_h^{n+1}) \\
&\le C\Delta t_n\sqrt{M_0}\|E(\tilde{\mathbf{u}}_h^n)\|_0^{\frac{1}{2}}\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0\|A_h E(\tilde{\mathbf{u}}_h^n)\|_0^{\frac{1}{2}}\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0 \\
&\le \varepsilon_1\nu\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \varepsilon_2\nu\Delta t_{n-1}\|A_h\tilde{\mathbf{u}}_h^n\|_0^2 + \varepsilon_3\nu\Delta t_{n-2}\|A_h\tilde{\mathbf{u}}_h^{n-1}\|_0^2 \\
&\quad + C\frac{1}{\nu^3}[(1+\omega_n)^2 + \omega_n^2\omega_{n-1}]\omega_n\Delta t_n M_0^2\|E(\tilde{\mathbf{u}}_h^n)\|_0^2\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^4.
\end{aligned} \tag{28}$$

Then, for the second term of the right-hand side of (27), applying the Hölder inequality and the Young inequality, we gain

$$|-\Delta t_n(\mathbf{f}(t_{n+1}), A_h\tilde{\mathbf{u}}_h^{n+1})| \le \frac{\nu}{8}\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{8}{\nu}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2. \tag{29}$$

Next, taking $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{8}$, and combining (27) with (28), (29) and Theorem 2 lead to

$$\begin{aligned} H_2(\nabla\tilde{\mathbf{u}}_h^{n+1}) + \frac{3}{4}\nu\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 &\leq \frac{\nu}{8}\Delta t_{n-1}\|A_h\tilde{\mathbf{u}}_h^{n}\|_0^2 + \frac{\nu}{8}\Delta t_{n-2}\|A_h\tilde{\mathbf{u}}_h^{n-1}\|_0^2 \\ &+ \frac{8}{\nu}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2 + C\frac{1}{\nu^3}[(1+\omega_n)^2+\omega_n^2\omega_{n-1}]\omega_n(1+\omega_n)^2\Delta t_n M_0^3(\|\nabla\tilde{\mathbf{u}}_h^n\|_0^2 \\ &+ \|\nabla\tilde{\mathbf{u}}_h^{n-1}\|_0^2)\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2. \end{aligned} \tag{30}$$

By the similar argument as proof of Theorem 2, invoking Lemma 1 and summing (30) from $n=1$ to $n=N-1$, we obtain

$$\begin{aligned} &C^1\left[A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right]\left(\|\nabla\mathbf{u}_h^N\|_0^2 - \|\nabla\mathbf{u}_h^1\|_0^2\right) + C^2\left[A_2(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right]\left(\|\nabla\mathbf{u}_h^{N-1}\|_0^2 - \|\nabla\mathbf{u}_h^0\|_0^2\right) \\ &+ C^3\frac{A_3(\tilde{\omega})}{2}\left[\left(\|\nabla\mathbf{u}_h^N\|_0 - \|\nabla\mathbf{u}_h^{N-1}\|_0\right)^2 - \left(\|\nabla\mathbf{u}_h^1\|_0 - \|\nabla\mathbf{u}_h^0\|_0\right)^2\right] + \frac{3}{4}\sum_{n=1}^{N-1}\nu\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 \\ &\leq \frac{\nu}{8}\sum_{n=1}^{N-1}\Delta t_{n-1}\|A_h\tilde{\mathbf{u}}_h^{n}\|_0^2 + \frac{\nu}{8}\sum_{n=1}^{N-1}\Delta t_{n-2}\|A_h\tilde{\mathbf{u}}_h^{n-1}\|_0^2 + \frac{8}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2 \\ &+ C\frac{1}{\nu^3}\sum_{n=1}^{N-1}[(1+\omega_n)^2+\omega_n^2\omega_{n-1}]\omega_n(1+\omega_n)^2\Delta t_n M_0^3(\|\nabla\tilde{\mathbf{u}}_h^n\|_0^2 + \|\nabla\tilde{\mathbf{u}}_h^{n-1}\|_0^2)\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2, \end{aligned}$$

which, combines with

$$\begin{aligned} &\frac{\nu}{8}\sum_{n=1}^{N-1}(\Delta t_{n-1}\|A_h\tilde{\mathbf{u}}_h^{n}\|_0^2 + \Delta t_{n-2}\|A_h\tilde{\mathbf{u}}_h^{n-1}\|_0^2) \leq \frac{\nu}{4}\sum_{n=1}^{N-1}\Delta t_{n-1}\|A_h\tilde{\mathbf{u}}_h^{n}\|_0^2, \\ &\frac{\nu}{4}\sum_{n=1}^{N-1}(3\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 - \Delta t_{n-1}\|A_h\tilde{\mathbf{u}}_h^{n}\|_0^2) \\ &= \frac{\nu}{2}\sum_{n=1}^{N-1}\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{\nu}{4}\Delta t_{N-1}\|A_h\tilde{\mathbf{u}}_h^{N}\|_0^2 - \frac{\nu}{4}\Delta t_0\|A_h\tilde{\mathbf{u}}_h^{1}\|_0^2, \end{aligned}$$

and rearrange the ensuing inequality to arrive at

$$\begin{aligned} &\|\nabla\mathbf{u}_h^N\|_0^2 + \frac{1}{2\Pi}\sum_{n=1}^{N-1}\nu\Delta t_n\|A_h\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{\nu}{4\Pi}\Delta t_{N-1}\|A_h\tilde{\mathbf{u}}_h^{N}\|_0^2 + C^3\frac{A_3(\tilde{\omega})}{2\Pi}\left(\|\nabla\mathbf{u}_h^N\|_0 - \|\nabla\mathbf{u}_h^{N-1}\|_0\right)^2 \\ &\leq \frac{1}{\Pi}\left(C\frac{1}{\nu^3}\sum_{n=1}^{N-1}[(1+\omega_n)^2+\omega_n^2\omega_{n-1}]\omega_n(1+\omega_n)^2\Delta t_n M_0^3\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2(\|\nabla\tilde{\mathbf{u}}_h^n\|_0^2 + \|\nabla\tilde{\mathbf{u}}_h^{n-1}\|_0^2)\right. \\ &+ \frac{8}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2 + \frac{\nu}{4}\Delta t_0\|A_h\tilde{\mathbf{u}}_h^{1}\|_0^2 + C^1\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right)\|\nabla\mathbf{u}_h^1\|_0^2 \\ &\left.+C^3\frac{A_3(\tilde{\omega})}{2}\left(\|\nabla\mathbf{u}_h^1\|_0 - \|\nabla\mathbf{u}_h^0\|_0\right)^2\right) + \alpha\|\nabla\mathbf{u}_h^{N-1}\|_0^2) \\ &\leq \frac{1}{\Pi(1-\alpha)}\left(C\frac{1}{\nu^3}\sum_{n=1}^{N-1}[(1+\omega_n)^2+\omega_n^2\omega_{n-1}]\omega_n(1+\omega_n)^2\Delta t_n M_0^3\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2(\|\nabla\tilde{\mathbf{u}}_h^n\|_0^2 + \|\nabla\tilde{\mathbf{u}}_h^{n-1}\|_0^2\right) \\ &+ \frac{8}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_0^2 + \frac{\nu}{4}\Delta t_0\|A_h\tilde{\mathbf{u}}_h^{1}\|_0^2 + C^1\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right)\|\nabla\mathbf{u}_h^1\|_0^2 \\ &\left.+C^3\frac{A_3(\tilde{\omega})}{2}\left(\|\nabla\mathbf{u}_h^1\|_0 - \|\nabla\mathbf{u}_h^0\|_0\right)^2\right) + \alpha^{N-1}\|\nabla\mathbf{u}_h^1\|_0^2, \end{aligned}$$

where $\Pi := C^1\left(A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right)$. Finally, applying the discrete Grönwall lemma and Theorem 2, we complete the proof.

□

**Theorem 4.** *Under assumptions of Theorem 3, if $\|A_h\mathbf{u}_h^1\|_0^2 < +\infty$, then for any $N \geq 2$, there exists a constant $M_2 > 0$ which is independent on $h$ and $\Delta t_n$ such that the approximate solution to (9)-(10) of the Navier-Stokes equations satisfies*

$$\|A_h\mathbf{u}_h^N\|_0^2 + \frac{1}{2}\sum_{n=0}^{N-1}\nu\Delta t_n\|A_h^{\frac{3}{2}}\tilde{\mathbf{u}}_h^{n+1}\|_0^2 \leq M_2.$$

*Proof.* Taking $\mathbf{v}_h = A_h^2\tilde{\mathbf{u}}_h^{n+1}\Delta t_n \in \mathbf{V}_h$ in the first equation of (9), we obtain

$$\begin{aligned} &H_3\left(A_h\tilde{\mathbf{u}}_h^{n+1}\right) + \nu\Delta t_n\|A_h^{\frac{3}{2}}\tilde{\mathbf{u}}_h^{n+1}\|_0^2 \\ =& -\Delta t_n Q^{n+1}\exp\left(\frac{t_{n+1}}{T}\right)b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), A_h^2\tilde{\mathbf{u}}_h^{n+1}) + \Delta t_n(\mathbf{f}(t_{n+1}), A_h^2\tilde{\mathbf{u}}_h^{n+1}), \end{aligned} \tag{31}$$

where $H_3(A_h\tilde{\mathbf{u}}_h^{n+1}) := \left(A_h(\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n), A_h\tilde{\mathbf{u}}_h^{n+1}\right)$. Then, utilizing (6), Theorem 2 and the Young inequality, we have

$$\begin{aligned} &|-\Delta t_n Q^{n+1}\exp\left(\frac{t_{n+1}}{T}\right)b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), A_h^2\tilde{\mathbf{u}}_h^{n+1})| \\ \leq& \frac{2}{3}C\Delta t_n|Q^{n+1}|(\|E(\tilde{\mathbf{u}}_h^n)\|_0^{\frac{1}{2}}\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^{\frac{3}{2}} + \|E(\tilde{\mathbf{u}}_h^n)\|_1\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0)\|A_h^2\tilde{\mathbf{u}}_h^{n+1}\|_{-1} \\ \leq& \frac{\nu}{4}\Delta t_n\|A_h^{\frac{3}{2}}\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{CM_0}{\nu}\Delta t_n(\|E(\tilde{\mathbf{u}}_h^n)\|_0\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0 + \|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2)\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^2, \end{aligned} \tag{32}$$

and

$$|\Delta t_n(\mathbf{f}(t_{n+1}), A_h^2\tilde{\mathbf{u}}_h^{n+1})| \leq \frac{\nu}{4}\Delta t_n\|A_h^{\frac{3}{2}}\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \frac{4}{\nu}\Delta t_n\|\mathbf{f}(t_{n+1})\|_1^2. \tag{33}$$

Now, by the similar argument as proof of Theorem 2, combining (31) with (32) and (33) and summing the ensuing inequality from $n = 1$ to $N-1$, we arrive at

$$\begin{aligned} &C^1\left[A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right]\left(\|A_h\mathbf{u}_h^N\|_0^2 - \|A_h\mathbf{u}_h^1\|_0^2\right) + C^2\left[A_2(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right]\left(\|A_h\mathbf{u}_h^{N-1}\|_0^2 - \|A_h\mathbf{u}_h^0\|_0^2\right) \\ &+ C^3\frac{A_3(\tilde{\omega})}{2}\left[\left(\|A_h\mathbf{u}_h^N\|_0 - \|A_h\mathbf{u}_h^{N-1}\|_0\right)^2 - \left(\|A_h\mathbf{u}_h^1\|_0 - \|A_h\mathbf{u}_h^0\|_0\right)^2\right] + \frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|A_h^{\frac{3}{2}}\tilde{\mathbf{u}}_h^{n+1}\|_0^2 \\ \leq& \frac{CM_0}{\nu}\sum_{n=1}^{N-1}\Delta t_n(\|E(\tilde{\mathbf{u}}_h^n)\|_0\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0 + \|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2)\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^2 + \frac{4}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_1^2. \end{aligned}$$

Rearranging and applying recursive method yields

$$\begin{aligned} &\|A_h\mathbf{u}_h^N\|_0^2 + \frac{1}{2\Pi}\sum_{n=1}^{N-1}\nu\Delta t_n\|A_h^{\frac{3}{2}}\tilde{\mathbf{u}}_h^{n+1}\|_0^2 + C^3\frac{A_3(\tilde{\omega})}{2\Pi}\left(\|A_h\mathbf{u}_h^N\|_0 - \|A_h\mathbf{u}_h^{N-1}\|_0\right)^2 \\ \leq& \frac{1}{\Pi}\left(\Pi\|A_h\mathbf{u}_h^1\|_0^2 + C^3\frac{A_3(\tilde{\omega})}{2}\left(\|A_h\mathbf{u}_h^1\|_0 - \|A_h\mathbf{u}_h^0\|_0\right)^2 + \frac{4}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_1^2\right. \\ &\left. + \frac{CM_0}{\nu}\sum_{n=1}^{N-1}\Delta t_n(\|E(\tilde{\mathbf{u}}_h^n)\|_0\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0 + \|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2)\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^2\right) + \alpha\|A_h\mathbf{u}_h^{N-1}\|_0^2 \\ \leq& \frac{1}{\Pi(1-\alpha)}\left(\Pi\|A_h\mathbf{u}_h^1\|_0^2 + C^3\frac{A_3(\tilde{\omega})}{2}\left(\|A_h\mathbf{u}_h^1\|_0 - \|A_h\mathbf{u}_h^0\|_0\right)^2 + \frac{4}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|\mathbf{f}(t_{n+1})\|_1^2\right. \end{aligned}$$

$$+\frac{CM_0}{\nu}\sum_{n=1}^{N-1}\Delta t_n(\|E(\tilde{\mathbf{u}}_h^n)\|_0\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0+\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2)\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^2\Bigg)+\alpha^{N-1}\|A_h\mathbf{u}_h^1\|_0^2.$$

With the help of the discrete Grönwall lemma, Theorem 2 and 3, we obtain the desired result. □

3.3. **Error analysis.** In this subsection, it is devoted to giving error equation of the variable time filter method (9)-(10), and to establishing error estimate.

Firstly, we define the error functions for the velocity, pressure and scalar auxiliary variable as followings:

$$\begin{aligned}
e_{\mathbf{u}}^{n+1}&=\mathbf{u}(t_{n+1})-\mathbf{u}_h^{n+1}=(\mathbf{u}(t_{n+1})-\bar{\mathbf{u}}^{n+1})+(\bar{\mathbf{u}}^{n+1}-\mathbf{u}_h^{n+1})=:\eta_{\mathbf{u}}^{n+1}+\phi_{\mathbf{u}}^{n+1},\\
e_p^{n+1}&=p(t_{n+1})-p_h^{n+1}=(p(t_{n+1})-\bar{p}^{n+1})+(\bar{p}^{n+1}-p_h^{n+1})=:\eta_p^{n+1}+\phi_p^{n+1},\\
\tilde{e}_{\mathbf{u}}^{n+1}&=\mathbf{u}(t_{n+1})-\tilde{\mathbf{u}}_h^{n+1}=(\mathbf{u}(t_{n+1})-\bar{\mathbf{u}}^{n+1})+(\bar{\mathbf{u}}^{n+1}-\tilde{\mathbf{u}}_h^{n+1})=:\eta_{\mathbf{u}}^{n+1}+\tilde{\phi}_{\mathbf{u}}^{n+1},\\
\tilde{e}_p^{n+1}&=p(t_{n+1})-\tilde{p}_h^{n+1}=(p(t_{n+1})-\bar{p}^{n+1})+(\bar{p}^{n+1}-\tilde{p}_h^{n+1})=:\eta_p^{n+1}+\tilde{\phi}_p^{n+1},\\
e_Q^{n+1}&=Q(t_{n+1})-Q^{n+1},
\end{aligned}$$

where $\bar{\mathbf{u}}^{n+1}$ and $\bar{p}^{n+1}$ are denoted as the interpolation of $\mathbf{u}(t_{n+1}),p(t_{n+1})$. Here, we set $\phi_{\mathbf{u}}^0=\phi_p^0=0$.

Secondly, we present the order of consistency errors, which is crucial in analyzing error bound. In fact, by Taylor's theorem with integral remainder and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\|\frac{\frac{1+2\omega_n}{1+\omega_n}\mathbf{u}(t_{n+1})-(1+\omega_n)\mathbf{u}(t_n)+\frac{\omega_n^2}{1+\omega_n}\mathbf{u}(t_{n-1})}{\Delta t_n}-\mathbf{u}_t(t_{n+1})\right\|_0^2\\
=&\Bigg\|\frac{1}{\Delta t_n}\Bigg[\frac{1+2\omega_n}{1+\omega_n}\mathbf{u}(t_{n+1})-\Delta t_n\mathbf{u}_t(t_{n+1})\\
&-(1+\omega_n)\left(\mathbf{u}(t_{n+1})-\Delta t_n\mathbf{u}_t(t_{n+1})+\frac{(\Delta t_n)^2}{2}\mathbf{u}_{tt}(t_{n+1})+\frac{1}{2}\int_{t_{n+1}}^{t_n}\mathbf{u}_{ttt}(t)(t_n-t)^2\mathrm{d}t\right)\\
&+\frac{\omega_n^2}{1+\omega_n}\left(\mathbf{u}(t_{n+1})-(\Delta t_n+\Delta t_{n-1})\mathbf{u}_t(t_{n+1})+\frac{(\Delta t_n+\Delta t_{n-1})^2}{2}\mathbf{u}_{tt}(t_{n+1})\right.\\
&\left.+\frac{1}{2}\int_{t_{n+1}}^{t_{n-1}}\mathbf{u}_{ttt}(t)(t_{n-1}-t)^2\mathrm{d}t\right)\Bigg]\Bigg\|_0^2\\
=&\left\|\frac{1}{\Delta t_n}\left(\frac{\omega_n^2}{2(1+\omega_n)}\int_{t_{n+1}}^{t_{n-1}}\mathbf{u}_{ttt}(t)(t_{n-1}-t)^2\mathrm{d}t-\frac{1+\omega_n}{2}\int_{t_{n+1}}^{t_n}\mathbf{u}_{ttt}(t)(t_n-t)^2\mathrm{d}t\right)\right\|_0^2\\
\le&\Big(\frac{\omega_n^2}{2(1+\omega_n)}\Big)^2\frac{1}{(\Delta t_n)^2}\left\|\frac{1}{5}(\Delta t_n+\Delta t_{n-1})^{5/2}\left(\int_{t_{n+1}}^{t_{n-1}}\mathbf{u}_{ttt}^2(t)\mathrm{d}t\right)^{1/2}\right\|_0^2\\
&+\Big(\frac{1+\omega_n}{2}\Big)^2\frac{1}{(\Delta t_n)^2}\left\|\frac{1}{5}(\Delta t_n)^{5/2}\left(\int_{t_{n+1}}^{t_n}\mathbf{u}_{ttt}^2(t)\mathrm{d}t\right)^{1/2}\right\|_0^2
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{1}{25}\Big(\frac{\omega_n^2}{2(1+\omega_n)}\Big)^2(\Delta t_n+\Delta t_{n-1})^3\Big\|\left(\int_{t_{n+1}}^{t_{n-1}}\mathbf{u}_{ttt}^2(t)\mathrm{d}t\right)^{1/2}\Big\|_0^2 \\
& + \frac{1}{25}\Big(\frac{1+\omega_n}{2}\Big)^2(\Delta t_n)^3\Big\|\left(\int_{t_{n+1}}^{t_n}\mathbf{u}_{ttt}^2(t)\mathrm{d}t\right)^{1/2}\Big\|_0^2, \qquad (34)
\end{aligned}
$$

and

$$
\begin{aligned}
& \Big\|\frac{\frac{1+2\omega_n}{1+\omega_n}\eta_{\mathbf{u}}^{n+1}-(1+\omega_n)\eta_{\mathbf{u}}^n+\frac{\omega_n^2}{1+\omega_n}\eta_{\mathbf{u}}^{n-1}}{\Delta t_n}\Big\|_0^2 \\
= & \frac{1}{(\Delta t_n)^2}\Big\|\frac{1+2\omega_n}{1+\omega_n}(\eta_{\mathbf{u}}^{n+1}-\eta_{\mathbf{u}}^n)-\frac{\omega_n^2}{1+\omega_n}(\eta_{\mathbf{u}}^n-\eta_{\mathbf{u}}^{n-1})\Big\|_0^2 \\
\leq & \frac{2}{(\Delta t_n)^2}\Big(\frac{1+2\omega_n}{1+\omega_n}\Big)^2\Big\|\int_{t_{n-1}}^{t_{n+1}}\eta_{\mathbf{u}_t}\mathrm{d}t\Big\|_0^2+\frac{2}{(\Delta t_n)^2}\Big(\frac{\omega_n^2}{1+\omega_n}\Big)^2\Big\|\int_{t_{n-1}}^{t_n}\eta_{\mathbf{u}_t}\mathrm{d}t\Big\|_0^2 \\
\leq & \frac{2}{(\Delta t_n)^2}\Big(\frac{1+2\omega_n}{1+\omega_n}\Big)^2(\Delta t_n+\Delta t_{n-1})\Big\|\left(\int_{t_{n-1}}^{t_{n+1}}\eta_{\mathbf{u}_t}^2\mathrm{d}t\right)^{1/2}\Big\|_0^2 \\
& + \frac{2}{(\Delta t_n)^2}\Big(\frac{\omega_n^2}{1+\omega_n}\Big)^2\Delta t_n\Big\|\left(\int_{t_{n-1}}^{t_n}\eta_{\mathbf{u}_t}^2\mathrm{d}t\right)^{1/2}\Big\|_0^2,
\end{aligned} \qquad (35)
$$

where the constant $C(\omega_n)$ depends on $\omega_n$.

**Lemma 2.** *(Consistency Error). Under the assumptions of Theorem 1, the following inequalities hold*

$$
\begin{aligned}
& \Big\|\frac{\frac{1+2\omega_n}{1+\omega_n}\mathbf{u}(t_{n+1})-(1+\omega_n)\mathbf{u}(t_n)+\frac{\omega_n^2}{1+\omega_n}\mathbf{u}(t_{n-1})}{\Delta t_n}-\mathbf{u}_t(t_{n+1})\Big\|_0^2 \\
\leq & C(\omega_n)(\Delta t_n+\Delta t_{n-1})^3\int_{t_{n-1}}^{t_{n+1}}\|\mathbf{u}_{ttt}\|_0^2\mathrm{d}t, \\
& \Big\|\frac{\frac{1+2\omega_n}{1+\omega_n}\eta_{\mathbf{u}}^{n+1}-(1+\omega_n)\eta_{\mathbf{u}}^n+\frac{\omega_n^2}{1+\omega_n}\eta_{\mathbf{u}}^{n-1}}{\Delta t_n}\Big\|_0^2\leq\frac{C(\omega_n)}{\Delta t_n}\int_{t_{n-1}}^{t_{n+1}}\|(\eta_{\mathbf{u}})_t\|_0^2\mathrm{d}t, \\
& \big\|\mathbf{u}(t_{n+1})-E(\mathbf{u}(t_n))\big\|_0^2\leq C(\omega_n)(\Delta t_n+\Delta t_{n-1})^3\int_{t_{n-1}}^{t_{n+1}}\|\mathbf{u}_{tt}\|_0^2\mathrm{d}t, \\
& \big\|D\mathbf{u}(t_{n+1})-\mathbf{u}(t_{n+1})\big\|_0^2\leq C(\omega_n)(\Delta t_n+\Delta t_{n-1})^3\int_{t_{n-1}}^{t_{n+1}}\|\mathbf{u}_{tt}\|_0^2\mathrm{d}t,
\end{aligned}
$$

*where* $D\mathbf{u}(t_{n+1}):=\frac{1+2\omega_n}{1+\omega_n}\mathbf{u}(t_{n+1})-\omega_n\mathbf{u}(t_n)+\frac{\omega_n^2}{1+\omega_n}\mathbf{u}(t_{n-1})$.

*Proof.* The first and second bounds follow from (34) and (35), respectively. Additionally, the third and fourth bounds are the application of the integration by parts. In fact, one notices that

$\big\|\mathbf{u}(t_{n+1})-E(\mathbf{u}(t_n))\big\|_0^2=\big\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)-\omega_n(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}))\big\|_0^2=\big\|(1+\omega_n)\int_{t_n}^{t_{n+1}}\mathbf{u}_t\mathrm{d}(t-t_n)-\omega_n\int_{t_{n-1}}^{t_{n+1}}\mathbf{u}_t\mathrm{d}(t-t_{n-1})\big\|_0^2$ and $\big\|D\mathbf{u}(t_{n+1})-\mathbf{u}(t_{n+1})\big\|_0^2=\big\|\frac{\omega_n}{1+\omega_n}(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))-\frac{\omega_n^2}{1+\omega_n}(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}))\big\|_0^2=\big\|\frac{\omega_n}{1+\omega_n}\int_{t_n}^{t_{n+1}}\mathbf{u}_t\mathrm{d}(t-t_{n+1})-\frac{\omega_n^2}{1+\omega_n}\int_{t_{n-1}}^{t_n}\mathbf{u}_t\mathrm{d}(t-t_{n-1})\big\|_0^2$. Then, the results follow from the integration by parts and the Cauchy-Schwarz inequality. □

Additionally, we notice that

$$\tilde{\phi}_{\mathbf{u}}^{n+1}=\bar{\mathbf{u}}^{n+1}-D\bar{\mathbf{u}}^{n+1}+D\phi_{\mathbf{u}}^{n+1}=-(\eta_{\mathbf{u}}^{n+1}-D\eta_{\mathbf{u}}^{n+1})+(\mathbf{u}(t_{n+1})-D\mathbf{u}(t_{n+1}))+D\phi_{\mathbf{u}}^{n+1},$$

Therefore, according to Lemma 2, we get

$$\begin{aligned}(\tilde{\phi}_{\mathbf{u}}^{n+1},\tilde{\phi}_{\mathbf{u}}^{n+1})\leq&2\|D\phi_{\mathbf{u}}^{n+1}\|_0^2+C(\Delta t_n+\Delta t_{n-1})^3\Big(\|(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\\&+\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\Big).\end{aligned}\tag{36}$$

Moreover, set $(\mathbf{v},q)=(\mathbf{v}_h,q_h)$ in (8). Then the following error equations are obtained by substituting (10) into (9), and subtracting from (8) at the $(n+1)$-th time level,

$$\begin{cases}\Big(\frac{\frac{1+2\omega_n}{1+\omega_n}e_{\mathbf{u}}^{n+1}-(1+\omega_n)e_{\mathbf{u}}^n+\frac{\omega_n^2}{1+\omega_n}e_{\mathbf{u}}^{n-1}}{\Delta t_n},\mathbf{v}_h\Big)+\nu(\nabla\tilde{e}_{\mathbf{u}}^{n+1},\nabla\mathbf{v}_h)\\\quad+\exp\left(\frac{t_{n+1}}{T}\right)Q(t_{n+1})b(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h)-\exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1}b(E(\tilde{\mathbf{u}}_h^n),E(\tilde{\mathbf{u}}_h^n),\mathbf{v}_h)\\=(p(t_{n+1})-\tilde{p}_h^{n+1},\nabla\cdot\mathbf{v}_h)-(\nabla\cdot(\mathbf{u}(t_{n+1})-\tilde{\mathbf{u}}_h^{n+1}),q_h)\\\quad+\Big(\frac{\frac{1+2\omega_n}{1+\omega_n}\mathbf{u}(t_{n+1})-(1+\omega_n)\mathbf{u}(t_n)+\frac{\omega_n^2}{1+\omega_n}\mathbf{u}(t_{n-1})}{\Delta t_n}-\mathbf{u}_t(t_{n+1}),\mathbf{v}_h\Big),\\\frac{\frac{1+2\omega_n}{1+\omega_n}e_Q^{n+1}-(1+\omega_n)e_Q^n+\frac{\omega_n^2}{1+\omega_n}e_Q^{n-1}}{\Delta t_n}=-\frac{1}{T}e_Q^{n+1}+\exp\left(\frac{t_{n+1}}{T}\right)b(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}))\\\quad-\exp\left(\frac{t_{n+1}}{T}\right)b(E(\tilde{\mathbf{u}}_h^n),E(\tilde{\mathbf{u}}_h^n),\tilde{\mathbf{u}}_h^{n+1})\\\quad+\frac{\frac{1+2\omega_n}{1+\omega_n}Q(t_{n+1})-(1+\omega_n)Q(t_n)+\frac{\omega_n^2}{1+\omega_n}Q(t_{n-1})}{\Delta t_n}-Q_t(t_{n+1}).\end{cases}\tag{37}$$

Now, we show error estimates of the variable time filter method (9)-(10).

**Theorem 5.** *Assume that the true solution* $(\mathbf{u},p)$ *satisfies Theorem 1 and the following regularity* $\mathbf{u}\in L^\infty(0,T;H^{k+1}(\Omega)^2)$, $\mathbf{u}_{tt}\in L^2(0,T;H^1(\Omega)^2)$, $\mathbf{u}_{ttt}\in L^2(0,T;L^2(\Omega)^2)$, $p\in L^2(0,T;H^k(\Omega))$. *For* $(\mathbf{u}_h^{n+1},p_h^{n+1})$ *satisfying (9)-(10), if*

$$\|e_{\mathbf{u}}^1\|_0^2+\|\nabla e_{\mathbf{u}}^1\|_0^2+|e_Q^1|^2\leq C\left((\overline{\Delta t})^4+h^{2k}\right),$$

*where* $\overline{\Delta t}=\frac{1}{N}\sum_{n=1}^N\Delta t_n$, *then the following estimate holds*

$$\|e_{\mathbf{u}}^N\|_0^2+\frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|\nabla(De_{\mathbf{u}}^{n+1})\|_0^2+\frac{1}{2}|e_Q^N|^2+\frac{1}{2}\sum_{n=1}^{N-1}\frac{\Delta t_n}{T}|e_Q^{n+1}|^2\leq C\left((\overline{\Delta t})^4+h^{2k}\right).$$

*Proof.* We will prove this theorem by the following steps.

*Step* 1. Consider the error equation concerning the velocity. Selecting $\mathbf{v}_h = \Delta t_n D\phi_{\mathbf{u}}^{n+1} \in \mathbf{V}_h$ and $q_h = 0$ in the first equation of (37), we arrive at

$$\begin{aligned}
&H_4(\phi_{\mathbf{u}}^{n+1}) + \nu\Delta t_n(\nabla\tilde{\phi}_{\mathbf{u}}^{n+1}, \nabla(D\phi_{\mathbf{u}}^{n+1})) + \Delta t_n b(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), D\phi_{\mathbf{u}}^{n+1}) \\
&- \Delta t_n \exp\left(\frac{t_{n+1}}{T}\right) Q^{n+1} b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), D\phi_{\mathbf{u}}^{n+1}) + \nu\Delta t_n(\nabla\eta_{\mathbf{u}}^{n+1}, \nabla(D\phi_{\mathbf{u}}^{n+1})) \\
&= \Delta t_n(p(t_{n+1}) - \tilde{p}_h^{n+1}, \nabla\cdot D\phi_{\mathbf{u}}^{n+1}) - \Delta t_n\big(\frac{\frac{1+2\omega_n}{1+\omega_n}\eta_{\mathbf{u}}^{n+1} - (1+\omega_n)\eta_{\mathbf{u}}^{n} + \frac{\omega_n^2}{1+\omega_n}\eta_{\mathbf{u}}^{n-1}}{\Delta t_n}, D\phi_{\mathbf{u}}^{n+1}\big) \\
&+ \Delta t_n\big(\frac{\frac{1+2\omega_n}{1+\omega_n}\mathbf{u}(t_{n+1}) - (1+\omega_n)\mathbf{u}(t_n) + \frac{\omega_n^2}{1+\omega_n}\mathbf{u}(t_{n-1})}{\Delta t_n} - \mathbf{u}_t(t_{n+1}), D\phi_{\mathbf{u}}^{n+1}\big),
\end{aligned} \tag{38}$$

where $H_4(\phi_{\mathbf{u}}^{n+1}) := (D\phi_{\mathbf{u}}^{n+1} - \phi_{\mathbf{u}}^{n}, D\phi_{\mathbf{u}}^{n+1})$ and

$$\begin{aligned}
(\nabla\tilde{\phi}_{\mathbf{u}}^{n+1}, \nabla(D\phi_{\mathbf{u}}^{n+1})) = &(\nabla(D\eta_{\mathbf{u}}^{n+1} - \eta_{\mathbf{u}}^{n+1}), \nabla(D\phi_{\mathbf{u}}^{n+1})) + (\nabla(D\phi_{\mathbf{u}}^{n+1}), \nabla(D\phi_{\mathbf{u}}^{n+1})) \\
&+ (\nabla(\mathbf{u}(t_{n+1}) - D\mathbf{u}(t_{n+1})), \nabla(D\phi_{\mathbf{u}}^{n+1})).
\end{aligned} \tag{39}$$

With the help of Lemma 2, we have

$$\begin{aligned}
&\big|-\nu\Delta t_n(\nabla(D\eta_{\mathbf{u}}^{n+1} - \eta_{\mathbf{u}}^{n+1}), \nabla(D\phi_{\mathbf{u}}^{n+1})) - \nu\Delta t_n(\nabla(\mathbf{u}(t_{n+1}) - D\mathbf{u}(t_{n+1})), \nabla(D\phi_{\mathbf{u}}^{n+1}))\big| \\
&\le \frac{\nu}{12}\Delta t_n\|\nabla(D\phi_{\mathbf{u}}^{n+1})\|_0^2 + C(\omega_n)\Delta t_n(\Delta t_n + \Delta t_{n-1})^3\Big(\|(\nabla\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} \\
&\quad + \|\nabla\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\Big).
\end{aligned}$$

Next, we rewrite the trilinear terms in (38) as:

$$\begin{aligned}
&\Delta t_n\left(b(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), D\phi_{\mathbf{u}}^{n+1}) - \exp(\frac{t_{n+1}}{T})Q^{n+1}b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), D\phi_{\mathbf{u}}^{n+1})\right) \\
&= \Delta t_n b(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}) - E(\mathbf{u}(t_n)), D\phi_{\mathbf{u}}^{n+1}) + \Delta t_n b(\mathbf{u}(t_{n+1}), E(\mathbf{u}(t_n)) - E(\tilde{\mathbf{u}}_h^n), D\phi_{\mathbf{u}}^{n+1}) \\
&\quad + \Delta t_n b(\mathbf{u}(t_{n+1}) - E(\mathbf{u}(t_n)), E(\tilde{\mathbf{u}}_h^n), D\phi_{\mathbf{u}}^{n+1}) + \Delta t_n b(E(\mathbf{u}(t_n)) - E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), D\phi_{\mathbf{u}}^{n+1}) \\
&\quad + \Delta t_n \exp(\frac{t_{n+1}}{T})e_Q^{n+1}b(E(\tilde{\mathbf{u}}_h^n), E(\tilde{\mathbf{u}}_h^n), D\phi_{\mathbf{u}}^{n+1}) =: \sum_{k=1}^{5} I_k.
\end{aligned} \tag{40}$$

Thanks to Lemma 2, the Hölder inequality and the Young inequality, the terms $I_k$ in (40) are respectively bounded.

$$\begin{aligned}
I_1 + I_3 \le &\frac{\nu}{12}\Delta t_n\|\nabla(D\phi_{\mathbf{u}}^{n+1})\|_0^2 + \frac{1}{\nu}C(\omega_n)\Delta t_n(\Delta t_n + \Delta t_{n-1})^3\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} \\
&\times \big(\|A\mathbf{u}(t_{n+1})\|_0^2 + \|A_h E(\tilde{\mathbf{u}}_h^n)\|_0^2\big),
\end{aligned} \tag{41}$$

as well as:

$$\begin{aligned}
I_2 + I_4 \le &\frac{\nu}{12}\Delta t_n\|\nabla(D\phi_{\mathbf{u}}^{n+1})\|_0^2 + \frac{C}{\nu}\Delta t_n(\|E(\eta_{\mathbf{u}}^n)\|_0^2 + \|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2)\big(\|A\mathbf{u}(t_{n+1})\|_0^2 \\
&+ \|A_h E(\tilde{\mathbf{u}}_h^n)\|_0^2\big),
\end{aligned} \tag{42}$$

where we have used the Poincaré inequality, (2) and (3).

Moreover, combining (38) with (39)-(42), we obtain

$$\begin{aligned}
&H_4(\phi_{\mathbf{u}}^{n+1}) + \frac{\nu}{2}\Delta t_n\|\nabla(D\phi_{\mathbf{u}}^{n+1})\|_0^2 + I_5 \\
&\le \frac{C(\omega_n)}{\nu}\Delta t_n(\Delta t_n + \Delta t_{n-1})^3\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}(\|A\mathbf{u}(t_{n+1})\|_0^2 + \|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^2) \\
&\quad + \frac{C(\omega_n)}{\nu}\Delta t_n(\Delta t_n + \Delta t_{n-1})^3\Big(\|\mathbf{u}_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} + \nu\|(\nabla\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} \\
&\quad + \nu\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\Big) + \frac{C(\omega_n)}{\nu}\|(\eta_{\mathbf{u}})_t\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} \\
&\quad + \frac{C}{\nu}\Delta t_n(\|A\mathbf{u}(t_{n+1})\|_0^2 + \|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^2)\big(\|E(\eta_{\mathbf{u}}^n)\|_0^2 + E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2\big) \\
&\quad + \frac{C}{\nu}\Delta t_n(\|\nabla\eta_{\mathbf{u}}^{n+1}\|_0^2 + \|\eta_p^{n+1}\|_0^2),
\end{aligned} \tag{43}$$

where we have applied the Young and Poincaré inequality and Lemma 2 for the other terms of (38).

Finally, by the similar argument as proof of Theorem 2, summing (43) from $n = 1$ to $N-1$ leads to

$$\begin{aligned}
&C^1\left[A_1(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right](\|\phi_{\mathbf{u}}^N\|_0^2 - \|\phi_{\mathbf{u}}^1\|_0^2) + C^2\left[A_2(\tilde{\omega}) - \frac{A_3(\tilde{\omega})}{2}\right](\|\phi_{\mathbf{u}}^{N-1}\|_0^2 - \|\phi_{\mathbf{u}}^0\|_0^2) \\
&\quad + C^3\frac{A_3(\tilde{\omega})}{2}\left[(\|\phi_{\mathbf{u}}^N\|_0 - \|\phi_{\mathbf{u}}^{N-1}\|_0)^2 - (\|\phi_{\mathbf{u}}^1\|_0 - \|\phi_{\mathbf{u}}^0\|_0)^2\right] + \frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|\nabla(D\phi_{\mathbf{u}}^{n+1})\|_0^2 + \sum_{n=1}^{N-1} I_5 \\
&\le \sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\Delta t_n(\Delta t_n + \Delta t_{n-1})^3\Big(\|\mathbf{u}_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} + \|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} \\
&\quad + \nu\|(\nabla\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} + \nu\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\Big) + \frac{C}{\nu}\sum_{n=1}^{N-1}\Delta t_n(\|\nabla\eta_{\mathbf{u}}^{n+1}\|_0^2 + \|\eta_p^{n+1}\|_0^2) \\
&\quad + \sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\|(\eta_{\mathbf{u}})_t\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)} + \frac{C}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2 + \frac{C}{\nu}\sum_{n=1}^{N-1}\Delta t_n\|E(\eta_{\mathbf{u}}^n)\|_0^2,
\end{aligned} \tag{44}$$

where we have used Theorem 1, 3 and 4.

*Step* 2. Consider the error equation on the scalar auxiliary variable. Multiplying both sides of the second equation in (37) by $\Delta t_n e_Q^{n+1}$, we have

$$\begin{aligned}
&\frac{1}{2}|\tilde{e}_Q^{n+1}|^2 - \frac{1}{2}|\tilde{e}_Q^n|^2 + \frac{1}{2}|\tilde{e}_D^{n+1}|^2 + \frac{1}{T}\Delta t_n|e_Q^{n+1}|^2 \\
&= \Delta t_n e_Q^{n+1}\exp(\frac{t_{n+1}}{T})\big(b(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1})) - b(E(\tilde{\mathbf{u}}_h^n),E(\tilde{\mathbf{u}}_h^n),\tilde{\mathbf{u}}_h^{n+1})\big) \\
&\quad + \Delta t_n e_Q^{n+1}\Big(\frac{\frac{1+2\omega_n}{1+\omega_n}Q(t_{n+1}) - (1+\omega_n)Q(t_n) + \frac{\omega_n^2}{1+\omega_n}Q(t_{n-1})}{\Delta t_n} - Q_t(t_{n+1})\Big),
\end{aligned} \tag{45}$$

where $|\tilde{e}_Q^n|^2 := |\frac{\sqrt{r_2}-\sqrt{r_2-4r_3}}{2}e_Q^n|^2 + |\sqrt{r_2}e_Q^n - \frac{\sqrt{r_2}-\sqrt{r_2-4r_3}}{2}e_Q^{n-1}|^2$ and $\tilde{e}_D^{n+1} := \frac{2r_3}{\sqrt{r_2}-\sqrt{r_2-4r_3}}e_Q^{n+1} - \sqrt{r_2}e_Q^n + \frac{\sqrt{r_2}-\sqrt{r_2-4r_3}}{2}e_Q^{n-1}$.

For the trilinear terms in (45), we rewrite them as:

$$\begin{aligned}
&\Delta t_n e_Q^{n+1}\exp(\frac{t_{n+1}}{T})\big(b(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}))-b(E(\tilde{\mathbf{u}}_h^n),E(\tilde{\mathbf{u}}_h^n),\tilde{\mathbf{u}}_h^{n+1})\big)\\
&=\Delta t_n e_Q^{n+1}\exp(\frac{t_{n+1}}{T})\Big(b(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1})-E(\mathbf{u}(t_n)),\mathbf{u}(t_{n+1}))\\
&\quad+b(\mathbf{u}(t_{n+1}),E(\mathbf{u}(t_n))-E(\tilde{\mathbf{u}}_h^n),\mathbf{u}(t_{n+1}))\\
&\quad+b(\mathbf{u}(t_{n+1})-E(\mathbf{u}(t_n)),E(\tilde{\mathbf{u}}_h^n),\mathbf{u}(t_{n+1}))+b(E(\mathbf{u}(t_n))-E(\tilde{\mathbf{u}}_h^n),E(\tilde{\mathbf{u}}_h^n),\mathbf{u}(t_{n+1}))\\
&\quad+b(E(\tilde{\mathbf{u}}_h^n),E(\tilde{\mathbf{u}}_h^n),\eta_{\mathbf{u}}^{n+1})\Big)+\Delta t_n e_Q^{n+1}\exp(\frac{t_{n+1}}{T})b(E(\tilde{\mathbf{u}}_h^n),E(\tilde{\mathbf{u}}_h^n),\tilde{\phi}_{\mathbf{u}}^{n+1})=:\sum_{k=1}^{6}J_k.
\end{aligned}\tag{46}$$

Next, based on Lemma 2, the Hölder inequality and the Young inequality, we bound $J_k$ in (46)

$$\begin{aligned}
J_1+J_3\le{}&\varepsilon_4\frac{\Delta t_n}{T}|e_Q^{n+1}|^2+C(\omega_n)\Delta t_n(\Delta t_n+\Delta t_{n-1})^3\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\|\nabla\mathbf{u}(t_{n+1})\|_0^2\\
&\times\Big(\|A\mathbf{u}(t_{n+1})\|_0^2+\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2\Big),
\end{aligned}$$

and

$$J_2+J_4\le\varepsilon_5\frac{\Delta t_n}{T}|e_Q^{n+1}|^2+C\Delta t_n\|A\mathbf{u}(t_{n+1})\|_0^2(\|E(\eta_{\mathbf{u}}^n)\|_0^2+\|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2)(\|\nabla\mathbf{u}(t_{n+1})\|_0^2+\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2),$$

as well as

$$J_5\le\varepsilon_6\frac{\Delta t_n}{T}|e_Q^{n+1}|^2+C\Delta t_n\|A_hE(\tilde{\mathbf{u}}_h^n)\|_0^2\|\nabla E(\tilde{\mathbf{u}}_h^n)\|_0^2\|\eta_{\mathbf{u}}^{n+1}\|_0^2.$$

For the last term on the right-hand side of (45), we have

$$\begin{aligned}
&\big|\Delta t_n e_Q^{n+1}\big(\frac{\frac{1+2\omega_n}{1+\omega_n}Q(t_{n+1})-(1+\omega_n)Q(t_n)+\frac{\omega_n^2}{1+\omega_n}Q(t_{n-1})}{\Delta t_n}-Q_t(t_{n+1})\big)\big|\\
&\le\varepsilon_7\frac{\Delta t_n}{T}|e_Q^{n+1}|^2+C(\omega_n)\Delta t_n(\Delta t_n+\Delta t_{n-1})^3\|Q_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}.
\end{aligned}\tag{47}$$

Finally, combining (45) and (46)-(47) with $\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7=\frac{1}{2}$, summing it from $n=1$ to $N-1$, we gain

$$\begin{aligned}
&\frac{1}{2}|\tilde{e}_Q^N|^2+\frac{1}{2}\sum_{n=1}^{N-1}|\tilde{e}_D^{n+1}|^2+\frac{1}{2}\sum_{n=1}^{N-1}\frac{\Delta t_n}{T}|e_Q^{n+1}|^2\\
&\le\sum_{n=1}^{N-1}J_6+\sum_{n=1}^{N-1}C(\omega_n)\Delta t_n(\Delta t_n+\Delta t_{n-1})^3\big(\|Q_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\big)\\
&\quad+C\sum_{n=1}^{N-1}\Delta t_n\|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2+C\sum_{n=1}^{N-1}\Delta t_n\|E(\eta_{\mathbf{u}}^n)\|_0^2+C\sum_{n=1}^{N-1}\Delta t_n\|\eta_{\mathbf{u}}^{n+1}\|_0^2+\frac{1}{2}|\tilde{e}_Q^1|^2,
\end{aligned}\tag{48}$$

where we have applied Theorem 1, 3 and 4.

*Step* 3. Combine the error equations and apply (44) and (48). Note that $I_5 = J_6$. Then, combining (44) with (48), we arrive at

$$
\begin{aligned}
&C^1\left[A_1(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}\right](\|\phi_{\mathbf{u}}^N\|_0^2-\|\phi_{\mathbf{u}}^1\|_0^2)+C^2\left[A_2(\tilde{\omega})-\frac{A_3(\tilde{\omega})}{2}\right](\|\phi_{\mathbf{u}}^{N-1}\|_0^2-\|\phi_{\mathbf{u}}^0\|_0^2)\\
&\quad+C^3\frac{A_3(\tilde{\omega})}{2}\left[(\|\phi_{\mathbf{u}}^N\|_0-\|\phi_{\mathbf{u}}^{N-1}\|_0)^2-(\|\phi_{\mathbf{u}}^1\|_0-\|\phi_{\mathbf{u}}^0\|_0)^2\right]+\frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|\nabla(D\phi_{\mathbf{u}}^{n+1})\|_0^2\\
&\quad+\frac{1}{2}|\tilde{e}_Q^N|^2+\frac{1}{2}\sum_{n=1}^{N-1}|\tilde{e}_D^{n+1}|^2+\frac{1}{2}\sum_{n=1}^{N-1}\frac{\Delta t_n}{T}|e_Q^{n+1}|^2\\
&\le\sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\Delta t_n(\Delta t_n+\Delta t_{n-1})^3\Big\{(1+\nu)\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\|\mathbf{u}_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\\
&\quad+\nu\|Q_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\nu\|(\nabla\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\nu\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\Big\}\\
&\quad+\sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\|(\eta_{\mathbf{u}})_t\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+C\sum_{n=1}^{N-1}(1+\frac{1}{\nu})\Delta t_n\|E(\eta_{\mathbf{u}}^n)\|_0^2+C\sum_{n=1}^{N-1}\Delta t_n\|\eta_{\mathbf{u}}^{n+1}\|_0^2\\
&\quad+C\sum_{n=1}^{N-1}(1+\frac{1}{\nu})\Delta t_n\|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2+\frac{C}{\nu}\sum_{n=1}^{N-1}\Delta t_n(\|\nabla\eta_{\mathbf{u}}^{n+1}\|_0^2+\|\eta_p^{n+1}\|_0^2)+\frac{1}{2}|\tilde{e}_Q^1|^2.
\end{aligned}
$$

Moreover, rearranging the above inequality and utilizing recursive method yield

$$
\begin{aligned}
&\|\phi_{\mathbf{u}}^N\|_0^2+C^3\frac{A_3(\tilde{\omega})}{2\Pi}(\|\phi_{\mathbf{u}}^N\|_0-\|\phi_{\mathbf{u}}^{N-1}\|_0)^2\\
&\quad+\frac{1}{\Pi}\Big\{\frac{1}{2}\sum_{n=1}^{N-1}\nu\Delta t_n\|\nabla(D\phi_{\mathbf{u}}^{n+1})\|_0^2+\frac{1}{2}|\tilde{e}_Q^N|^2+\frac{1}{2}\sum_{n=1}^{N-1}|\tilde{e}_D^{n+1}|^2+\frac{1}{2}\sum_{n=1}^{N-1}\frac{\Delta t_n}{T}|e_Q^{n+1}|^2\Big\}\\
&\le\frac{1}{\Pi}\Bigg\{\sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\Delta t_n(\Delta t_n+\Delta t_{n-1})^3\Big\{(1+\nu)\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\|\mathbf{u}_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\\
&\quad+\nu\|Q_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\nu\|(\nabla\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\nu\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\Big\}\\
&\quad+\sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\|(\eta_{\mathbf{u}})_t\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+C\sum_{n=1}^{N-1}(1+\frac{1}{\nu})\Delta t_n\|E(\eta_{\mathbf{u}}^n)\|_0^2+C\sum_{n=1}^{N-1}\Delta t_n\|\eta_{\mathbf{u}}^{n+1}\|_0^2\\
&\quad+C\sum_{n=1}^{N-1}(1+\frac{1}{\nu})\Delta t_n\|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2+\frac{C}{\nu}\sum_{n=1}^{N-1}\Delta t_n(\|\nabla\eta_{\mathbf{u}}^{n+1}\|_0^2+\|\eta_p^{n+1}\|_0^2)+\frac{1}{2}|\tilde{e}_Q^1|^2\\
&\quad+C^3\frac{A_3(\tilde{\omega})}{2}(\|\phi_{\mathbf{u}}^1\|_0-\|\phi_{\mathbf{u}}^0\|_0)^2\Bigg\}+\|\phi_{\mathbf{u}}^1\|_0^2+\alpha\|\phi_{\mathbf{u}}^{N-1}\|_0^2\\
&\le\frac{1}{\Pi(1-\alpha)}\Bigg\{\sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3\Big\{(1+\nu)\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\\
&\quad+\|\mathbf{u}_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\nu\|Q_{ttt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}+\nu\|(\nabla\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\\
&\quad+\nu\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}\Big\}+\sum_{n=1}^{N-1}\frac{C(\omega_n)}{\nu}\|(\eta_{\mathbf{u}})_t\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega)^2)}
\end{aligned}
$$

$$+C\sum_{n=1}^{N-1}(1+\frac{1}{\nu})\Delta t_n\|E(\eta_{\mathbf{u}}^n)\|_0^2+C\sum_{n=1}^{N-1}\Delta t_n\|\eta_{\mathbf{u}}^{n+1}\|_0^2+C\sum_{n=1}^{N-1}(1+\frac{1}{\nu})\Delta t_n\|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2$$
$$+\frac{C}{\nu}\sum_{n=1}^{N-1}\Delta t_n(\|\nabla\eta_{\mathbf{u}}^{n+1}\|_0^2+\|\eta_p^{n+1}\|_0^2)+\frac{1}{2}|\tilde{e}_Q^1|^2+\Pi\|\phi_{\mathbf{u}}^1\|_0^2+C^3\frac{A_3(\tilde{\omega})}{2}(\|\phi_{\mathbf{u}}^1\|_0-\|\phi_{\mathbf{u}}^0\|_0)^2\Big\}$$
$$+\alpha^{N-1}\|\phi_{\mathbf{u}}^1\|_0^2.$$

In addition, using (36), we have

$$C\sum_{n=1}^{N-1}(1+\frac{1}{\nu})\Delta t_n\|E(\tilde{\phi}_{\mathbf{u}}^n)\|_0^2\le C\sum_{n=1}^{N-1}\Delta t_n(\|\tilde{\phi}_{\mathbf{u}}^n\|_0^2+\|\tilde{\phi}_{\mathbf{u}}^{n-1}\|_0^2)$$
$$\le C\sum_{n=1}^{N-1}\Delta t_n(\|D\phi_{\mathbf{u}}^n\|_0^2+\|D\phi_{\mathbf{u}}^{n-1}\|_0^2)+C\sum_{n=1}^{N-1}\Delta t_n(\Delta t_{n-1}+\Delta t_{n-2})^3\Big(\|(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-2},t_n;L^2(\Omega)^2)}$$
$$+\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-2},t_n;L^2(\Omega)^2)}\Big)+C\sum_{n=1}^{N-1}\Delta t_n(\Delta t_{n-2}+\Delta t_{n-3})^3\Big(\|(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t_{n-3},t_{n-1};L^2(\Omega)^2)}$$
$$+\|\mathbf{u}_{tt}\|^2_{L^2(t_{n-3},t_{n-1};L^2(\Omega)^2)}\Big).$$

Finally, by using the discrete Grönwall lemma, the property of the finite element interpolation approximation and the triangle inequality, we end the proof. □

## 4. Numerical experiments

In this section, we perform several numerical experiments to test the effectiveness of the proposed method (9)-(10). In first example, we check the energy dissipation of the variable time filter approach. The second test confirms the convergence rate predicted in Theorem 5 for an analytical test problem with a known solution. In the final test, we investigate a classical vortex shedding problem to demonstrate the effectiveness of the proposed method. All simulations are carried out with the Taylor-Hood finite element for velocity and pressure on conforming triangular grid.

4.1. **Stability.** In this subsection, we will test the stability of the presented numerical method. Let $\Omega=[0,\,1]\times[0,\,1]$. Set homogeneous Dirichlet boundary condition, and zero forcing in this example. Choose initial values to ensure that the divergence-free condition is satisfied:

$$u_1=\sin(2\pi y)\sin^2(\pi x),\ u_2=\sin(2\pi x)\cos^2(\pi y).$$

Give the time step ratio $\omega=\frac{10}{13}$. The initial time step is determined by $\Delta t_0=\frac{1-1.3}{1-1.3^N}$. For the time step at the case $N=16$, we divide the time step size sequence $\Delta t_n$ into two halves: $\Delta t_0,\Delta t_1,\cdots,\Delta t_7$ and $\Delta t_8,\Delta t_9,\cdots,\Delta t_{15}$. In the first half, the time step $\Delta t_n$ is chosen to be $\frac{1}{\omega}$ times the previous time step size, i.e., $\Delta t_1=\frac{1}{\omega}\Delta t_0,\Delta t_2=\frac{1}{\omega}\Delta t_1,\cdots,\Delta t_7=\frac{1}{\omega}\Delta t_6$. In the second half, $\Delta t_n$ is chosen to be $\omega$ times the previous time step size, i.e., $\Delta t_9=\omega\Delta t_8,\Delta t_{10}=\omega\Delta t_9,\cdots,\Delta t_{15}=\omega\Delta t_{14}$. Additionally, we set $\Delta t_8=\Delta t_7$.

Fix the mesh size $h=\frac{1}{64}$ and set the final time $T=5.0$. In Fig.1, we can see that all energy curves obtained by the presented method (9)-(10) always show monotonic decays, which illustrates its energy stability.

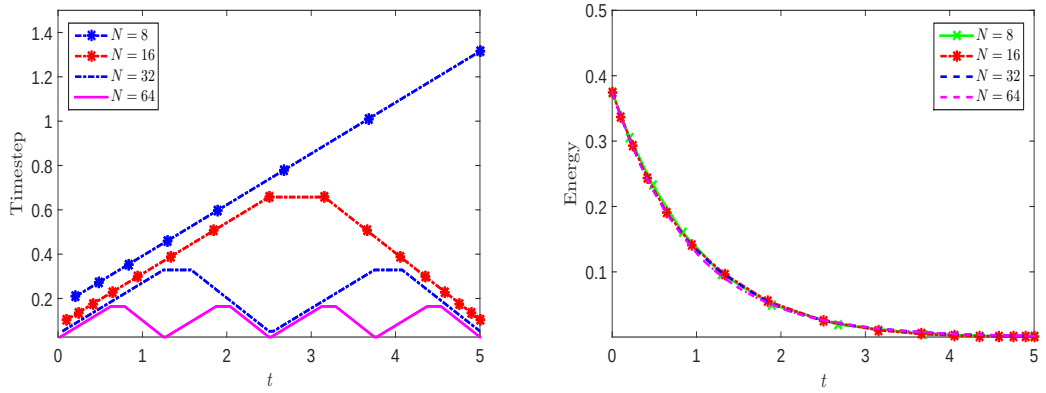



FIGURE 1. (a) The strategy of the variable time step; (b) Time evolution of the energy by the proposed method.

4.2. **Convergence rate.** In this experiment, firstly, we use the constant time step to test the second-order convergence of the proposed method. Let the computational domain $\Omega = [0, 1]\times[0, 1]$. The exact solution of the problem (1) is presented as follows

$$\begin{aligned} u_1(t,x,y) &= \nu \exp(-2t)x^2(x-1)^2y(y-1)(2y-1), \\ u_2(t,x,y) &= -\nu \exp(-2t)x(x-1)(2x-1)y^2(y-1)^2, \\ p(t,x,y) &= \exp(-2t)(2x-1)(2y-1). \end{aligned}$$

The right-hand side $\mathbf{f} = (f_1(t,x,y), f_2(t,x,y))$ of (1) is obliged to satisfy that $(\mathbf{u}, p)$ is the solution of the original problem.

Set the parameter $\nu = 0.1$. The space mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and $\frac{1}{64}$ are picked to test the convergence rate with respect to $h$. Here, we choose $\Delta t_n = 0.001$. Additionally, to confirm the convergence rate of $\Delta t_n$, the mesh size is selected as $h = \frac{1}{120}$ with $T = 1.0$. Temporal and Spatial errors and convergence rates are listed in Table 1 and Table 2, respectively. It can be seen from these tables that the velocity and pressure of the numerical method (9)-(10) achieve the optimal rates for a constant time step.

TABLE 1. The errors and convergence order for the proposed method with varying $\Delta t_n$ and fixed $h = \frac{1}{120}$ at $T = 1.0$.

| $\Delta t_n$ | $\|\mathbf{u}(t_N) - \mathbf{u}_h^N\|_0$ | Rate | $\|\nabla(\mathbf{u}(t_N) - \mathbf{u}_h^N)\|_0$ | Rate | $\|p(t_N) - p_h^N\|_0$ | Rate |
|---|---|---|---|---|---|---|
| 1/4 | 2.1242E−5 | — | 1.5573E−4 | — | 1.5195E−2 | — |
| 1/8 | 4.2584E−6 | 2.32 | 3.0950E−5 | 2.33 | 1.5681E−3 | 3.28 |
| 1/16 | 8.9410E−7 | 2.25 | 6.5072E−6 | 2.25 | 3.9657E−4 | 1.98 |
| 1/32 | 2.1077E−7 | 2.08 | 1.5360E−6 | 2.08 | 9.3659E−5 | 2.08 |
| 1/64 | 5.1291E−8 | 2.04 | 3.8158E−7 | 2.01 | 2.2846E−5 | 2.04 |

Secondly, we test the convergence of the presented method in the case of variable time step. Here, we refer to time adaptive algorithm in [25] and give the presented method (9)-(10) based on this time adaptive algorithm.

**Algotirhm 1.** ***Step*** *I: Given initial guesses and the tolerance TOL.*
***Step*** *II: Solve (9) and (10) to get* $\mathbf{u}_h^{n+1}$ *and* $Q^{n+1}$. *Repeat the following step for* $n = 1, 2, \ldots, N-1$.

TABLE 2. The errors and convergence order for the proposed method with varying $h$ and fixed $\Delta t_n = 0.001$ at $T = 0.1$.

| $h$ | $\|\mathbf{u}(t_N) - \mathbf{u}_h^N\|_0$ | Rate | $\|\nabla(\mathbf{u}(t_N) - \mathbf{u}_h^N)\|_0$ | Rate | $\|p(t_N) - p_h^N\|_0$ | Rate |
|---|---|---|---|---|---|---|
| 1/4 | 1.1744E−5 | — | 3.8906E−4 | — | 1.3212E−2 | — |
| 1/8 | 1.5221E−6 | 2.95 | 1.0438E−4 | 1.90 | 3.3030E−3 | 2.00 |
| 1/16 | 1.9089E−7 | 3.00 | 2.6715E−5 | 1.97 | 8.2576E−4 | 2.00 |
| 1/32 | 2.3950E−8 | 2.99 | 6.7251E−6 | 1.99 | 2.0644E−4 | 2.00 |
| 1/64 | 3.0986E−9 | 2.95 | 1.6845E−6 | 2.00 | 5.1613E−5 | 2.00 |

***Step** III: Adapt the time step $\Delta t_n$. Compute the estimators $EST$ and $EST_Q$ and check whether the tolerance $TOL$ is satisfied. The estimators are defined by*

$$EST^{n+1} = \frac{\omega_{n-1}\omega_n(1+\omega_n)}{1+2\omega_n+\omega_{n-1}(1+4\omega_n+3\omega_n^2)}\|\mathbf{D}^{n+1}\|_0,$$

$$EST_Q^{n+1} = \frac{\omega_{n-1}\omega_n(1+\omega_n)}{1+2\omega_n+\omega_{n-1}(1+4\omega_n+3\omega_n^2)}|D_Q^{n+1}|,$$

*where* $\mathbf{D}^{n+1} = \mathbf{u}_h^{n+1} - \frac{(1+\omega_n)(1+\omega_{n-1}(1+\omega_n))}{1+\omega_{n-1}}\mathbf{u}_h^n + \omega_n(1+\omega_{n-1}(1+\omega_n))\mathbf{u}_h^{n-1} - \frac{\omega_{n-1}^2\omega_n(1+\omega_n)}{1+\omega_{n-1}}\mathbf{u}_h^{n-2}$, *and* $D_Q^{n+1} = Q^{n+1} - \frac{(1+\omega_n)(1+\omega_{n-1}(1+\omega_n))}{1+\omega_{n-1}}Q^n + \omega_n(1+\omega_{n-1}(1+\omega_n))Q^{n-1} - \frac{\omega_{n-1}^2\omega_n(1+\omega_n)}{1+\omega_{n-1}}Q^{n-2}$. *If* $\max\{EST^{n+1}, EST_Q^{n+1}\} < \frac{TOL}{3}$, *then* $\Delta t_{n+1} = 0.9\Delta t_n \min\left\{2, (\frac{TOL}{EST^{n+1}})^{\frac{1}{3}}, (\frac{TOL}{EST_Q^{n+1}})^{\frac{1}{3}}\right\}$; *Else if* $\frac{TOL}{3} \le \max\{EST^{n+1}, EST_Q^{n+1}\} \le TOL$, *then* $\Delta t_{n+1} = 0.9\Delta t_n \min\left\{1, (\frac{TOL}{EST^{n+1}})^{\frac{1}{3}}, (\frac{TOL}{EST_Q^{n+1}})^{\frac{1}{3}}\right\}$; *Otherwise repeat Step II after reset* $\Delta t_n = 0.7\Delta t_n \min\left\{(\frac{TOL}{EST^{n+1}})^{\frac{1}{3}}, (\frac{TOL}{EST_Q^{n+1}})^{\frac{1}{3}}\right\}$.

Then, set $\nu = 0.5$ and $T = 1.0$. We provide the errors of the numerical solutions with the fixed mesh size $h = \frac{1}{100}$ and different tolerance errors $7.0E-4 \le TOL \le 5.0E-3$. For the sake of simplicity, we denote the errors

$$Err(\mathbf{u}) = \Big(\sum_{n=1}^{N-1}\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_0^2\Big)^{\frac{1}{2}}, \quad Err(p) = \Big(\sum_{n=1}^{N-1}\Delta t_n\|(p(t_n) - p_h^n)\|_0^2\Big)^{\frac{1}{2}}.$$

Since the time step is variable, the average time step $\overline{\Delta t}$ is considered for convergence rate. The corresponding errors and convergence rates produced by Algorithm 1 are shown in Fig.2, and it can be clearly found that the convergence rates are consistent with the results from theoretical analysis.

4.3. **Flow past a cylinder.** Now, we use the Benchmark problem of flow around a cylinder, originally proposed in [32], to test the performance of the time adaptive algorithm. This problem has also been used as a benchmark in [5,7,26]. The computational domain follows the paper [32],

$$\Omega = \{(x,y)|0 < x < 2.2,\ 0 < y < 0.41 \text{ and } (x-0.2)^2 + (y-0.2)^2 > 0.05^2\},$$

i.e., a channel with a cylindrical cutout. The time dependent inflow and outflow velocity profiles are given by:

$$\mathbf{u}_1(0,y,t) = \mathbf{u}_1(2.2,y,t) = \frac{6}{0.41^2}\sin(\frac{\pi t}{8})y(0.41-y), \quad \mathbf{u}_2(0,y,t) = \mathbf{u}_2(2.2,y,t) = 0.$$

No-slip boundary conditions are enforced at the cylinder and walls. The initial condition is $\mathbf{u}_h^0 = 0$. Since the method can not be self-starting, we generate the second and third initial values $\mathbf{u}_h^1, \mathbf{u}_h^2$, by taking two steps backward-Euler with $\Delta t = \frac{1}{256}$ before switching to Algorithm 1. Moreover, there is no external force acting on the flow $\mathbf{f} \equiv \mathbf{0}$.

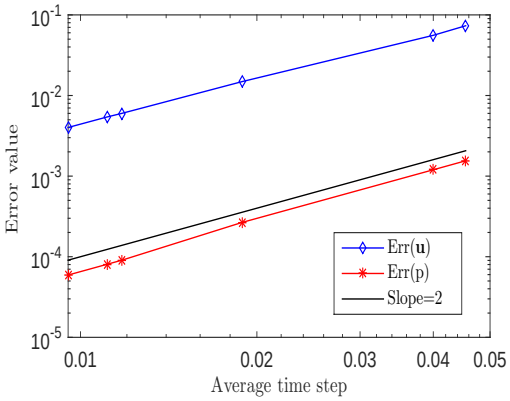



FIGURE 2. Errors and convergence rates with respect to $\overline{\Delta t}$.

Apply a spatial discretization with 10676 degrees of freedom with 2739 vertices on the boundary of the cylinder. Firstly, we tested three different viscosities of $\frac{1}{600}$, $\frac{1}{800}$ and $\frac{1}{1000}$ over a time interval of 6. Fig.3 shows snapshot of the velocity at different viscosities with $TOL = 5.0E-3$ and $5.0E-5$. From this figure, we can see that the algorithm with $TOL = 5.0E-5$ can capture vortex street well.

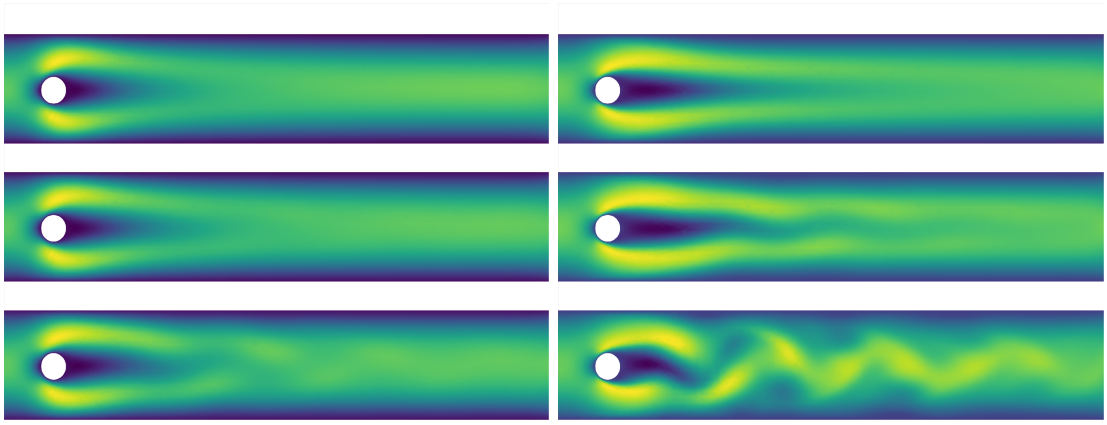


FIGURE 3. Velocity at time t = 6.0 for $\nu = \frac{1}{600}$ (top), $\frac{1}{800}$ and $\frac{1}{1000}$ (bottom) generated by Algorithm 1 with (a) $TOL = 5.0E-3$ and (b) $TOL = 5.0E-5$.

The evolution of the time step is shown in Fig.4. We can see that as the tolerance error $TOL$ decreases, the time step gradually tends a stable value. Besides, we notice that there is a severe oscillation at roughly $t = 1$ before the fluid collides with the cylinder. Further, for the small tolerance, the time step will be approximately 0.001, and for the small viscosity, it will take little time for the time step to reach a stable value.

## 5. Conclusion

In this paper, a variable time filter approach for the incompressible Navier-Stokes equations based on a scalar auxiliary variable. Compared with the work in [10], the main work of this paper is to theoretically prove that the numerical accuracy of the variable time filter is improved from the first order to the second order without increasing the computational complexity. Stability and convergence results are provided, which have been tested by a series of numerical

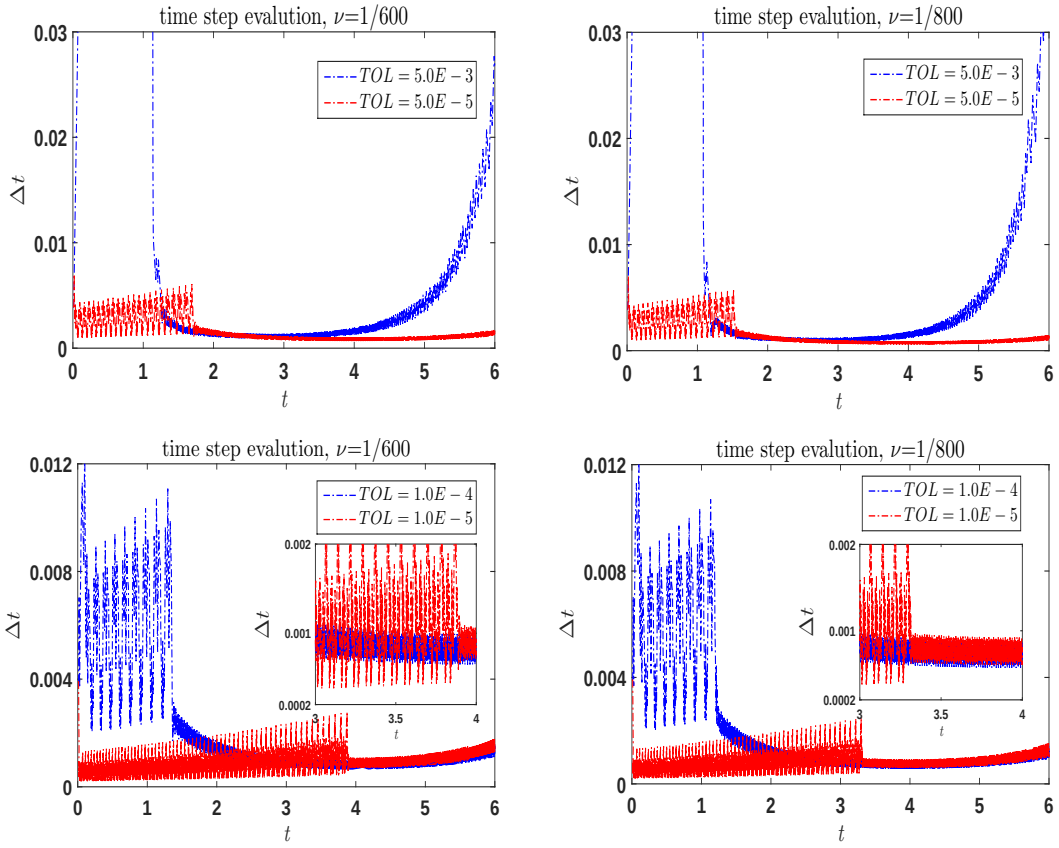



FIGURE 4. The time step evolution of Algorithm 1 under different values of $TOL$ and $\nu$.

experiments. All computational results support the theoretical analysis and demonstrate the effectiveness of the proposed approach.

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