Appl. Comput. Math., Vol.24, No.3, 2025, pp.486-509
DOI: 10.30546/1683-6154.24.3.2025.486

ANALYSIS OF STABILITY AND CONVERGENCE OF A MODIFIED THREE-STEP SPLITTING SCHEME FOR NAVIER-STOKES EQUATIONS

B. ZHUMAGULOV[1], N. TEMIRBEKOV[1,2], D. TAMABAY[1,2,*]

Abstract. The article presents numerical solution for the Navier-Stokes equations and the atmospheric boundary layer equations using a modified difference splitting scheme. It examines the widely used the Marker-and-Cell (MAC) method and evaluates its effectiveness. The study investigates stability and convergence within the "large particles" method, emphasizing a three-step splitting scheme. Research provides a priori estimates for the second derivative of velocity and pressure gradient. Proposed method computes auxiliary velocity fields at boundary nodes and solves a pressure equation, differing from techniques using Neumann conditions, which may not reflect the physical problem. Numerical simulations are performed for convective flow in a rectangular cavity. A numerical comparison is conducted between the classical two-step splitting method and the modified three-step scheme. Method's applicability is shown through modeling pollutant dispersion in the atmosphere of an industrial city.

Keywords: Navier-Stokes Equations, "Large Particles" Method, Splitting, Physical Processes, Stability and Convergence, Atmospheric Boundary Layer Model.

AMS Subject Classification: 65M06, 65M12, 65M22, 76D05.

1. Introduction

The atmospheric boundary layer is governed by the Navier-Stokes (N-S) equations, which describe the dynamics of incompressible fluid flows. These equations form the mathematical foundation for many commercial and open-source software packages used in the numerical modeling of atmospheric and environmental processes. Over the past several decades, a variety of numerical methods have been developed to solve the N-S equations, including finite difference methods (FDM), the finite element method (FEM), and the finite volume method (FVM). Among them, the finite difference method on staggered (spaced) grids commonly known as the MAC method stands out for its simplicity and efficiency in handling incompressible flows.

Originally proposed in the 1960s, the staggered-grid FDM was designed as a stable and robust scheme for incompressible fluid dynamics and has since been widely applied in engineering problems [10, 14, 36]. The mathematical convergence properties of this method have been rigorously analyzed in a number of works [12, 15, 29] and references therein, demonstrating its applicability to a broad range of incompressible flow models.

Specifically, in [19] it is proved that $O(h)$ convergence for both vorticity and pressure, while Samelson et al. established higher-order convergence rates: $O(h^2)$ for velocity in the PE formulation [24] and $O(h^4)$ for temperature in the PGE formulation [23]. Further development includes convergence estimates for fully discrete schemes using time discretizations of first and

[1]National Engineering Academy of the Republic of Kazakhstan, Bogenbai Batyr Str. 80, 050010, Almaty, Kazakhstan
[2]Faculty of Mechanics and Mathematics, Al-Farabi Kazakh National University, Al-Farabi Ave., 71, 050040, Almaty, Kazakhstan, e-mail: temirbekov@rambler.ru
*Corresponding author e-mail: dtamabay@gmail.com
Manuscript received 22 November 2024.

second order, combined with explicit treatment of nonlinear convective terms [11,35]. Superconvergence results on non-uniform rectangular grids for staggered-grid FDM applied to the N-S equations were subsequently reported in works [15, 21].

Moreover, it has been shown in [18] that certain finite difference schemes can be interpreted as discrete Galerkin systems. The MAC scheme has also been analyzed within the frameworks of finite element and discontinuous Galerkin methods [12, 19, 28].

Despite significant progress, relatively few studies have addressed the unconditional stability of fully discrete schemes for the N-S equations using both time and space discretization on staggered grids [22, 26]. A review article [5] discusses three categories of schemes: A linear implicit scheme, a pressure-correction scheme, and a pressure-stabilization scheme all using staggered grids and Neumann boundary conditions for pressure [4, 9, 25, 37].

In a broader context, the review [6] surveys 157 sources related to essentially non-oscillatory finite difference schemes (WENO), while ENO schemes have been applied to biomedical modeling, particularly in simulations of arterial diseases [16, 17].

The study [20] models blood flow using the N-S equations with Dirichlet boundary conditions, focusing on coupling 1D and 3D representations in the circulatory system. In [1], the authors investigate the regularity of weak solutions for micropolar fluid flows, employing the Littlewood-Paley decomposition and energy methods to enhance known regularity results.

Additional progress includes the use of the finite element method to solve problems with slip boundary conditions, where the stability of the numerical method is proved through variational inequalities [35]. Furthermore, effective algorithms have been proposed for fictitious domain methods in complex geometries, utilizing staggered grid difference schemes and optimizing boundary conditions on physical boundaries [31, 32].

In [38], the authors present improved defect-correction algorithms for the stationary N-S equations under small-viscosity conditions. Their finite element-based methods incorporate grad-div stabilization and error correction techniques, yielding faster convergence and robust error control, supported by a comprehensive stability analysis and numerical validation.

As is known, in the main version of the MAC method [10, 37], as a result of the divergence operation, the Poisson equation for determining pressure with Neumann boundary conditions is obtained, which is associated with computational difficulties.

In [27], the equation

$$\overrightarrow{u}^{n+1} + \tau \nabla_h P^{n+1} = \overrightarrow{u}^{n+1/2},$$
$$div_h \overrightarrow{u}^{n+1} = 0$$

by introducing a new variable ψ-the current function at the nodes $(x_{1,i+1/2}, x_{2,j+1/2})$ is reduced to the equation for the current function

$$\Delta_h \psi^{n+1} = rot_h \overrightarrow{u}_{n+1/2},$$

where $rot_h \overrightarrow{u}_{n+1/2} = \left(u^{n+1/2}_{i+1/2,j}\right)_{x_2} - \left(v^{n+1/2}_{i,j+1/2}\right)_{x_1}$.

This approach avoids solving the Neumann problem for pressure.

In [7], two-dimensional N-S equations in alternating current function, velocity vortex are considered as a system of operator-difference equations of the following form

$$J(\omega,\psi) + A\omega + B\psi = f,$$
$$A\psi = \omega,$$

where the operators A, B and the nonlinear form $J(\omega,\psi)$ are considered to be set over the entire space of H.

The following iterative process is constructed for the numerical implementation of the solution of the above system of equations:

$$\frac{\omega^{n+1} - \omega^n}{\tau} + J\left(\omega^n, \psi^{n+\frac{1}{2}}\right) + A\omega^n + B\psi^{n+\frac{1}{2}} = f,$$

$$A\psi^{n+1/2}=\omega^{n+1/2},$$
$$\frac{\omega^{n+1}-\omega^{n+1/2}}{\tau}+A\left(\omega^{n+1}-\omega^{n}\right)=0,$$
$$A\psi^{n+1}=\omega^{n+1}.$$

For the first time, for the N-S difference equations in the variables current function, velocity vortex, an estimate of the stability of solutions according to the initial conditions and the right side is obtained.

The study presented in [8] highlights iterative schemes characterized by three parameters and used to solve the Navier-Stokes grid equations. Within the framework of this work, we analyze three-parameter iterative methods designed to effectively solve equations describing the flow of a viscous incompressible fluid.

This paper explores stability and convergence issues of numerical solutions using "large particles" methods for the Navier-Stokes equations, which are part of the system of equations for the atmospheric boundary layer.

A three-step scheme of splitting by physical processes is considered. The difference from the classical splitting scheme is the use of implicit difference schemes in the first and second steps. This approach is effective for numerical implementation and obtaining a priori estimates of the second derivative of the velocity vector and pressure gradient. This assessment ensures the stability of the three-stage difference scheme.

In many works, the Neumann condition is set for the numerical solution of the elliptic equation for pressure by the iterative method, which does not quite correctly describe the physical formulation of the problem. In this work, the definition of auxiliary velocity fields in boundary nodes and the equation for pressure are used.

Numerical calculations for the convective flow test problem in a rectangular cavity have been conducted.

A series of numerical experiments was conducted to compare the classical two-stage splitting scheme with the proposed three-stage splitting scheme based on physical processes. The obtained results are presented and discussed, demonstrating the effectiveness of the modified approach.

A three-step splitting scheme is used to simulate the spread of harmful impurities in the atmospheric air of the industrial city of Ust-Kamenogorsk.

2. Formulation of the initial boundary value problem

Suppose that the area $\Omega \in R^2$ is a unit square. In the domain of Ω, consider the system of N-S equations

$$\begin{gathered}\frac{\partial \overrightarrow{V}}{\partial t}+(\overrightarrow{V}\cdot\nabla)\overrightarrow{V}+\nabla p=\nu\Delta\overrightarrow{V}+f,\\ div\,\overrightarrow{V}=0,\end{gathered}\tag{1}$$

here $\overrightarrow{V}$ is velocity vector field with u, v components, t is time variable, $(\overrightarrow{V}\cdot\nabla)\overrightarrow{V}$ is convective term, ∇p is pressure gradient force, $\nu\Delta\overrightarrow{V}$ is viscous term, where ν is the kinematic viscosity, f is external force acting on the fluid. System of equations is considered with the following initial and boundary conditions and condition for unambiguous determination of pressure

$$\begin{gathered}\overrightarrow{V}(x_1,x_2,0)=\overrightarrow{V}_0(x_1,x_2),\\ \overrightarrow{V}\,|_{\partial\Omega}=0,\\ \int\limits_{\Omega} p dx=0.\end{gathered}\tag{2}$$

2.1. **Construction of finite difference schemes.** Approximation of the problem (1)-(2) on a chess grid $\Omega_h = \Omega_{0h} \cup \Omega_{1h} \cup \Omega_{2h}$,

$$\begin{gathered}\Omega_{0h} = \{(x_{1i}, x_{2j}) \mid i = 1, ..., N_1, j = 1, ..., N_2; \\ x_{1i} = ih_1, x_{2j} = jh_2, h_1 = 1/N_1, h_2 = 1/N_2\},\end{gathered}$$

$$\begin{gathered}\Omega_{1h} = \{(x_{1i+1/2}, x_{2j}) \mid i = 0, ..., N_1, j = 1, ..., N_2; \\ x_{1i+1/2} = (i + 1/2)h_1, x_{2j} = jh_2, h_1 = 1/N_1, h_2 = 1/N_2\},\end{gathered} \tag{3}$$

$$\begin{gathered}\Omega_{2h} = \{(x_{1i}, x_{2j+1/2}) \mid i = 1, ..., N_1, j = 0, ..., N_2; \\ x_{1i} = ih_1, x_{2j+1/2} = (j + 1/2)h_2, h_1 = 1/N_1, h_2 = 1/N_2\}.\end{gathered}$$

As in the monograph [1], the components of the velocity vector $\overrightarrow{V}$ are defined at the nodes of the grid Ω_{1h} and Ω_{2h}, respectively, i.e. $u_{i+1/2,j} \in \Omega_{1h}$,$v_{i,j+1/2} \in \Omega_{2h}$ the pressure is defined in Ω_{0h}. The differential equations (1) are approximated by the following difference scheme using the integro-interpolation method. The first equation is integrated by x_1 from $x_{1,i}$ to $x_{1,i+1}$; by x_2 from $x_{2,j-1/2}$ to $x_{2,j+1/2}$, the second equation is similar by x_1 from $x_{1,i-1/2}$ to $x_{1,i+1/2}$; by x_2 from $x_{2,j}$ to $x_{2,j+1}$, and the third equation by x_1 from $x_{1,i-1/2}$ to $x_{1,i+1/2}$; by x_2 from $x_{2,j-1/2}$ to $x_{2,j+1/2}$.

$$\begin{aligned}&\frac{u^{n+1}_{i+1/2,j} - u^n_{i+1/2,j}}{\tau} + \frac{1}{4h_1}[(u^n_{i+3/2,j} + u^n_{i+1/2,j})^2 - (u^n_{i+1/2,j} + u^n_{i-1/2,j})^2] \\ &+ \frac{1}{4h_2}[(u^n_{i+1/2,j+1} + u^n_{i+1/2,j})(v^n_{i+1,j+1/2} + v^n_{i,j+1/2}) \\ &- (u^n_{i+1/2,j} + u^n_{i+1/2,j-1})(v^n_{i+1,j-1/2} + v^n_{i,j-1/2})] + \frac{P^{n+1}_{i+1,j} - P^{n+1}_{i,j}}{h_1} \\ &= \nu(\frac{u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{h_1^2} + \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{h_2^2}) + f^{(1)}_{i,j},\end{aligned} \tag{4}$$

for $i = 1, ..., N_1 - 1, j = 2, ..., N_2 - 1$,

$$\begin{aligned}&\frac{v^{n+1}_{i,j+1/2} - v^n_{i,j+1/2}}{\tau} + \frac{1}{4h_1}[(u^n_{i+1/2,j+1} + u^n_{i+1/2,j})(v^n_{i+1,j+1/2} + v^n_{i,j+1/2}) - (u^n_{i-1/2,j+1} \\ &+ u^n_{i-1/2,j})(v^n_{i,j+1/2} + v^n_{i-1,j+1/2})] \\ &+ \frac{1}{4h_2}[(v^n_{i,j+3/2} + v^n_{i,j+1/2})^2 - (v^n_{i,j+1/2} + v^n_{i,j-1/2})^2] + \frac{P^{n+1}_{i,j+1} - P^{n+1}_{i,j}}{h_2} \\ &= \nu(\frac{v^n_{i,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i,j-1/2}}{h_2^2} + \frac{v^n_{i+1,j+1/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j+1/2}}{h_1^2}) + f^{(2)}_{i,j},\end{aligned} \tag{5}$$

for $i = 2, ..., N_1 - 1, j = 1, ..., N_2 - 1$,

$$\frac{u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}}{h_1} + \frac{v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}}{h_2} = 0. \tag{6}$$

For the numerical implementation of the difference scheme (4)-(6), we use a splitting scheme for physical processes. The first half step is implicit:

$$\frac{u^{n+1/2}_{i+1/2,j} - u^n_{i+1/2,j}}{\tau} + L_{1h}(u^n, v^n) + (P^n_{ij})_{x_1} = \nu((u^{n+1/2}_{i+1/2,j})_{x_1\overline{x}_1} + (u^n_{i+1/2,j})_{x_2\overline{x}_2}) + f^{(1)}_{i+1/2,j},$$

for $i = 1, ..., N_1 - 1, j = 2, ..., N_2 - 1$,

$$\frac{v^{n+1/2}_{i,j+1/2} - v^n_{i,j+1/2}}{\tau} + L_{2h}(u^n, v^n) + (P^n_{ij})_{x_2} = \nu((v^{n+1/2}_{i,j+1/2})_{x_1\overline{x}_1} + v^n_{i,j+1/2})_{x_2\overline{x}_2}) + f^{(2)}_{i,j+1/2} \tag{7}$$

for $i = 2, ..., N_1 - 1, j = 1, ..., N_2 - 1$, here $(P^n_{i,j})_{x_1} = \frac{P^n_{i+1,j} - P^n_{i,j}}{h_1}$ and $(P^n_{i,j})_{x_2} = \frac{P^n_{i,j+1} - P^n_{i,j}}{h_2}$.

In the second half step, following equations are solved:

$$\frac{\overline{u}^{n+1/2}_{i+1/2,j} - u^{n+1/2}_{i+1/2,j}}{\tau} = \nu\big(\overline{u}^{n+1/2}_{i+1/2,j}\big)_{x_2\overline{x}_2} - \nu\big(u^{n}_{i+1/2,j}\big)_{x_2,\overline{x}_2},$$

for $i = 1, ..., N_1 - 1, j = 2, ..., N_2 - 1$,

$$\frac{\overline{v}^{n+1/2}_{i,j+1/2} - v^{n+1/2}_{i,j+1/2}}{\tau} = \nu\big(\overline{v}^{n+1/2}_{i,j+1/2}\big)_{x_2\overline{x}_2} - \nu\big(v^{n}_{i,j+1/2}\big)_{x_2,\overline{x}_2}, \tag{8}$$

for $i = 2, ..., N_1 - 1, j = 1, ..., N_2 - 1$.

In the third half-step, following equations are solved:

$$\frac{u^{n+1}_{i+1/2,j} - \overline{u}^{n+1/2}_{i+1/2,j}}{\tau} + \big(P^{n+1}_{ij} - P^{n}_{ij}\big)_{x_1} = 0, \text{for } i = 1, ..., N_1 - 1, j = 2, ..., N_2 - 1,$$

$$\frac{v^{n+1}_{i,j+1/2} - \overline{v}^{n+1/2}_{i,j+1/2}}{\tau} + \big(P^{n+1}_{ij} - P^{n}_{ij}\big)_{x_2} = 0, \text{for } i = 2, ..., N_1 - 1, j = 1, ..., N_2 - 1, \tag{9}$$

$$div_h V^{n+1} = \frac{u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}}{h_1} + \frac{v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}}{h_2} = 0. \tag{10}$$

The splitting scheme (7)-(10) is closed by approximating the boundary conditions (2). Boundary conditions are set at the following lower boundary

$$u^{n+1}_{i+1/2,1/2} = 0, v^{n+1}_{i,1/2} = 0, i = 1, ..., N_1. \tag{11}$$

On the left boundary, the discrete boundary conditions have the form

$$u^{n+1}_{1/2,j} = 0, v^{n+1}_{1/2,j+1/2} = 0, j = 1, ..., N_2. \tag{12}$$

At the upper boundary of the square is given by:

$$u^{n+1}_{i+1/2,N_2+1/2} = 0, v^{n+1}_{i,N_2+1/2} = 0, i = 1, ..., N_1. \tag{13}$$

On the right boundary, the boundary conditions have the form

$$u^{n+1}_{N_1+1/2,j} = 0, v^{n+1}_{N_1+1/2,j+1/2} = 0, i = 1, ..., N_2. \tag{14}$$

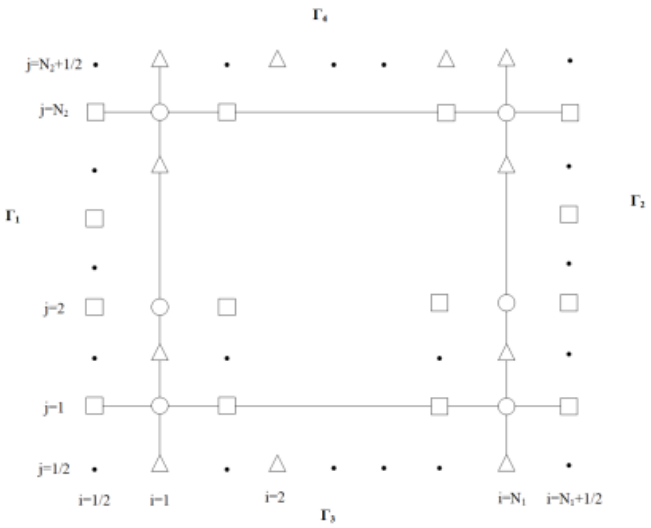


FIGURE 1. Computational grid template.

From the approximation of boundary conditions (11)-(14), it can be seen that the condition for the component normal to the boundary is fulfilled identically, and the tangent component is not defined at these grid nodes (Fig.1). There are no boundary conditions for pressure on all parts of the bounds. To close the system of equations (7)-(10), it is necessary to determine the boundary conditions

$$\begin{aligned} &u^{n+1}_{i+1/2,1}, \quad u^{n+1}_{i+1/2,N_2}, \qquad \text{when } i=1,\dots,N_1-1,\\ &v^{n+1}_{1,j+1/2}, \quad v^{n+1}_{N_1,j+1/2}, \qquad \text{when } i=1,\dots,N_2-1 \end{aligned} \tag{15}$$

We show for the lower bound an approximation of the boundary condition $u^{n+1}_{i+1/2,1}$. Decompose the Taylor series of the function $u^{n+1}_{i+1/2,1/2}$ in the vicinity of the nodes $(i+1/2,1)$

$$u^{n+1}_{i+1/2,1/2}=u^{n+1}_{i+1/2,1}-\frac{h_2}{2}\Big(\frac{\partial u^{n+1}}{\partial x_2}\Big)_{i+1/2,1}+\frac{h_2^2}{8}\Big(\frac{\partial^2 u^{n+1}}{\partial x_2^2}\Big)_{i+1/2,1}+O(h_2^3). \tag{16}$$

Hence, since $u^{n+1}_{i+1/2,1/2}=0$, from (16), we have

$$u^{n+1}_{i+1/2,1}=\frac{h_2}{2}\Big(\frac{\partial u^{n+1}}{\partial x_2}\Big)_{i+1/2,1}-\frac{h_2^2}{8}\Big(\frac{\partial^2 u^{n+1}}{\partial x_2^2}\Big)_{i+1/2,1}+O(h_2^3). \tag{17}$$

Projecting the first equation of motion onto the Ox_1 axis, we have

$$\Big(\frac{\partial P^{n+1}}{\partial x_1}\Big)_{i+1/2,1}=\nu\frac{\partial^2 u^{n+1}}{\partial x_2^2}\mid_{x_1,i+1/2;x_2,1}. \tag{18}$$

From (17) and (18), we have

$$u^{n+1}_{i+1/2,1}=\frac{h_2}{2}\Big(\frac{\partial u^{n}}{\partial x_2}\Big)_{i+1/2,1}-\frac{h_2^2}{8\nu}\Big(\frac{\partial P^{n+1}}{\partial x_1}\Big)_{i+1/2,1}. \tag{19}$$

We approximate the derivative $(\frac{\partial u^n}{\partial x_2})_{i+1/2,1}$ in (19) through the discrete values of the function u^n at the nodes $(i+1/2,1/2),(i+1/2,1),(i+1/2,2)$. To do this, we use the following Taylor series expansions

$$\begin{aligned} 0\equiv u^{n+1}_{i+1/2,1/2}&=u^{n+1}_{i+1/2,1}-\frac{h_2}{2}\Big(\frac{\partial u^n}{\partial x_2}\Big)_{i+1/2,1}+\frac{h_2^2}{8}\Big(\frac{\partial^2 u^n}{\partial x_2^2}\Big)_{i+1/2,1}+O(h_2^3),\\ u^{n+1}_{i+1/2,1}&=u^{n+1}_{i+1/2,1},\\ u^{n+1}_{i+1/2,2}&=u^{n+1}_{i+1/2,1}+h_2\Big(\frac{\partial u^n}{\partial x_2}\Big)_{i+1/2,1}+\frac{h_2^2}{2}\Big(\frac{\partial^2 u^n}{\partial x_2^2}\Big)_{i+1/2,1}+O(h_2^3). \end{aligned} \tag{20}$$

Using the method of undefined coefficients, we determine the second-order approximation for $(\frac{\partial u^n}{\partial x_2})_{i+1/2,1}$ through the values of the function at three points. The first equation of (20) is multiplied by -4, the second by 3, the third by 1 and

$$3h_2\Big(\frac{\partial u^n}{\partial x_2}\Big)_{i+1/2,1}=-4u^n_{i+1/2,1/2}+3u^n_{i+1/2,1}+u^n_{i+1/2,2}+O(h_2^3). \tag{21}$$

From (21), we have the formula of the first difference derivative of the second order

$$\Big(\frac{\partial u^n}{\partial x_2}\Big)_{i+1/2,1}\approx\frac{3u^n_{i+1/2,1}+u^n_{i+1/2,2}}{3h_2}+O(h_2^2). \tag{22}$$

Substituting (22) into (19), we get

$$u^{n+1}_{i+1/2,1}=\frac{3u^n_{i+1/2,1}+u^n_{i+1/2,2}}{6}-\frac{h_2^2}{8\nu}\frac{P^{n+1}_{i+1,1}-P^{n+1}_{i,1}}{h_1}. \tag{23}$$

Comparing (23) with the first formula (9) for $j=1$, we have that

$$\overline{u}^{n+1/2}_{i+1/2,1}=\frac{3u^n_{i+1/2,1}+u^n_{i+1/2,2}}{6},\tau=\tau_0=\frac{h_2^2}{8\nu}. \tag{24}$$

Reasoning similarly, we obtain for the upper bound

$$\overline{u}^{n+1/2}_{i+1/2,N_2} = \frac{3u^n_{i+1/2,N_2} + u^n_{i+1/2,N_2-1}}{6}, i = 1, ..., N_1 - 1. \tag{25}$$

On the left bound, we get

$$\overline{v}^{n+1/2}_{1,j+1/2} = \frac{3v^n_{1,j+1/2} + v^n_{2,j+1/2}}{6}, j = 1, ..., N_2 - 1. \tag{26}$$

On the right bound, we get

$$\overline{u}^{n+1/2}_{N_1,j+1/2} = \frac{3v^n_{N_1,i+1/2} + v^n_{N_1-1,i+1/2}}{6}, j = 1, ..., N_2 - 1. \tag{27}$$

To find the pressure in the inner nodes of the grid region Ω_{0h} from (9)-(10) we obtain an elliptical type difference scheme

$$\left(P^{n+1}_{i,j}\right)_{x_1,\overline{x_1}} + \left(P^{n+1}_{i,j}\right)_{x_2,\overline{x_2}} = \Delta_h P^n_{i,j} + \frac{1}{\tau} div_h \overrightarrow{V}^{n+1/2}, i = 2, ..., N_1 - 2; j = 2, ..., N_2 - 2, \tag{28}$$

here $\left(P^{n+1}_{i,j}\right)_{x_1,\overline{x_1}} = \frac{P^{n+1}_{i+1,j} - 2P^{n+1}_{i,j} + P^{n+1}_{i-1,j}}{h_1^2}$ and $\left(P^{n+1}_{i,j}\right)_{x_2,\overline{x_2}} = \frac{P^{n+1}_{i,j+1} - 2P^{n+1}_{i,j} + P^{n+1}_{i,j-1}}{h_2^2}$.

Due to the absence of boundary conditions for pressure in the boundary nodes , it is necessary to construct equations based on the relations of the form (23)-(27) for solving (28). For example, for the lower boundary, we have the continuity equation at $j = 1$:

$$\frac{u^{n+1}_{i+1/2,1} - u^{n+1}_{i-1/2,1}}{h_1} + \frac{v^{n+1}_{i,3/2} - v^{n+1}_{i,1/2}}{h_2} = 0, i = 1, ..., N_1. \tag{29}$$

Substituting the values in (29), we have

$$u^{n+1}_{i+1/2,1} = \overline{u}^{n+1/2}_{i+1/2,1} - \tau_0\Big(\frac{P^{n+1}_{i+1,1} - P^{n+1}_{i,1}}{h_1} - \frac{P^n_{i+1,1} - P^n_{i,1}}{h_1}\Big),$$

$$v^{n+1}_{i,3/2} = \overline{v}^{n+1/2}_{i,3/2} - \tau\Big(\frac{P^{n+1}_{i,2} - P^{n+1}_{i,1}}{h_2} - \frac{P^n_{i,2} - P^n_{i,1}}{h_2}\Big),$$

$$v^{n+1}_{i,1/2} \equiv 0,$$

thus, we obtain an equation for determining the pressure at the nodes $(i, 1), i = 2, ..., N_1 - 1$,

$$\begin{aligned} &\frac{\tau_0}{h_1^2}\left(P^{n+1}_{i+1,1} - 2P^{n+1}_{i,1} + P^{n+1}_{i-1,1}\right) + \frac{\tau}{h_2^2}\left(P^{n+1}_{i,2} - P^{n+1}_{i,1}\right) \\ &= \frac{\tau_0}{h_1^2}\left(P^n_{i+1,1} - 2P^n_{i,1} + P^n_{i-1,1}\right) + \frac{\tau}{h_2^2}\left(P^n_{i,2} - P^n_{i,1}\right) + \frac{\overline{u}^{n+1/2}_{i+1/2,1} - \overline{u}^{n+1/2}_{i-1/2,1}}{h_1} + \frac{\overline{v}^{n+1/2}_{i,3/2} - v^{n+1}_{i,1/2}}{h_2}, \end{aligned} \tag{30}$$

reasoning similarly at the upper bound for $j = N_2$, we have the following equation for determining pressure

$$\begin{aligned} &\frac{\tau_0}{h_1^2}\left(P^{n+1}_{i+1,N_2} - 2P^{n+1}_{i,N_2} + P^{n+1}_{i-1,N_2}\right) + \frac{\tau}{h_2^2}\left(P^{n+1}_{i,N_2} - P^{n+1}_{i,N_2-1}\right) \\ &= \frac{\tau_0}{h_1^2}\left(P^n_{i+1,N_2} - 2P^n_{i,N_2} + P^n_{i-1,N_2}\right) + \frac{\tau}{h_2^2}\left(P^n_{i,N_2} - P^n_{i,N_2-1}\right) \\ &+ \frac{\overline{u}^{n+1/2}_{i+1/2,N_2} - \overline{u}^{n+1/2}_{i-1/2,N_2}}{h_1} + \frac{v^{n+1}_{i,N_2+1/2} - \overline{v}^{n+1/2}_{i,N_2-1/2}}{h_2}, i = 2, ..., N_1 - 1. \end{aligned} \tag{31}$$

The continuity equation on the left boundary has the form

$$\frac{u^{n+1}_{3/2,j} - u^{n+1}_{1/2,j}}{h_1} + \frac{v^{n+1}_{1,j+1/2} - v^{n+1}_{1,j-1/2}}{h_2} = 0, j = 1, ..., N_2. \tag{32}$$

Substituting values in (32)

$$\begin{aligned} u_{3/2,j}^{n+1} &= \overline{u}_{3/2,j}^{n+1} - \frac{\tau}{h_1}[(P_{2,j}^{n+1} - P_{1,j}^{n+1}) - (P_{2,j}^{n} - P_{1,j}^{n})], \\ v_{1,j+1/2}^{n+1} &= \overline{v}_{1,j+1/2}^{n+1} - \frac{\tau}{h_2}[(P_{1,j+1}^{n+1} - P_{1,j}^{n+1}) - (P_{1,j+1}^{n} - P_{1,j}^{n})], \\ u_{1/2,j}^{n+1} &\equiv 0, \end{aligned} \tag{33}$$

from (32)-(33), we obtain the equations for determining at the boundary nodes $(1, j)$

$$\begin{aligned} &\frac{\tau}{h_1^2}(P_{2,j}^{n+1} - P_{1,j}^{n+1}) + \frac{\tau_1}{h_2^2}(P_{1,j+1}^{n+1} - 2P_{1,j}^{n+1} + P_{1,j-1}^{n+1}) \\ &= \frac{\tau}{h_1^2}(P_{2,j}^{n} - P_{1,j}^{n}) + \frac{\tau_1}{h_2^2}(P_{1,j+1}^{n} - 2P_{1,j}^{n} + P_{1,j-1}^{n}) \\ &+ \frac{\overline{u}_{3/2,j}^{n+1/2} - u_{1/2,j}^{n+1}}{h_1} + \frac{\overline{v}_{1,j+1/2}^{n+1/2} - \overline{v}_{1,j-1/2}^{n+1.2}}{h_2}, j = 2, ..., N_2 - 1. \end{aligned} \tag{34}$$

On the right boundary, a similar formula is obtained for finding $P_{N_1,j}^{n+1}$

$$\begin{aligned} &-\frac{\tau}{h_1^2}(P_{N_1,j}^{n+1} - P_{N_1-1,j}^{n+1}) + \frac{\tau_1}{h_2^2}(P_{N_1,j+1}^{n+1} - 2P_{N_1,j}^{n+1} + P_{N_1,j-1}^{n+1}) \\ &= -\frac{\tau}{h_1^2}(P_{N_1,j}^{n} - P_{N_1-1,j}^{n}) + \frac{\tau_1}{h_2^2}(P_{N_1,j+1}^{n} - 2P_{N_1,j}^{n} + P_{N_1,j-1}^{n}) \\ &+ \frac{u_{N_1+1/2,j}^{n+1} - \overline{u}_{N_1-1/2,j}^{n+1/2}}{h_1} + \frac{\overline{v}_{N_1,j+1/2}^{n+1/2} - \overline{v}_{N_1,j-1/2}^{n+1/2}}{h_2}, j = 2, ..., N_2 - 1. \end{aligned} \tag{35}$$

After making the approximations, the boundary conditions for the velocity components in fractional steps are as follows:
-at the lower bound of Γ_3:

$$\overline{u}_{i+1/2,1}^{n+1/2} = \psi_3(x_{1,i+1/2}), i = 1, ..., N_1 - 1, \overline{v}_{i,1/2}^{n+1/2} = 0. \tag{36}$$

-at the upper bound of Γ_4:

$$\overline{u}_{i+1/2,N_2}^{n+1/2} = \psi_4(x_{1,i+1/2}), i = 1, ..., N_1 - 1, \overline{v}_{i,N_2+1/2}^{n+1/2} = 0. \tag{37}$$

-at the left bound of Γ_1:

$$\overline{u}_{1/2,j}^{n+1/2} = 0, \overline{v}_{1,j+1/2}^{n+1/2} = \psi_1(x_{2,j+1/2}), j = 1, ..., N_2. \tag{38}$$

-at the right bound of Γ_2:

$$\overline{u}_{N_1+1/2,j}^{n+1/2} = 0, \overline{v}_{N_1,j+1/2}^{n+1/2} = \psi_2(x_{2,j+1/2}), j = 1, ..., N_2. \tag{39}$$

3. A priori estimation of the solution of the difference scheme

For an a priori estimation of the solution of the difference scheme (7)-(10) with boundary conditions (11)-(14), (30)-(31), (34)-(35), (36)-(39) the following theorem is formulated:

Theorem 1. *If condition*

$$\nu - \frac{C_1^2}{4}\frac{\tau}{h^2}\|\overrightarrow{V}^n\|^2 \geq 0.$$

is satisfied, then for the three-stage difference scheme (7)-(10), following a priori estimate is performed:

$$\|\overrightarrow{V}^{m+1}\|^2+\frac{1}{2}\sum_{n=0}^{m+1}\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2+\tau^2\|\nabla_h P^{m+1}\|^2+\tau\nu^2\|\overrightarrow{V}^{m+1}_{x\overline{x}}\|^2$$
$$\leq\|\overrightarrow{V}^{0}\|^2+\tau^2\|\nabla_h P^{0}\|^2+\tau\nu^2\|\overrightarrow{V}^{0}_{x\overline{x}}\|^2+\sum_{n=0}^{m}\|f^n\|^2.$$

Proof. From fractional steps (7)-(8) we obtain the relations containing $\overrightarrow{V}^{n+1/2}$ and $\overrightarrow{V}^{n}$. To do this, we summarize the first equation of (7) with the first equation of (8), similarly, we summarize the second with the second

$$\begin{aligned}&\frac{\overline{u}^{n+1/2}_{i+1/2,j}-u^n_{i+1/2,j}}{\tau}+L_{1h}(u^n,v^n)+(P^n_{ij})_{x_1}=\nu[(u^{n+1/2}_{i+1/2,j})_{x_1\overline{x}_1}+(\overline{u}^{n+1/2}_{i+1/2,j})_{x_2\overline{x}_2}]+f^{(1)}_{i+1/2,j},\\&\frac{\overline{v}^{n+1/2}_{i,j+1/2}-v^n_{i,j+1/2}}{\tau}+L_{2h}(u^n,v^n)+(P^n_{ij})_{x_2}=\nu[(v^{n+1/2}_{i,j+1/2})_{x_1\overline{x}_1}+(\overline{v}^{n+1/2}_{i,j+1/2})_{x_2\overline{x}_2}]+f^{(2)}_{i,j+1/2}.\end{aligned}\tag{40}$$

Now from (8), we find

$$\begin{aligned}u^{n+1/2}_{i+1/2,j}&=\overline{u}^{n+1/2}_{i+1/2,j}-\tau\nu(\overline{u}^{n+1/2}_{i+1/2,j})_{x_2\overline{x}_2}+\tau\nu(u^n_{i+1/2,j})_{x_2\overline{x}_2},\\v^{n+1/2}_{i,j+1/2}&=\overline{v}^{n+1/2}_{i,j+1/2}-\tau\nu(\overline{v}^{n+1/2}_{i,j+1/2})_{x_2\overline{x}_2}+\tau\nu(v^n_{i,j+1/2})_{x_2\overline{x}_2}.\end{aligned}\tag{41}$$

In (40), instead of $u^{n+1/2}_{i+1/2,j}$ and $v^{n+1/2}_{i,j+1/2}$ substituting (41) and we get

$$\begin{aligned}&\frac{\overline{u}^{n+1/2}_{i+1/2,j}-u^n_{i+1/2,j}}{\tau}+L_{1h}(u^n,v^n)+(P^n_{ij})_{x_1}\\&=\nu[(\overline{u}^{n+1/2}_{i+1/2,j})_{x_1\overline{x}_1}+(\overline{u}^{n+1/2}_{i+1/2,j})_{x_2\overline{x}_2}]\\&-\tau\nu^2((\overline{u}^{n+1/2}_{i+1/2,j})_{x_1\overline{x}_1x_2\overline{x}_2})-(u^n_{i+1/2,j})_{x_1\overline{x}_1x_2\overline{x}_2})+f^{(1)}_{i+1/2,j},\end{aligned}\tag{42}$$

$$\begin{aligned}&\frac{\overline{v}^{n+1/2}_{i,j+1/2}-v^n_{i,j+1/2}}{\tau}+L_{2h}(u^n,v^n)+(P^n_{ij})_{x_2}\\&=\nu[(\overline{v}^{n+1/2}_{i,j+1/2})_{x_1\overline{x}_1}+(\overline{v}^{n+1/2}_{i,j+1/2})_{x_2\overline{x}_2}]-\tau\nu^2(\overline{v}^{n+1/2}_{i,j+1/2})_{x_1\overline{x}_1x_2\overline{x}_2})-(v^n_{i,j+1/2})_{x_1\overline{x}_1x_2\overline{x}_2})+f^{(2)}_{i,j+1/2}.\end{aligned}\tag{43}$$

Consider (42)-(43) with initial boundary conditions (11)-(14), (36)-(39) and (2). In the future, we omit the "line" in (42)-(43). Multiply (42) by $2\tau u^{n+1/2}_{i+1/2,j}h_1$, (43) by $2\tau v^{n+1/2}_{i,j+1/2}h_1$ and sum by Ω_{1h} and Ω_{2h}, respectively, and as a result we get

$$\begin{aligned}&2\|\overrightarrow{V}^{n+1/2}\|^2-2(\overrightarrow{V}^{n+1/2},\overrightarrow{V}^{n})+2\tau(L_h\overrightarrow{V}^{n},\overrightarrow{V}^{n+1/2})\\&+\tau(\nabla_h P^n,\overrightarrow{V}^{n+1/2})+2\nu\tau\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2+\tau\nu^2(\|\overrightarrow{V}^{n+1/2}_{x\overline{x}}\|^2-\|\overrightarrow{V}^{n}_{x\overline{x}}\|^2)=(f^n,\overrightarrow{V}^{n+1/2}).\end{aligned}\tag{44}$$

By virtue of the identity

$$-2(\overrightarrow{V}^{n+1/2},\overrightarrow{V}^{n})=\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2-\|\overrightarrow{V}^{n+1/2}\|^2-\|\overrightarrow{V}^{n}\|^2,\tag{45}$$

and lemmas for nonlinear terms proved in [30], we have

$$(L_h\overrightarrow{V}^{n},\overrightarrow{V}^{n})=0.\tag{46}$$

Taking into account (45) and (46), from (44) we have:

$$\begin{aligned}&\|\overrightarrow{V}^{n+1/2}\|^2-\|\overrightarrow{V}^{n}\|^2+\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2+2\tau^2(L_h\overrightarrow{V}^{n},\frac{\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}}{\tau})\\&+2\tau(P^n,div_h\overrightarrow{V}^{n+1/2})+\tau\nu^2(\|\overrightarrow{V}^{n+1/2}_{x\overline{x}}\|^2\\&-|\overrightarrow{V}^{n}_{x\overline{x}}\|^2)+2\tau\nu\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2=(f^n,\overrightarrow{V}^{n+1/2}).\end{aligned}\tag{47}$$

Similarly, multiplying the first equation (9) by $2\tau u^{n+1}_{i+1/2,j}h_1$, the second equation by $2\tau v^{n+1}_{i,j+1/2}h_2$ and summing by Ω_{1h} and Ω_{2h}, respectively, we have

$$\|\overrightarrow{V}^{n+1}\|^2-\|\overrightarrow{V}^{n}\|^2+\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2+2\tau(\nabla(P^{n+1}-P^n),\overrightarrow{V}^{n+1})=0.\tag{48}$$

Summing up the integral identities (47)-(48), we provide

$$\begin{aligned}&\|\overrightarrow{V}^{n+1}\|^2-\|\overrightarrow{V}^{n}\|^2+\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2+\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2\\&+2\tau^2(L_h\overrightarrow{V}^{n},\frac{\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}}{\tau})+2\tau(\nabla_hP^n,\overrightarrow{V}^{n+1/2})+2\tau(\nabla_h(P^{n+1}-P^n),\overrightarrow{V}^{n+1})\\&+2\tau\nu\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2+\tau\nu^2(\|\overrightarrow{V}^{n+1/2}_{x\overline{x}}\|^2-|\overrightarrow{V}^{n}_{x\overline{x}}\|^2)=(f^n,\overrightarrow{V}^{n+1/2}).\end{aligned}\tag{49}$$

Let's consider separately the terms containing the pressure gradient form (49)

$$\begin{aligned}&2\tau(\nabla_hP^n,\overrightarrow{V}^{n+1/2})+2\tau(\nabla_h(P^{n+1}-P^n),\overrightarrow{V}^{n+1})\\&=-2\tau(\nabla_hP^n,\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2})+2\tau(\nabla_hP^{n+1},\overrightarrow{V}^{n+1}).\end{aligned}\tag{50}$$

Due to the continuity equation, the second term on the right side (50) is zero. Next, we convert

$$\begin{aligned}&2\tau(\nabla_hP^n,\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2})=2\tau(\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2},-\nabla_h(P^{n+1}-P^n)+\nabla_hP^{n+1})\\&=2\tau(\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2},\frac{\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}}{\tau}+\nabla_hP^{n+1})\\&=2\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2+2\tau(\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2},\nabla_hP^{n+1})\\&=2\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2+2\tau^2(-\nabla_h(P^{n+1}-P^n),\nabla_hP^{n+1})\\&=2\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2-2\tau^2(\nabla_hP^{n+1})^2+2\tau^2(\nabla_hP^n,\nabla_hP^{n+1})\\&=2\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2-2\tau^2(\nabla_hP^{n+1})^2-\tau^2\|\nabla_h(P^{n+1}-P^n)\|^2\\&-\tau^2\|\nabla_h(P^{n+1})\|^2+\tau^2\|\nabla_h(P^n)\|^2\\&=2\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2-\tau^2(\|\nabla_h(P^{n+1})\|^2-\|\nabla_h(P^n)\|^2)-\tau^2\|\nabla_h(P^{n+1}-P^n)\|^2\\&=\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2-\tau^2(\|\nabla_h(P^{n+1})\|^2-\|\nabla_h(P^n)\|^2).\end{aligned}\tag{51}$$

Substituting (51) into (49)

$$\begin{aligned}&\|\overrightarrow{V}^{n+1}\|^2-\|\overrightarrow{V}^{n}\|^2+\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2+\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2\\&+2\tau^2\Big(L_h\overrightarrow{V}^{n},\frac{\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}}{\tau}\Big)\\&-\|\overrightarrow{V}^{n+1}-\overrightarrow{V}^{n+1/2}\|^2+\tau^2\Big(\|\nabla_hP^{n+1}\|^2-\|\nabla_hP^n\|^2\Big)\\&+2\tau\nu\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2+\tau\nu^2\Big(\|\overrightarrow{V}^{n+1/2}_{x\overline{x}}\|^2-\|\overrightarrow{V}^{n}_{x\overline{x}}\|^2\Big)=(f^n,\overrightarrow{V}^{n+1/2}).\end{aligned}\tag{52}$$

After reducing same terms in (52), we get following inequality

$$\begin{aligned}&\|\overrightarrow{V}^{n+1}\|^2+\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2+2\tau\nu\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2+\tau^2\|\nabla_h P^{n+1}\|^2+\tau\nu^2\Big(\|\overrightarrow{V}^{n+1/2}_{x\overline{x}}\|^2\Big)\\&+2\tau^2\left|\Big(L_h\overrightarrow{V}^{n},\frac{\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}}{\tau}\Big)\right|\le\|\overrightarrow{V}^{n}\|^2+\tau^2\|\nabla_h P^{n}\|^2+\tau\nu^2\|\overrightarrow{V}^{n}_{x\overline{x}}\|^2+\|f^n\|^2.\end{aligned}\tag{53}$$

To evaluate nonlinear terms, we use the Cauchy-Bunyakovsky inequality

$$\begin{aligned}2\tau^2|(L_h\overrightarrow{V}^{n},\overrightarrow{V}^{n+1/2}_t)|&\le\frac{3\sqrt{2}\tau^2}{h}\left\{\sum_{\Omega_h}[u^{n^2}_{i+1/2,j}+v^{n}_{i,j+1/2^2}]h_1h_2\right\}^{1/2}\|\overrightarrow{V}^{n+1/2}_t\|\|\\&=\frac{3\sqrt{2}\tau^2}{h}\||\overrightarrow{V}^{n}|^2\|\cdot\|\overrightarrow{V}^{n+1/2}_t\|.\end{aligned}$$

The value of $\||\overrightarrow{V}^{n}|^2\|$ is estimated as in [13] such that

$$\||\overrightarrow{V}^{n}|^2\|\le\sqrt{2}\|\overrightarrow{V}^{n}\|\cdot\|\nabla_h\overrightarrow{V}^{n}\|,$$

then

$$\begin{aligned}2\tau^2|(L_h\overrightarrow{V}^{n},\overrightarrow{V}^{n+1/2}_t)|&\le\frac{6\tau^2}{h}\|\overrightarrow{V}^{n}\|\cdot\|\nabla_h\overrightarrow{V}^{n}\|\cdot\|\overrightarrow{V}^{n+1/2}_t\|\\&\le\frac{\tau^2}{2}\|\overrightarrow{V}^{n+1/2}_t\|^2+\frac{C_1^2}{2}(\frac{\tau}{h})^2\cdot\|\overrightarrow{V}^{n}\|^2\|\nabla_h\overrightarrow{V}^{n}\|^2.\end{aligned}\tag{54}$$

Further, from the third stage of the splitting scheme (9), we have

$$\overrightarrow{V}^{n+1/2}=\overrightarrow{V}^{n+1}+\tau\nabla(P^{n+1}-P^{n}).\tag{55}$$

Take the second derivative of (55), then

$$\overrightarrow{V}^{n+1/2}_{x\overline{x}}=\overrightarrow{V}^{n+1}_{x\overline{x}}+\tau\nabla(P^{n+1}_{x\overline{x}}-P^{n}_{x\overline{x}}).\tag{56}$$

Squaring (56) and using Young's inequality, we have

$$\|\overrightarrow{V}^{n+1/2}_{x\overline{x}}\|^2\le(1+\frac{\varepsilon}{2})\|\overrightarrow{V}^{n+1}_{x\overline{x}}\|^2+\tau(1+\frac{1}{2\varepsilon})\|\nabla(P^{n+1}_{x\overline{x}}-P^{n}_{x\overline{x}})\|^2.\tag{57}$$

for an arbitrary $\varepsilon>0$. Then, from (53), (54) and (57) we get

$$\begin{aligned}&\|\overrightarrow{V}^{n+1}\|^2+\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2+2\tau\cdot\nu\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2-\frac{1}{2}\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2\\&-\frac{C_1^2}{2}(\frac{\tau}{h})^2\|\overrightarrow{V}^{n}\|^2\cdot\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2+\tau^2\|\nabla_h P^{n+1}\|^2+\tau\nu^2(1+\frac{\varepsilon}{2})\|\overrightarrow{V}^{n+1}_{x\overline{x}}\|^2\\&+\tau^2\nu^2(1+\frac{1}{2\varepsilon})\|\nabla(P^{n+1}_{x\overline{x}}-P^{n}_{x\overline{x}})\|^2\le\|\overrightarrow{V}^{n}\|^2+\tau^2\|\nabla_h P^{n}\|^2+\tau\nu^2\|\overrightarrow{V}^{n}_{x\overline{x}}\|^2+\|f^n\|^2.\end{aligned}\tag{58}$$

Further, grouping similar elements in (58), we have

$$\begin{aligned}&\|\overrightarrow{V}^{n+1}\|^2+\frac{1}{2}\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2+2\tau(\nu-\frac{C_1^2}{4}\frac{\tau}{h^2}\|\overrightarrow{V}^{n}\|^2)\|\nabla_h\overrightarrow{V}^{n+1/2}\|^2\\&+\tau^2\|\nabla_h P^{n+1}\|^2+\tau\nu^2(1+\frac{\varepsilon}{2})\|\overrightarrow{V}^{n+1}_{x\overline{x}}\|^2+\tau^2\nu^2(1+\frac{1}{2\varepsilon})\|\nabla_h(P^{n+1}_{x\overline{x}}-P^{n}_{x\overline{x}})\|^2\\&\le\|\overrightarrow{V}^{n}\|^2+\tau^2\|\nabla_h P^{n}\|^2+\tau\nu^2\|\overrightarrow{V}^{n}_{x\overline{x}}\|^2+\|f^n\|^2.\end{aligned}\tag{59}$$

Suppose that

$$\nu-\frac{C_1^2}{4}\frac{\tau}{h^2}\|\overrightarrow{V}^{n}\|^2\ge0.$$

Then, from (59) we get

$$\begin{aligned}&\|\overrightarrow{V}^{n+1}\|^2+\frac{1}{2}\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2+\tau^2\|\nabla_h P^{n+1}\|^2+\tau\nu^2\|\overrightarrow{V}^{n+1}_{x\overline{x}}\|^2\\&\leq\|\overrightarrow{V}^{n}\|^2+\tau^2\|\nabla_h P^{n}\|^2++\tau\nu^2\|\overrightarrow{V}^{n}_{x\overline{x}}\|^2+\|f^n\|^2.\end{aligned}\tag{60}$$

Summing up n from 0 to m in the inequality (60), we have

$$\begin{aligned}&\|\overrightarrow{V}^{m+1}\|^2+\frac{1}{2}\sum_{n=0}^{m+1}\|\overrightarrow{V}^{n+1/2}-\overrightarrow{V}^{n}\|^2+\tau^2\|\nabla_h P^{m+1}\|^2\\&+\tau\nu^2\|\overrightarrow{V}^{m+1}_{x\overline{x}}\|^2\leq\|\overrightarrow{V}^{0}\|^2+\tau^2\|\nabla_h P^{0}\|^2+\tau\nu^2\|\overrightarrow{V}^{0}_{x\overline{x}}\|^2+\sum_{n=0}^{m}\|f^n\|^2.\end{aligned}$$

The theorem is proved. □

When choosing a coordinate grid in the form (3), the continuity equation is approximated with the second order of accuracy by h everywhere in $\Omega_{1h}\cup\Omega_{2h}$, where $h=max\,(h_3,h_2)$.

To establish the convergence of the difference problem (4)-(6), the following theorem is formulated:

Theorem 2. *If the solution of the problem (1)-(2) is sufficiently smooth and the condition*

$$\nu-\frac{C_1^2}{4}\frac{\tau}{h^2}\|\overrightarrow{\omega}^{n}\|^2\geq 0.$$

is fulfilled, then the solution of the difference problem (4)-(6) with boundary conditions (11)-(14) converges to the solution of problems (1)-(2) with a convergence rate of $O(\tau+h^2)$.

Proof. The errors of the solution are denoted by:

$$\begin{aligned}\omega^{n+1}_{1,i+1/2,j}&=u^{n+1}_{i+1/2,j}-u\left(t_{n+1},x_{1,i+\frac{1}{2}},x_{2,j}\right),\\\omega^{n+1}_{2,i,j+1/2}&=v^{n+1}_{i,j+1/2}-v\left(t_{n+1},x_{1,i},x_{2,j+\frac{1}{2}}\right),\\\pi^{n+1}_{i,j}&=P^{n+1}_{i,j}-P\left(t_{n+1},x_{1,i},x_{2,j}\right).\end{aligned}\tag{61}$$

Then, for the error of the solution for the linear equations corresponding to (4)-(6), we have the following relations

$$\begin{aligned}&\frac{\omega^{n+1}_{1,i+1/2,j}-\omega^{n}_{1,i+1/2,j}}{\tau}+\frac{\pi^{n+1}_{i+1,j}-\pi^{n+1}_{i,j}}{h_1}-\nu\left(\left(\omega^{n}_{1,i+\frac{1}{2},j}\right)_{x_1\overline{x}_1}+\left(\omega^{n}_{1,i+\frac{1}{2},j}\right)_{x_2\overline{x}_2}\right)=-\psi^{n}_{1h},\\&\frac{\omega^{n+1}_{2,i,j+1/2}-\omega^{n}_{2,i,j+1/2}}{\tau}+\frac{\pi^{n+1}_{i,j+1}-\pi^{n+1}_{i,j}}{h_2}-\nu\left(\left(\omega^{n}_{2,i,j+\frac{1}{2}}\right)_{x_1\overline{x}_1}+\left(\omega^{n}_{2,i,j+\frac{1}{2}}\right)_{x_2\overline{x}_2}\right)=-\psi^{n}_{2h},\\&\frac{\omega^{n+1}_{1,i+1/2,j}-\omega^{n+1}_{1,i-1/2,j}}{h_1}+\frac{\omega^{n+1}_{2,i,j+1/2}-\omega^{n+1}_{2,i,j-1/2}}{h_2}=-\rho^{n+1}_{h},\end{aligned}\tag{62}$$

where ψ^n_{1h}, psi^n_{2h}, ρ^{n+1}_h are the approximation errors of the difference scheme (4)-(6) on the exact solution of the differential problem (1)-(2).

Let's show that $\rho^n_h=O(h^2)$.

To do this, from (61) we find

$$\begin{aligned}u^{n}_{i+\frac{1}{2},j}&=\omega^{n+1}_{1,i+\frac{1}{2}j}+u\left(t_{n+1},x_{1,i+\frac{1}{2}},x_{2,j}\right),\\u^{n}_{i,j+\frac{1}{2}}&=\omega^{n+1}_{1,ij+\frac{1}{2}}+u\left(t_{n+1},x_{1,i},x_{2,j+\frac{1}{2}}\right).\end{aligned}\tag{63}$$

And substitute (63) into (6), we get

$$\begin{aligned}&\frac{\omega_{1,i+1/2,j}^{n+1}-\omega_{1,i-1/2,j}^{n+1}}{h_1}+\frac{\omega_{2,i,j+1/2}^{n+1}-\omega_{2,i,j-1/2}^{n+1}}{h_2}\\&+\frac{u\left(t_{n+1},x_{1,i+\frac{1}{2}},x_{2,j}\right)-u\left(t_{n+1},x_{1,i-\frac{1}{2}},x_{2,j}\right)}{h_1}\\&+\frac{v\left(t_{n+1},x_{1,i},x_{2,j+\frac{1}{2}}\right)-v\left(t_{n+1},x_{1,i},x_{2,j-\frac{1}{2}}\right)}{h_2}=0,\end{aligned}\tag{64}$$

so from (64)

$$\begin{aligned}\rho_h^{n+1}=&\frac{u\left(t_{n+1},x_{1,i+\frac{1}{2}},x_{2,j}\right)-u\left(t_{n+1},x_{1,i-\frac{1}{2}},x_{2,j}\right)}{h_1}\\&+\frac{v\left(t_{n+1},x_{1,i},x_{2,j+\frac{1}{2}}\right)-v\left(t_{n+1},x_{1,i},x_{2,j-\frac{1}{2}}\right)}{h_2},\end{aligned}\tag{65}$$

Now decompose into a Taylor series the function $u\left(t_{n+1},x_{1,i\pm\frac{1}{2}},x_{2,j}\right)$ in the vicinity of the point $x_{1,i}$ and the function $v\left(t_{n+1},x_{1,i},x_{2,j\pm\frac{1}{2}}\right)$ in the vicinity of the point $x_{2,j}$, as follows:

$$\begin{aligned}u\left(t_{n+1},x_{1,i\pm\frac{1}{2}},x_{2,j}\right)=&u\left(t_{n+1},x_{1,i},x_{2,j}\right)\pm\frac{h_1}{2}\left(\frac{\partial u\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_1}\right)\\&+\frac{h_1^2}{8}\left(\frac{\partial^2 u\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_1^2}\right)\pm\frac{h_1^3}{48}\left(\frac{\partial^3 u\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_1^3}\right)\pm O\left(h_1^4\right)),\end{aligned}\tag{66}$$

$$\begin{aligned}v\left(t_{n+1},x_{1,i},x_{2,j\pm\frac{1}{2}}\right)=&v\left(t_{n+1},x_{1,i},x_{2,j}\right)\pm\frac{h_2}{2}\left(\frac{\partial v\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_2}\right)\\&+\frac{h_2^2}{8}\left(\frac{\partial^2 v\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_2^2}\right)\pm\frac{h_2^3}{48}\left(\frac{\partial^3 v\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_2^3}\right)\pm O\left(h_2^4\right)),\end{aligned}\tag{67}$$

Substituting expansions (66)-(67) into (65), we have

$$\begin{aligned}\rho_h^{n+1}=&\frac{h_1}{2}\left(\frac{\partial u\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_1}\right)+\frac{h_1^3}{48}\left(\frac{\partial^3 u\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_1^3}\right)\\&+\frac{h_2}{2}\left(\frac{\partial v\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_2}\right)+\frac{h_2^3}{48}\left(\frac{\partial^3 v\left(t_{n+1},x_{1,i},x_{2,j}\right)}{\partial x_2^3}\right)+O\left(h^3\right).\end{aligned}$$

According to the continuity equation (1), we have that

$$\rho_h^{n+1}=O\left(h^2\right).$$

Similarly, it can be shown that

$$\psi_{1h}^n=O\left(\tau+h^2\right),\psi_{2h}^n=O\left(\tau+h^2\right).$$

In the linear case, the orders of residuals are determined very simply and, according to Lax's theorem, convergence follows from approximation and stability.

Consider the following nonlinear terms included in the first equation for the error of the solution

$$
\begin{aligned}
&\frac{1}{4h_1}\left\{\left[\omega^n_{1,i+\frac{3}{2},j}+u\left(t_n,x_{1,i+\frac{3}{2}},x_{2,j}\right)+\omega^n_{1,i+\frac{1}{2},j}+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right]^2\right.\\
&\left.-\left[\omega^n_{1,i+\frac{1}{2},j}+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)+\omega^n_{1,i-\frac{1}{2},j}+u\left(t_n,x_{1,i-\frac{1}{2}},x_{2,j}\right)\right]^2\right\}\\
&+\frac{1}{4h_2}\left\{\left[\omega^n_{1,i+\frac{1}{2},j+1}+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j+1}\right)+\omega^n_{1,i+\frac{1}{2},j}+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right]\right.\\
&\times\left[\omega^n_{2,i+1,j+\frac{1}{2}}+v\left(t_n,x_{1,i+1},x_{2,j+\frac{1}{2}}\right)+\omega^n_{2,i,j+\frac{1}{2}}+v\left(t_n,x_{1,i},x_{2,j+\frac{1}{2}}\right)\right]\\
&-\left[\omega^n_{1,i+\frac{1}{2},j}+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j+1}\right)+\omega^n_{1,i+\frac{1}{2},j-1}+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j-1}\right)\right]\\
&\left.\times\left[\omega^n_{2,i+1,j-\frac{1}{2}}+v\left(t_n,x_{1,i+1},x_{2,j-\frac{1}{2}}\right)+\omega^n_{2,ij-\frac{1}{2}}+v\left(t_n,x_{1,i},x_{2,j-\frac{1}{2}}\right)\right]\right\}\\
&=\frac{1}{4h_1}\left[\left(\omega^n_{1,i+\frac{3}{2},j}+\omega^n_{1,i+\frac{1}{2}j}\right)^2-\left(\omega^n_{1,i+\frac{1}{2}j}+\omega^n_{1,i-\frac{1}{2}j}\right)^2\right]\\
&+\frac{1}{4h_2}\left[\left(\omega^n_{1,i+\frac{1}{2},j+1}+\omega^n_{1,i+\frac{1}{2}j}\right)\left(\omega^n_{2,i+1,j+\frac{1}{2}}+\omega^n_{2,i,j+\frac{1}{2}}\right)\right.\\
&\left.-\left(\omega^n_{1,i+\frac{1}{2},j}+\omega^n_{1,i+\frac{1}{2},j-1}\right)\left(\omega^n_{2,i+1,j-\frac{1}{2}}+\omega^n_{2,i,j-\frac{1}{2}}\right)\right]\\
&+\frac{1}{2}\left[\left(\omega^n_{1,i+\frac{1}{2}j}+\omega^n_{1,i-\frac{1}{2}j}\right)\left(u^n_{i+\frac{1}{2}j}+u^n_{i-\frac{1}{2},j}\right)\right]_{x_1}\\
&+\frac{1}{4}\left[\left(\omega^n_{1,i+\frac{1}{2}j+1}+\omega^n_{1,i+\frac{1}{2}j}\right)\left(v^n_{i+1j+\frac{1}{2}}+v^n_{ij+\frac{1}{2}}\right)\right.\\
&\left.+\left(u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j+1}\right)+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right)\left(\omega^n_{2,i+1,j+\frac{1}{2}}+\omega^n_{2,i,j+\frac{1}{2}}\right)\right]_{\overline{x}_2}\\
&+\frac{1}{4h_1}\left[\left(u\left(t_n,x_{1,i+\frac{3}{2}},x_{2,j}\right)+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right)^2-\left(u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right.\right.\\
&\left.\left.+\ u\left(t_n,x_{1,i-\frac{1}{2}},x_{2,j}\right)\right)^2\right]\\
&+\frac{1}{4}\left[\left(u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j+1}\right)+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right)\right.\\
&\left.\times\left(v\left(t_n,x_{1,i+1},x_{2,j+\frac{1}{2}}\right)c+v\left(t_n,x_{1,i},x_{2,j+\frac{1}{2}}\right)\right)\right]_{\overline{x}_2}.
\end{aligned}
\tag{68}
$$

The last two terms in (68) form the approximation error of the nonlinear terms

$$
\begin{aligned}
\tilde{r}^n_{1,h,\ i+\frac{1}{2j}}&=\frac{1}{4}\left[\left(u\left(t_n,x_{1,i+\frac{3}{2}},x_{2,j}\right)-u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right)^2\right]_{x_1}\\
&+\frac{1}{4}\left[\left(u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j+1}\right)+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right)\cdot\left(u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j+1}\right)+u\left(t_n,x_{1,i+\frac{1}{2}},x_{2,j}\right)\right)\right]_{\overline{x}_2}
\end{aligned}
$$

and has a second approximation order, i.e. $\widetilde{\psi}^n_{1,h,i+\frac{1}{2}j}=O\left(h^2\right)$. Similarly, the order of approximation of the nonlinear terms for the second equation is obtained as: $\widetilde{\psi}^n_{2,h,ij+\frac{1}{2}}=O\left(h^2\right)$.

There are two terms left that contain the error of the solution and the solution itself

$$\tilde{L}_1\omega_1^n = \frac{1}{2}\Big[\Big(\omega^n_{1,i+\frac{1}{2}j} + \omega^n_{1,i-\frac{1}{2}j}\Big)\Big(u\Big(t_n, x_{1,i+\frac{1}{2}}, x_{2,j}\Big) - u\Big(t_n, x_{1,i-\frac{1}{2}}, x_{2,j}\Big)\Big)\Big]_{x_1}$$
$$+ \frac{1}{4}[\Big(v\Big(t_n, x_{1,i+1}, x_{2,j+\frac{1}{2}}\Big) + v\Big(t_n, x_{1,i}, x_{2,j+\frac{1}{2}}\Big)\Big)\cdot\Big(\omega^n_{1,i+\frac{1}{2}\ j+1} - \omega^n_{1,i+\frac{1}{2}j}\Big)$$
$$+ \Big(u\Big(t_n, x_{1,i+\frac{1}{2}}, x_{2,j+1}\Big) + u\Big(t_n, x_{1,i+\frac{1}{2}}, x_{2,j}\Big)\Big)\Big(\omega^n_{2,i+1\mathrm{J}+\frac{1}{2}} - \omega^n_{2,i\mathrm{J}+\frac{1}{2}}\Big)]_{\overline{x}_2}. \quad (69)$$

Taking into account the above nonlinear terms, we consider the finite difference relations for the error of the solution in the following form:

$$\Big(\omega^{n+1}_{1,i+\frac{1}{2},j}\Big)_{\overline{t}} + \frac{1}{4}\Big(\Big(\omega^n_{1,i+\frac{1}{2},j} + \omega^n_{1,i-\frac{1}{2}j}\Big)^2\Big)_{x_1} + \frac{1}{2}\Big[\Big(\omega^n_{1,i+\frac{1}{2},j} + \omega^n_{1,i-\frac{1}{2}j}\Big)\Big(u^n_{i+\frac{1}{2}j} + u^n_{i+\frac{1}{2},j}\Big)\Big]_{x_1}$$
$$+ \frac{1}{4}\Big[\Big(\omega^n_{2,i+1\ j+\frac{1}{2}} + \omega^n_{2,i,j+\frac{1}{2}}\Big)\Big(\omega^n_{1,i+\frac{1}{2}\ j+1} - \omega^n_{1,i+\frac{1}{2}j}\Big)\Big]_{\overline{x}_2}$$
$$+ \frac{1}{4}[\Big(v\Big(t_n, x_{1,i+1}, x_{2,j+\frac{1}{2}}\Big) + v\Big(t_n, x_{1,i}, x_{2,j+\frac{1}{2}}\Big)\Big)\Big(\omega^n_{1,i+\frac{1}{2}\ j+1} - \omega^n_{1,i+\frac{1}{2}j}\Big)$$
$$+ \Big(\omega^n_{2,i+1\ j+\frac{1}{2}} + \omega^n_{2,i,j+\frac{1}{2}}\Big)\Big(u\Big(t_n, x_{1,i+\frac{1}{2}}, x_{2,j+1}\Big) + u\Big(t_n, x_{1,i+\frac{1}{2}}, x_{2,j}\Big)\Big)]_{\overline{x}_2}$$
$$+ \pi^{n+1}_{x_1,i,j} = 2\Big(\mu_{i,j}\omega^n_{1,\ \overline{x}_1,i+\frac{1}{2},j}\Big)_{x_1} + \Big(a_{i+\frac{1}{2},j+\frac{1}{2}}\Big(\omega^n_{1,\ \overline{x}_2,i+\frac{1}{2},j} + \omega^n_{2,\ \overline{x}_1,i+\frac{1}{2}j}\Big)\Big)_{\overline{x}_2} - \psi_{1,i+\frac{1}{2},j} - \widetilde{\psi}_{1,i+\frac{1}{2},j},$$

$$\Big(\omega^{n+1}_{2,i,j+\frac{1}{2}}\Big)_{\overline{t}} + \frac{1}{4}\Big(\Big(\omega^n_{1,i+\frac{1}{2},j} + \omega^n_{1,i-\frac{1}{2},j}\Big)^2\Big)_{x_1} + \frac{1}{2}\Big[\Big(\omega^n_{1,i+\frac{1}{2},j} + \omega^n_{1,i-\frac{1}{2},j}\Big)\Big(u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j}\Big)\Big]_{x_1}$$
$$+ \frac{1}{4}\Big[\Big(\omega^n_{2,i+,j+\frac{1}{2}} + \omega^n_{2,i,j+\frac{1}{2}}\Big)\Big(\omega^n_{1,i+\frac{1}{2},j+1} - \omega^n_{1,i+\frac{1}{2},j}\Big)\Big]_{\overline{x}_2}$$
$$+ \frac{1}{4}[\Big(v\Big(t_n, x_{1,i+1}, x_{2,j+\frac{1}{2}}\Big) + v\Big(t_n, x_{1,i}, x_{2,j+\frac{1}{2}}\Big)\Big)\Big(\omega^n_{1,i+\frac{1}{2}\ j+1} - \omega^n_{1,i+\frac{1}{2},j}\Big)$$
$$\Big(\omega^n_{2,i+1\ j+\frac{1}{2}} + \omega^n_{2,ij+\frac{1}{2}}\Big)\Big(u\Big(t_n, x_{1,i+\frac{1}{2}}, x_{2,j+1}\Big) + u\Big(t_n, x_{1,i+\frac{1}{2}}, x_{2,j}\Big)\Big)]_{\overline{x}_2}$$
$$+ \pi^{n+1}_{x_2,i,j} = 2\Big(\mu_{i,j}\omega^n_{2,\ \overline{x}_2,i,j+\frac{1}{2}}\Big)_{x_2} + \Big(a_{i+\frac{1}{2},j+\frac{1}{2}}\Big(\omega^n_{1,\ \overline{x}_2,i+\frac{1}{2},j} + \omega^n_{2,\ \overline{x}_1,i+\frac{1}{2},j}\Big)\Big)_{\overline{x}_1} - \psi_{2,i,j+\frac{1}{2}} - \widetilde{\psi}_{2,i,j+\frac{1}{2}}.$$

Suppose that

$$div_h\overrightarrow{\omega}^{n+1}_h = 0, \quad \overrightarrow{\omega}^{n+1}_h|_{\gamma h} = 0, \quad \overrightarrow{\omega}^0_h = 0,$$

where $\widetilde{\psi}_{1,i+\frac{1}{2},j}$, $\widetilde{\psi}_{2,i,j+\frac{1}{2}}$ is the approximation error of nonlinear terms and have the order is $O\left(h^2\right)$.

A priori estimates for the problem (63), taking into account the nonlinear terms (69) and approximation errors $\psi_{1,i+\frac{1}{2},j}$, $\widetilde{\psi}_{1,i+\frac{1}{2},j}$, $\psi_{2,i,j+\frac{1}{2}}$, $\widetilde{\psi}_{2,i,j+\frac{1}{2}}$ are obtained as the first a priori estimate (62). Naturally, to fulfill this inequality, it is necessary to set a condition similar to (60)

$$\nu - \frac{C_1^2}{4}\frac{\tau}{h^2}\|\overrightarrow{\omega}^n\|^2 \geq 0.$$

Then, based on the obtained a priori estimate, convergence theorem has been proved. □

4. Computational experiments and analisys

The numerical implementation was performed for the problem of flow in a cavity using a three-stage difference scheme. A set of application programs for the numerical solution of difference schemes for the Navier-Stokes equations has been developed and debugged. To perform the calculations, a script was written in the high-level Python programming language. With the

help of a ready-made set of application programs, it is possible to visualize the numerical results obtained for different numbers of grid nodes and for different values of the iterative parameter τ and the small parameter ε. The calculation took the area $[0,1] \times [0,1]$ and the parameters $N_1 = N_2 = 20$, $\tau = 10^{-2}$ and $\varepsilon = 10^{-5}$. Visualizations of the results for different values of the Reynolds number are presented (Figs.3,5,7). A comparative numerical experiment was conducted between the classical two-stage splitting method (Figs.2,4,6) and the modified three-stage splitting method by physical processes.

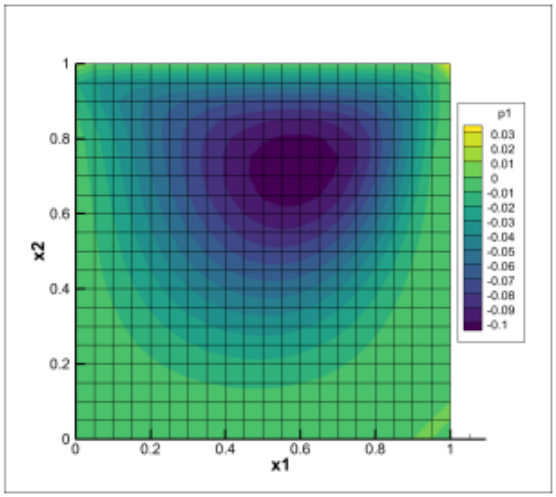


FIGURE 2. Isolines of stationary flow in a cavity with Re=100 in the case of a classical two-step splitting scheme.

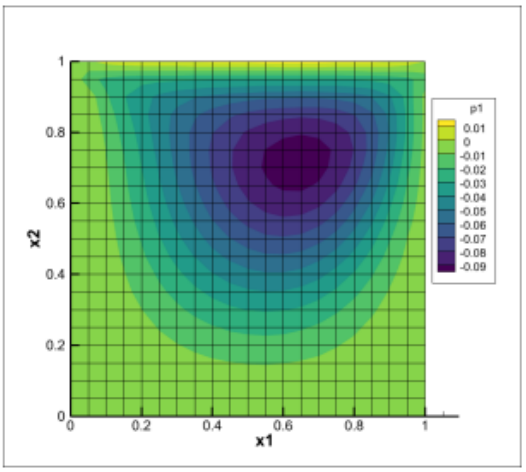


FIGURE 3. Isolines of stationary flow in a cavity with Re=100 in the case of a modified three-step splitting scheme.

Figs.3,5,7 illustrate the behavior of the system under different Reynolds numbers, providing insights into its performance across various flow regimes. Consequently, these findings demonstrate that the three-stage difference scheme effectively approximates the problem described by Eqns. (1)-(2) with a high degree of accuracy [10].

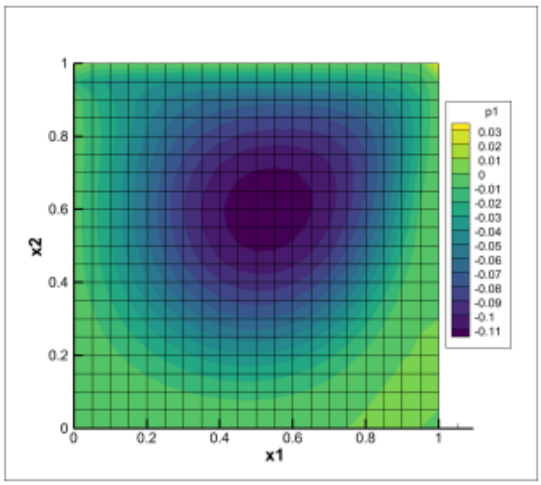


FIGURE 4. Isolines of stationary flow in a cavity with Re=400 in the case of a classical two-step splitting scheme.

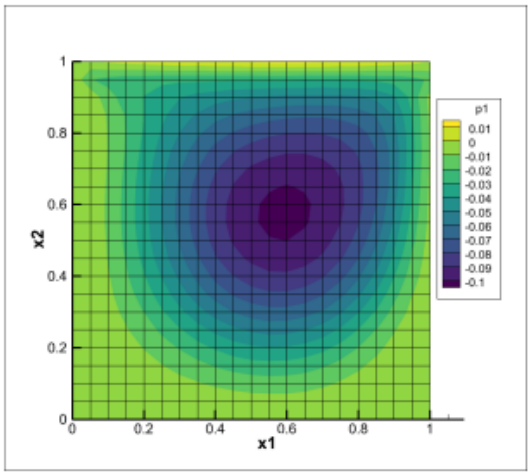


FIGURE 5. Isolines of stationary flow in a cavity with Re=400 in the case of a modified three-step splitting scheme.

The developed three-step splitting scheme allows the first two steps to be implemented implicitly. It follows that numerical calculations are performed with large time steps. In the classical version of the splitting scheme proposed in the works [3, 10, 36], the test problem was solved for $\tau \leq 0.001$. However, the computational experiment for the test problem using the three-step scheme was conducted with $\tau = 0.05$. Numerical calculations confirm the theoretically obtained results on the correctness of the difference scheme.

At the first step, the boundary conditions for $u_{i-1/2,j}^{n+1/2}$, $v_{i,j-1/2}^{n+1/2}$ were set as the values of the velocity components at the n-th time layer. Unlike the classical splitting scheme, at the first step, the pressure gradient at the n-th time layer is taken into account.

At the second step, the boundary conditions for $\bar{u}_{i-1/2,j}^{n+1/2}$, $\bar{v}_{i,j-1/2}^{n+1/2}$ are obtained from the Taylor series expansion of the boundary conditions for the velocity components and the projection of equations onto the boundaries. These conditions allow the modeling of flows near boundaries with second-order accuracy.

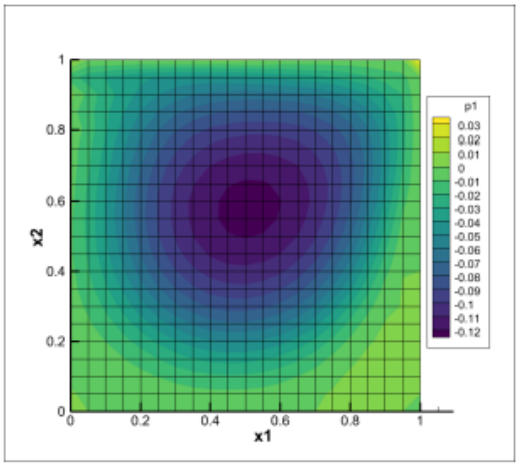


FIGURE 6. Isolines of stationary flow in a cavity with Re=800 in the case of a classical two-step splitting scheme

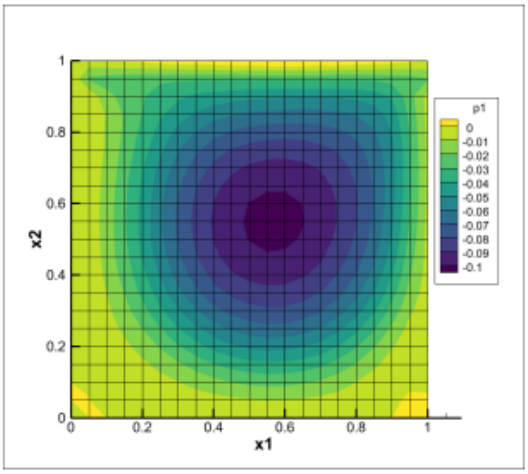


FIGURE 7. Isolines of stationary flow in a cavity with Re=800 in the case of a modified three-step splitting scheme

Then, at the third step, $u^{n+1}_{i-1/2,j}$, $v^{n+1}_{i,j-1/2}$ are determined through the gradients of the pressure difference over the full step. At the same time, conditions in the form of equations are obtained for determining the pressure at boundary grid nodes. These conditions are of the Robin type, so the pressure is computed iteratively up to a constant. By fixing the pressure value at one internal node, a unique determination can be achieved.

5. Algorithm for numerical implementation of the atmospheric boundary layer problem

As a model task for studying the influence of anthropogenic heat sources, harmful substances and the heterogeneity of the underlying surface into the atmosphere of an industrial city in a three-dimensional area $\Omega = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$, a model of the atmospheric boundary layer is considered [30, 33, 34]. The approximation of the problem is implemented on the grid $\Omega_h = \{\Omega_{0h} \cup \Omega_{1h} \cup \Omega_{2h} \cup \Omega_{3h}\}$ where

$$\Omega_{1h} = \left\{ \left(x_{1_i}, x_{2_j}, x_{3_k}\right) = (ih_1, jh_2, kh_3),\ i = \overline{0, N_1}, j = \overline{0, N_2},\ k = \overline{0, N_3} \right\},$$

$$\Omega_{1h} = \left\{ \left(x_{1_{i-\frac{1}{2}}}, x_{2_j}, x_{3_k}\right) = \left(\left(i - \frac{1}{2}\right) h_1, jh_2, kh_3\right),\ i = \overline{1, N_1}, j = \overline{0, N_2},\ k = \overline{0, N_3} \right\},$$

$$\Omega_{2h} = \left\{ \left(x_{1_i}, x_{2_{j-\frac{1}{2}}}, x_{3_k}\right) = \left(ih_1, \left(j - \frac{1}{2}\right) h_2, kh_3\right),\ i = \overline{0, N_1}, j = \overline{1, N_2},\ k = \overline{0, N_3} \right\},$$

$$\Omega_{3h} = \left\{ \left(x_{1_i}, x_{2_j}, x_{3_{k-\frac{1}{2}}}\right) = \left(ih_1, jh_2, \left(k - \frac{1}{2}\right) h_3\right),\ i = \overline{0, N_1}, j = \overline{0, N_2},\ k = \overline{1, N_3} \right\}.$$

here $h_1 = \frac{1}{N_1},\ h_2 = \frac{1}{N_2}, h_3 = \frac{1}{N_3}$.

When implementing the calculation, the components of the wind speed $\overrightarrow{V}$ are determined in the grid nodes Ω_{1h}, Ω_{2h}, Ω_{3h}, respectively. The temperature θ is determined at the nodes of the grid Ω_{3h}, the pressure p and the concentration of substances φ_q are determined at the nodes of the grid Ω_{0h}.

Assume that the velocity field $\vec{V}_h = (u_h, v_h, \omega_h)$, the pressure field P_h^n, the concentration of harmful substances $\varphi_{q,h}^n$, and the temperature θ_h^n be known at time t^n at the nodal points of the grid regions $\Omega_{x,h}, \Omega_{y,h}, \Omega_{z,h}, \Omega_h$. To calculate the unknown fields of velocity, pressure, temperature, and concentration at time t^{n+1}, a splitting scheme is used, which consists of the following stages:

(1) The definition of intermediate values of velocity fields $\overrightarrow{V}_h^{n+\frac{1}{4}}$ is performed by the following difference schemes

$$\frac{\overrightarrow{V}_h^{n+\frac{1}{4}} - \overrightarrow{V}_h^{n}}{\tau} = -L_h \overrightarrow{V}_h^{n} - P_h^n + \nu \left(\left(\overrightarrow{V}_h^{n+\frac{1}{4}} \right)_{x_1 \overline{x}_1} + \left(\overrightarrow{V}_h^{n} \right)_{x_2 \overline{x}_2} + \left(\overrightarrow{V}_h^{n} \right)_{x_3 \overline{x}_3} \right) + \overrightarrow{G}_h^n,$$

where $\overrightarrow{G}_h^n = (lv_h^n, -lu_h^n, \lambda\theta_h^n)$ terms describing the influence of coriolis forces and temperature.

(2)

$$\frac{\overrightarrow{V}_h^{n+\frac{1}{2}} - \overrightarrow{V}_h^{n+\frac{1}{4}}}{\tau} = \nu \left((\overrightarrow{V}_h^{n+\frac{1}{2}})_{x_2 \overline{x}_2} - (\overrightarrow{V}_h^{n})_{x_2 \overline{x}_2} \right),$$

$$\frac{\overrightarrow{V}_h^{n+\frac{3}{4}} - \overrightarrow{V}_h^{n+\frac{1}{2}}}{\tau} = \nu \left((\overrightarrow{V}_h^{n+\frac{3}{4}})_{x_3 \overline{x}_3} - (\overrightarrow{V}_h^{n})_{x_3 \overline{x}_3} \right).$$

(3) Using the values of the found $\overrightarrow{V}_h^{n+\frac{3}{4}}$, we find the pressure value from the following equation

$$\frac{\overrightarrow{V}_h^{n+1} - \overrightarrow{V}_h^{n+\frac{3}{4}}}{\tau} = -\nabla(P_h^{n+1} - P_h^n).$$

(4) Based on the found values of the velocity fields, temperature transfer and diffusion are calculated as follows:

$$\frac{\theta_h^{n+1} - \theta_h^n}{\tau} = -L_h \theta_h^n + \Lambda \theta_h^n.$$

(5) The transformation of the proportion of harmful substances into impurities is calculated according to the following scheme:

$$\frac{\varphi_{q,h}^{n+1} - \varphi_{q,h}^n}{\tau} = -L_h \varphi_{q,h}^n + \Lambda \varphi_{q,h}^n + \alpha_q \varphi_{q,h}^n + \beta_q + f_q^n,$$

where $\varphi_{q,h}^n$ the proportion of the concentration of the harmful substance q, the coefficients α_q , β_q are determined from differential equations as in [30].

The following initial boundary conditions are met for the above schemes:

$$V = V^0(x_1, x_2, x_3), \quad \theta = \theta^0(x_1, x_2, x_3), \quad \varphi_q = \varphi_q^0(x_1, x_2, x_3) \text{ when } t = 0,$$

$$u_1 = u'_{11}(x_2, x_3, t), \quad u_2 = u'_{21}(x_2, x_3, t), \quad u_3 = 0, \quad \frac{\partial\theta}{\partial x_1} = 0, \quad \varphi_q = \varphi_q^0$$

$$\text{when } x_1 = 0, \quad 0 \le x_2 \le Y, \frac{\partial u_1}{\partial x_1} = 0, \quad u_2 = 0, \quad u_3 = 0,$$

$$\frac{\partial\theta}{\partial x_1} = 0, \quad \frac{\partial\varphi_q}{\partial x_1} = 0 \text{ when } x_1 = X, \quad 0 \le x_2 \le Y,$$

$$u_1 = u'_{12}(x, z, t), \quad u_2 = u'_{22}(x, z, t), \quad u_3 = 0, \quad \frac{\partial\theta}{\partial x_2} = 0, \quad \varphi_q = \varphi_q^0$$

$$\text{when } x_2 = 0, \quad 0 \le x_1 \le X, u_1 = 0, \quad \frac{\partial u_2}{\partial x_2} = 0, \quad u_3 = 0,$$

$$\frac{\partial\theta}{\partial x_2} = 0, \quad \frac{\partial\varphi_q}{\partial x_2} = 0 \text{ when } x_2 = Y, \quad 0 \le x_1 \le X,$$

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \quad \theta = 0, \quad p = 0, \quad \varphi_q = 0 \text{ when } x_3 = \eta,$$

$$u_3 = 0, \quad h\frac{\partial u_1}{\partial x_3} = a_{(u_1)}u_1, \quad h\frac{\partial u_2}{\partial x_3} = a_{(u_1)}u_2, \quad h\frac{\partial\theta}{\partial x_3} = a_\theta(\theta - \theta_0),$$

$$\varphi_{(q,0)} = \frac{(f_q + a_\theta\varphi_{(q,h)}\nu)}{(\beta + a_\theta\nu_h)} \text{ when } x_3 = h.$$

In this context, H represents the height of the atmospheric boundary layer, while X and Y denote the lateral boundaries of the region. $\varphi_{(q,0)}$ and $\varphi_{(q,h)}$ refer to the concentrations of matter q at the level of roughness and the surface layer, respectively. θ_0 signifies the temperature at the roughness level. The parameter $a_{(u_1)}$ emerges from the interaction between air flows and the friction of the underlying surface, calculated as $\frac{\psi_{(u_1)}(\zeta_h)}{\eta_{(u_1)}(\zeta_h,\zeta_0)}$. Similarly, a_θ denotes a turbulent heat exchange parameter, expressed as $\frac{\psi_\theta(\zeta_h)}{\eta_\theta(\zeta_h,\zeta_0)}$. β represents a velocity-based value characterizing the interaction between impurities and the underlying surface, while v_h and v_d stand for turbulence coefficients at the altitude $x_3 = h$ and $x_3 = d$, respectively.

5.1. **Air pollution modeling results in Ust-Kamenogorsk which is an industrial city.** The atmospheric air of Ust-Kamenogorsk is one of the most heavily polluted industrial cities in Kazakhstan [2]. To simulate the atmospheric air pollution of the city, the following parameters were considered: The size of the considered area was 35×35 kilometers, and the height of the surface layer was fixed and is 3500 meters, the convection parameter $\lambda = 0$, $16\text{m}/(\text{s} \cdot \text{deg})$, the parameter of S - stratification in the physical sense determines temperature changes with height, therefore, in calculations it was calculated using a vertical temperature gradient, and the Coriolis force $l = 1,4 \cdot 10^{-5}s^{-1}$.

The values of horizontal μ_x, μ_y and vertical ν coefficients of turbulent exchange were taken from $30\text{m}/\text{s}^2$ to $6 \cdot 10^3\text{m}/\text{s}^2$. The following numerical values were used as a characteristic scale of length, time, velocity, density, pressure difference, temperature, stratification: $L = 35000\text{m}$, $T = 86400\text{s}$, $U^* = 10\text{m}/\text{s}$, $\rho^* = 1225\text{g}/\text{m}^3$, $\pi^* = 3000\text{g}/(\text{m} \cdot \text{s}^2)$, $\theta^* = 20^0\text{C}$, $S = 3 \cdot 10^{-3}\text{deg}/\text{m}$. In the simulation, the chemicals specified in [33] were considered as harmful impurities. Figs.8, 9 shows the distribution of the chemical substances CO and SO_2.

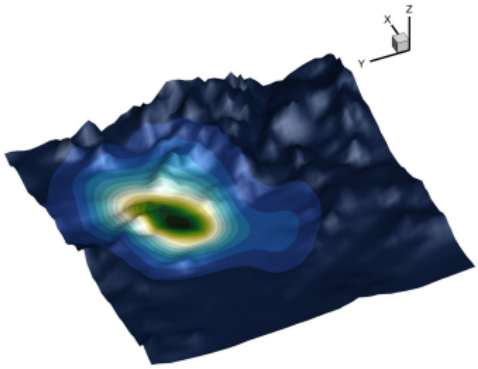


FIGURE 8. Distribution of substance CO in the atmospheric air of Ust-Kamenogorsk.

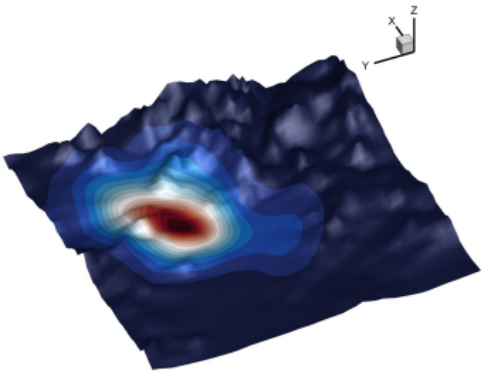


FIGURE 9. Distribution of substance SO_2 in the atmospheric air of Ust-Kamenogorsk.

The presented graphs (Figs.8,9) of the distribution of harmful substances in the atmosphere emphasize the need for emission control, focusing on air pollution problems and highlighting potential threats to the environment and human health, which emphasizes the importance of developing effective pollution management and reduction strategies.

6. Conclusion

This study thoroughly examines the stability issues associated with difference methods for solving equations governing the atmospheric boundary layer, particularly within the framework of the "large particles" method. The analysis focuses on a three-stage splitting scheme of physical processes, which modifies the classical splitting approach by incorporating implicit difference schemes in the first and second stages. The findings demonstrate that this method is highly effective for numerical implementation, offering a robust tool for obtaining a priori estimates of the second derivative of the velocity vector and pressure gradient, while ensuring the stability and convergence of the three-stage difference scheme.

It is important to highlight that in many previous studies, the numerical solution of the elliptic pressure equation using iterative methods often relied on Neumann boundary conditions. This paper proposes an alternative approach, defining auxiliary velocity fields at boundary nodes and introducing pressure equations, which appears to be a more accurate and physically justified method for the given problem.

Numerical simulations were conducted for a test problem involving convective flow within a rectangular cavity, and comparative results from computational experiments were presented.

To assess the effectiveness of the proposed approach, we performed a series of numerical experiments comparing the classical two-stage splitting scheme with the modified three-stage splitting scheme by physical processes. The comparative analysis of the obtained numerical results demonstrated the differences in solution accuracy, stability properties, and computational cost between the two methods, confirming the advantages of the proposed modification.

Additionally, the proposed scheme was applied to model the dispersion of harmful substances in the atmosphere of Ust-Kamenogorsk within the boundary layer model. The results emphasize the significance of the proposed approach in achieving more precise numerical modeling of atmospheric processes under boundary layer conditions.

7. Acknowledgments

This research is funded by the Science Committee of the Ministry of Higher Education and Science of the Republic of Kazakhstan (grant No. AP23490027 "Development of an application program sets for the numerical solution of Navier-Stokes equations in complex domains").

References

[1] Agarwal, R.P., Alghamdi, A.M., Gala, S., Ragusa, M.A. On the regularity for weak solutions to the micropolar fluid flows, *Appl. Comput. Math.*, Vol.23, No.4, 2024, pp.558-569.

[2] Assanov D., Radelyuk, I., Perederiy O., Galkin, S., Maratova, G., Zapasnyi, V., Klemeš, J.J. Spatiotemporal patterns of air pollution in an industrialised city: A case study of Ust-Kamenogorsk, Kazakhstan, *Atmosphere*, Vol.13, No.12, 2022, Article No. 1956.

[3] Belotserkovskii, O.M. *Numerical Modeling in Continuum Mechanics*, Moscow: Phiz.-mat. lit., 1994. 441 p. (in Russian).

[4] Cao, W.H., Rehman, S.U., Asjad, M.I., Inc, M. Numerical study of bioconvection and Cattaneo-Christov heat flux model in mhd maxwell nanofluid flow over a variable thickness elastic surface, *Appl. Comput. Math.*, Vol.23, No.2, 2024, pp.182-200.

[5] Chen, H., Sun, Sh., Zhang, T. Energy stability analysis of some fully discrete numerical schemes for incompressible Navier-Stokes equations on staggered grids, *J. Sci. Comput.*, Vol.75, No.1, 2017, pp.427-456.

[6] Cheng, J., Shu, C.W. High order schemes for CFD: A review, *Chin. J. Comput. Phys.*, Vol.26, No.5, 2009, pp.633-655.

[7] Danaev, N.T., Smagulov, S. Some methods for solving the Navier-Stokes equations for an incompressible fluid, *Preprint of NEA RK*, Almaty, 1995, 11 p. (in Russian).

[8] Danaev, N.T., Urmashev, B.A. Three-parameter iterative schemes for solving Navier-Stokes grid equations, *Proc. Int. Conf. RDAMM*, Vol.6, No.2, 2001, pp.253-257.

[9] Djoko, J., Koko, J. Numerical methods for the stokes and Navier-Stokes equations driven by threshold slip boundary conditions, *Comput. Methods Appl. Mech. Eng.*, Vol.305, 2016, pp.936-958.

[10] Harlow, F.H., Welch, J.E. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface, *Phys. Fluids*, Vol.8, No.12, 1965, pp.2182-2189.

[11] He, Y., Huang, P., Li, J. H2-stability of some second order fully discrete schemes for the Navier-Stokes equations, *Discrete Contin. Dyn. Syst. B*, Vol.24, No.6, 2019, pp.2745-2780.

[12] Kanschat, G. Divergence-free discontinuous Galerkin schemes for the stokes equations and the MAC scheme, *Int. J. Numer. Methods Fluids*, Vol.56, No.7, 2008, pp.941-950.

[13] Ladyzhenskaya, O.A. *Mathematical Problems of Dynamics of a Viscous Incompressible Liquid*, Moscow, Nauka, 1970, 288 p. (in Russian).

[14] Lebedev, V.L. Difference analogues of orthogonal decompositions, fundamental differential operators and certain boundary-value problems of mathematical physics, *USSR Comput. Math. Math. Physics,* Vol.4, No.3, 1964, pp.69-92.

[15] Li, J., Sun, S. The superconvergence phenomenon and proof of the MAC scheme for the Stokes equations on non-uniform rectangular meshes, *J. Sci. Comput.*, Vol.65, 2015, pp.341-362.

[16] Maussumbekova, S.D., Beketaeva, A.O. Numerical modeling of the atheroslerotic plaques effect in damaged vessels on coagulation dynamics, *Int. Rev. Model. Simul.*, Vol.15, No.5, 2022, pp.319-331.

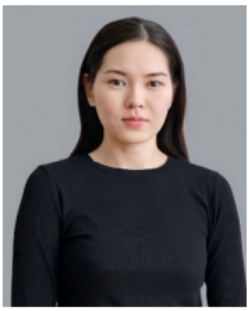
[17] Maussumbekova, S.D., Duysenali, A. Numerical simulation of boiling liquid outflows process, *Int. J. Math. Phys.*, Vol.12, No.1, 2021, pp.40-47.
[18] Minev, P.D. Remarks on the links between low-order DG methods and some finite-difference schemes for the stokes problem, *Int. J. Numer. Methods Fluids*, Vol.58, No.3, 2008, pp.307-317.
[19] Nicolaides, R.A., Wu, X. Analysis and convergence of the MAC scheme II, Navier-Stokes equations, *Math. Comput.*, Vol.65, No.213, 1996, pp.29-44.
[20] Nurjabova, D.S. Using numerical solution of nonlinear Navier-Stokes equations fluid model for blood vessel walls, *Harvard Educ. Sci. Rev.*, Vol.1, No.1, 2021, pp.96-106.
[21] Rui, H., Li, X. Stability and superconvergence of MAC scheme for Stokes equations on nonuniform grids, *SIAM J. Numer. Anal.*, Vol.55, No.3, 2017, pp.1135-1158.
[22] Sabzaliev, M.M., Sabzalieva, I.M., On a boundary value problem for a bisingularly perturbed quasilinear elliptic equation, *TWMS J. Pure Appl. Math.,* Vol.15, No.2, 2024, pp.318-326.
[23] Samelson, R., Temam, R., Wang, C., Wang, S. A fourth-order numerical method for the planetary geostrophic equations with inviscid geostrophic balance, *Numer. Math.*, Vol.107, 2007, pp.669-705.
[24] Samelson, R., Temam, R., Wang, C., Wang, S. Surface pressure Poisson equation formulation of the primitive equations: Numerical schemes, *SIAM J. Numer. Anal.*, Vol.41, No.3, 2003, pp.1163-1194.
[25] Sevdimaliyev, Y.M. The influence of imperfect contact conditions on the natural frequencies of a three-layer hollow sphere for high-order harmonics, *Appl. Comput. Math.,* Vol.22, No.4, 2023, pp.480-494
[26] Shah, K., Abdeljawad, T., Ahmad, I. On nonlinear fractional order coupled pantograph differential equations under nonlocal boundary conditions, *TWMS J. Pure Appl. Math.*, Vol.15, No.1, 2024, pp.65-78.
[27] Smagulov, Sh., Danaev, N.T., Temirbekov, N.M. Numerical solution of the Navier-Stokes equations with discontinuous coefficients, *Preprint of the CC SB of the USSR Acad. Sci.*, Krasnoyarsk, 1989, 21 p. (in Russian)
[28] Stephen, A.B. A finite difference Galerkin formulation for the incompressible Navier-Stokes equations, *J. Comput. Phys.*, Vol.53, No.1, 1984, pp.152-172.
[29] Temam, R. *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea Publishing, Vol.343, 1984, 408 p.
[30] Temirbekov, A.N., Danaev, N.T., Malgazhdarov, E.A. Modeling of pollutants in the atmosphere based on photochemical reactions, *Eurasian Chem. Technol. J.*, Vol.16, No.1, 2014, pp.61-71.
[31] Temirbekov, A.N., Temirbekova, L.N., Zhumagulov, B.T. Fictitious domain method with the idea of conjugate optimization for non-linear Navier-Stokes equations, *Appl. Comput. Math.*, Vol.22, No.2, 2023, pp.172-188.
[32] Temirbekov, A., Zhaksylykova, Z., Malgazhdarov, Y., Kasenov, S. Application of the fictitious domain method for Navier-Stokes equations, *Comput. Mater. Continua*, Vol.73, No.1, 2022, pp.2035-2055.
[33] Temirbekov, N., Malgazhdarov, Y., Tamabay, D., Temirbekov, A. Atmospheric modelling of photochemical transformations of pollutants: Impact of weather conditions and diurnal cycle, *Math. Model. Eng. Probl.*, Vol.10, No.5, 2023, pp.1699-1705.
[34] Temirbekov, N., Malgazhdarov, Y., Tamabay, D., Temirbekov, A. Mathematical and computer modeling of atmospheric air pollutants transformation with input data refinement, *Indones. J. Electr. Eng. Comput. Sci.*, Vol.32, No.3, 2023, pp.1405-1416.
[35] Weinen, E., Liu, J.-G. Gauge method for viscous incompressible flows, *Commun. Math. Sci.*, Vol.1, No.2, 2003, pp.317-332.
[36] Welch, J. E., Harlow, F.H., Shannon, J.P., Daly, B.J. The MAC *Method-A Computing Technique for Solving Viscous, Incompressible, Transient Fluid-Flow Problems Involving Free Surfaces,* Los Alamos Sci. Lab., Univ. of California, N. Mex., LA-3425, 1965, 150 p.
[37] Yücel, M., Mukhtarov, O.S. A new algorithm for solving two-linked boundary value problems with impulsive conditions, *TWMS J. Pure Appl. Math.*, Vol.15, No.2, 2024, pp.174-182.
[38] Zhang, Q., Huang, P. Improved defect-correction algorithms for the Navier-Stokes equations at small viscosity, *Phys. Fluids*, Vol.36, No.8, 2024, Articel No. 085136.



Bakytzhan Zhumagulov - is a Doctor, Professor, President of the National Engineering Academy of the Republic of Kazakhstan, President of the Mathematical Society of the Turkic World (TWMS), and the Kazakhstan Mathematical Society. He is an expert in computational mathematics, focusing on developing and applying information technologies, mathematical modeling, and solving hydrodynamics problems and challenges in the oil and gas industry.



Nurlan Temirbekov - is Doctor, Professor, Vice President of the National Engineering Academy of the Republic of Kazakhstan, academician of the National Engineering Academy of the Republic of Kazakhstan. He is engaged in research of the well-posedness of mathematical models of nonlinear physical processes, development of the difference schemes, and effective numerical algorithms.



Dinara Tamabay - is a doctoral student at Al-Farabi Kazakh National University. She received the B.Sc. and M.Sc. degrees, in 2018 and 2020, respectively. Her research interests include computational sciences, numerical analysis, computational fluid dynamics, and the investigation of atmospheric boundary layer models.