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A LINEARIZED STABILIZER FREE WEAK GALERKIN METHOD AND ITS OPTIMAL L^2 ERROR ESTIMATIONS FOR THE HIGH-FIELD MODEL

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Abstract. This paper concerns a linearized stabilizer-free weak Galerkin finite element method (SFWGFEM) for the two-dimensional high-field (HF) model. The proposed method not only offers the flexibility to approximate functions and is compatible with meshes including hanging nodes as WG-FEM, but also maintains a simple format similar to the traditional finite element method. Optimal error estimations are obtained in both L^2 and energy norms through the introduction of a weak Galerkin Ritz projection operator, where the associated Ritz projection error is derived by a dual problem. The superconvergent results between the Ritz projection of the exact function and the numerical solution in the energy norm lead to optimal L^2 estimations. Numerical experiments are conducted to demonstrate the theoretical analysis.

Keywords: High-Field Model, Stabilizer-Free, Weak Galerkin Finite Element Method, Ritz Projection, Optimal Error Estimations.

AMS Subject Classification: 35J20, 65N15, 65N30.

1. Introduction

The HF model describes the time evolution of concentrations and electric potential by neglecting the electron-electron interactions in the classical Boltzmann-Poisson problem and under the assumption of high electric field effects, which has important applications in semiconductor [8, 30].

The HF model is described as

$$n_t + \nabla \cdot \left(\left(\frac{3\tau\mu^2 q}{\varepsilon} n_d \nabla\varphi + \mu\nabla\varphi + \frac{\tau\mu q\boldsymbol{\omega}}{\varepsilon} \right) n - \frac{2\tau\mu^2 q}{\varepsilon} \nabla\varphi n^2 \right) - \nabla \cdot \big((\tau\theta + \tau\mu^2 \nabla\varphi \otimes \nabla\varphi) \nabla n \big) = 0, \tag{1}$$

$$-\frac{\varepsilon}{q}\Delta\varphi + n = n_d, \tag{2}$$

where $\otimes$ denotes a tensor product [7, 9, 10]. The unknown variables are the electron concentration n and the electric potential φ. By setting $C_1 = \dfrac{\tau\mu q}{\varepsilon}, C_2 = \dfrac{\tau\mu^2 q}{\varepsilon} = \mu C_1, \mathbf{C_3} = \dfrac{\tau\mu q\boldsymbol{\omega}}{\varepsilon} = \boldsymbol{\omega} C_1$, we rewrite (1)-(2) as

$$n_t + \nabla \cdot \big((3C_2 n_d \nabla\varphi + \mu\nabla\varphi + \mathbf{C_3})\, n - 2C_2 \nabla\varphi\, n^2 \big) - \nabla \cdot \big((\tau\theta + \tau\mu^2 \nabla\varphi \otimes \nabla\varphi) \nabla n \big) = 0, \tag{3}$$

$$-\frac{\varepsilon}{q}\Delta\varphi + n = n_d. \tag{4}$$

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where Ω is a polygonal domain in $\mathbb{R}^2$, $\boldsymbol{\omega} = \mu n \mathbf{E}$, and $\mathbf{E} = -\nabla\varphi$ represents the electric field. The unknown functions in the system are the electron concentration, denoted as n, and the electric potential, represented by φ. Mobility is symbolized by μ, and combined with the effective electron mass m and the electron charge q, it defines the relaxation parameter $\tau = \frac{m\mu}{q}$. The parameter $\theta = \frac{k}{m}T_0$ refers to the thermal energy, with k as the Boltzmann constant and T_0 as the lattice temperature. Other factors include dielectric permittivity ε and a specified doping function $n_d(x,t)$. The boundary and initial conditions are set by Dirichlet constraints for both n and φ on the boundary $\partial\Omega$ with given functions g_n and g_φ, and for n at the initial time within the domain with $n(\mathbf{x},0) = n_0(\mathbf{x})$.

Due to its coupled nonlinearities, nonstationarities, and complexities of tensor product, it is challenging to derive mathematical analytical solutions to the HF model. Numerical methods are therefore employed to explore the model. However, research on numerical computations and priori error estimations for the HF model is limited. In [9], the authors used the fifth-order weighted essentially non-oscillatory (WENO) schemes developed in [17] to simulate the one-dimensional HF model. Finite difference methods and other methods can be found in [1, 5, 6, 19, 20, 28, 31, 32, 35, 37].

The weak Galerkin finite element method (WG-FEM) is introduced in [33] to address the second-order elliptic problem. It employs weak differential operators to approximate functions with discontinuities across element boundaries and uses stabilizers to maintain weak continuity. The method supports polygonal or polyhedral meshes and discontinuous finite element functions, offering flexibility in mesh generation and function space selection. WG-FEM has since seen application in various problems [13, 16, 18, 23, 26, 29]. For the HF model, existing literature using WG-FEM was reviewed. In [25], the authors developed a backward Euler WG-FEM for the one-dimensional HF model without stabilizers, using elements P_k, P_0, P_{k+1} corresponding to interior, endpoints, and discrete weak gradient with $k \geq 0$. They established a priori error estimations that exhibit $O(\Delta t + h^{k+1})$ convergence rates in both L^2 and energy norms. However, generalizing the method mentioned above in [25] to multidimensional cases is challenging. The construction of stabilizer-free weak Galerkin finite element schemes, especially concerning the degree of the discrete weak gradient, correlates with the maximum number of element edges/faces in 2D/3D. This challenge elicits interest in adapting the WG-FEM without stabilizers to two-dimensional problems, leading to the development and examination of SFWG-FEM for the HF model.

The stabilizer-free weak Galerkin finite element method (SFWG-FEM), referenced in literature [38], is a variant of the WG-FEM. The method possesses not only the capability to generate approximation functions and flexibility in mesh generation, but also the size and global sparsity pattern of the stiffness matrix. In addition, the design without stabilizers simplifies the complexities encountered in analyzing and programming processes. Therefore, the SFWG-FEM has been applied to address many interesting problems [2, 11, 21, 22, 27, 39]. Different from the one-dimensional cases [25], numerical experiments for SFWG-FEM with $k \geq 1$ show that the errors between L^2 projection of the exact functions and the numerical solutions in energy norm achieve error rates $O(k)$. This phenomenon results in suboptimal L^2 priori error estimations for both unknown functions. To overcome the obstacle, we introduce a non-standard Ritz projection operator similar to our previous work [24] for the drift-diffusion model, called weak Galerkin Ritz projection, which plays a key role in deriving the optimal L^2 error estimations. The difference between the weak Galerkin Ritz projection operators arises from the varying models being addressed, with the operator in this article exhibiting a higher level of complexity than the one in the work [24] due to the involvement of the tensor product within the HF model. A rigorous proof of Ritz projection error is given by employing a dual problem, which is further intricate by the fact that the HF model contains a greater number and variety of nonlinear terms compared to the drift-diffusion model [24]. The numerical experiments illustrate the theoretical findings.

The structure of the paper is as follows: Section 2 introduces the semi-discrete SFWG finite element scheme for the HF model. The preliminaries, including the weak Galerkin Ritz projection along with its approximations, as well as the optimal error estimations in both the L^2 and energy norms are given. Section 3 delineates the fully discrete SFWG finite element scheme and establishes the optimal convergence rates. Section 4 presents a series of numerical experiments conducted to validate our theoretical analysis. Finally, Section 5 concludes and summarizes the findings of the paper.

2. Semi-discrete SFWG finite element method

Let the partition $\mathcal{T}_h$ satisfy the regularity assumption and consistency assumption postulated in Theorem 2.2 as referenced in [4] and assumptions A1-A4 in paper [34]. The collection of all edges is represented by $\mathcal{E}_h$, with the subset of interior edges designated as $\mathcal{E}_h^0$ and defined as $\mathcal{E}_h$ excluding the boundary $\partial\Omega$. The diameter of each polygonal element K within $\mathcal{T}_h$ is labeled as h_K, and we define the mesh size h as the maximum diameter across all such elements in $\mathcal{T}_h$.

Furthermore, we define $P_k(K)$ to represent the set of polynomial functions of up to degree k on each element K. Assuming k is an integer greater than or equal to 1, the corresponding weak Galerkin finite element space formed by the mesh $\mathcal{T}_h$ is denoted by:

$$V_h := \{v = \{v_0, v_b\} : v_0|_K \in P_k(K), v_b|_e \in P_k(e), K \in \mathcal{T}_h, e \subset \partial K\}.$$

The subsets $V_h^0, V_h^n, V_h^\varphi$ of the weak Galerkin finite element space V_h are represented by:

$$\begin{aligned} V_h^0 &:= \{v \in V_h : v_b|_{\partial\Omega} = 0\}, \\ V_h^n &:= \{v \in V_h : v_b|_{\partial\Omega} = g_n\}, \\ V_h^\varphi &:= \{v \in V_h : v_b|_{\partial\Omega} = g_\varphi\}. \end{aligned}$$

Definition 1. (The discrete weak gradient) For any $v = \{v_0, v_b\}$ within the space V_h, the discrete weak gradient, denoted as $\nabla_w v$ and belonging to $[P_j(K)]^2$, satisfies

$$(\nabla_w v, \mathbf{q})_K = -(v_0, \nabla \cdot \mathbf{q})_K + \langle v_b, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial K}, \quad \forall\, \mathbf{q} \in [P_j(K)]^2,$$

where $\mathbf{n}$ represents the unit outward normal vector along the edge of ∂K.

For the sake of clarity and simplicity, we employ a set of simplified notations throughout this paper as follows.

$$(v, w) = \sum_{K\in\mathcal{T}_h} (v, w)_K = \sum_{K\in\mathcal{T}_h} \int_K v\, w\, dx, \quad \langle v, w\rangle = \sum_{K\in\mathcal{T}_h} \langle v, w\rangle_{\partial K} = \sum_{K\in\mathcal{T}_h} \int_{\partial K} v\, w\, ds.$$

2.1. Semi-discrete scheme.

The semi-discrete method for the HF model detailed in Eqns. (3)-(4) seeks to solve for (n_h, φ_h) within the function spaces $V_h^n \times V_h^\varphi$, satisfying

$$\begin{aligned} &((n_{h0})_t, v_0) - \left((3C_2 n_d \nabla_w\varphi_h + \mu \nabla_w\varphi_h + \mathbf{C_3})\, n_{h0} - 2C_2 \nabla_w\varphi_h\, n_{h0}^2, \nabla_w v\right) \\ &\quad + ((\tau\theta + \tau\mu^2 \nabla_w\varphi_h \otimes \nabla_w\varphi_h) \nabla_w n_h, \nabla_w v) = 0, \qquad \forall\, v \in V_{h0}, \quad (5) \\ &\frac{\varepsilon}{q}(\nabla_w\varphi_h, \nabla_w w) + (n_{h0}, w_0) = (n_d, w_0), \qquad \forall\, w \in V_{h0}. \quad (6) \end{aligned}$$

Concomitantly, we introduce an energy norm $|||\cdot|||$ applicable to any element v from the space V_h, expressed as:

$$|||v|||^2 = (\nabla_w v, \nabla_w v).$$

In addition, we define a discrete H^1 norm given by:

$$\|v\|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} \left(\|\nabla v_0\|_K^2 + h_K^{-1} \|v_0 - v_b\|_{\partial K}^2\right).$$

By Lemma 3.1 as seen in [38], we designate $j_0 = k + m - 2$ or alternatively, $j_0 = k + m - 3$ if each side of element K possesses a parallel counterpart, where m represents the number of edges of element K. Also, let $deg\nabla_w v$ symbolize the degree of the discrete weak gradient $\nabla_w v$ such that $deg\nabla_w v = j \geq j_0$. Consequently, the energy norm and H^1 norm are equivalent, i.e., there exist two constants $C_1, C_2 \geq 0$, such that for each $v \in V_h$,

$$C_1 \|v\|_{1,h} \leq |||v||| \leq C_2 \|v\|_{1,h}.$$

Remark 1. The condition $j_0 = k + m - 2$ or $j_0 = k + m - 3$ is a sufficient condition to eliminate the stabilizers for the SFWG-FEM. The numerical tests, especially for two-dimensional polygon meshes with a minimum of four edges, indicate that the degree of the discrete weak gradient within the SFWG-FEM is generally less than j_0.

2.2. **Preliminaries.**

Definition 2. For each element $K \in \mathcal{T}_h$, denote the L^2 projections onto the associated local polynomial spaces by:

$$\begin{aligned} &Q_0 : L^2(K) \to P_k(K), \\ &Q_b : L^2(e) \to P_k(e), \\ &\mathbb{Q}_h : [L^2(K)]^2 \to [P_j(K)]^2. \end{aligned}$$

Define Q_h by $Q_h u = \{Q_0 u, Q_b u\} \in V_h$ for $u \in L^2(\Omega)$.

Lemma 1. *(cf. [38] Lemma 2.1) Let $u \in H^1(\Omega)$, then for any $K \in \mathcal{T}_h$, there is $\nabla_w u = \mathbb{Q}_h(\nabla u)$.*

Remark 2. The above equation signifies that for any function $u \in H^1(\Omega)$, the term $\nabla_w u$ serves as an approximation of the conventional gradient ∇u.

Lemma 2. *Let $u \in H^1(\Omega)$ and $A \in L^2(\Omega) \cup [L^2(\Omega)]^{2\times 2}$, then for all $v \in V_h$, there is*

$$(A\nabla u, \nabla v_0) = (A\nabla_w(Q_h u), \nabla_w v) + \langle \mathbb{Q}_h(A\nabla u) \cdot \mathbf{n}, v_0 - v_b \rangle + (\mathbb{Q}_h(A\nabla u) - A\nabla_w Q_h u, \nabla_w v).$$

Proof. For any $\mathbf{q} \in [P_j(K)]^2$, there holds

$$\begin{aligned} (\mathbf{q}, \nabla v_0)_K &= -(\nabla \cdot \mathbf{q}, v_0)_K + \langle \mathbf{q} \cdot \mathbf{n}, v_0 \rangle_{\partial K}, \\ (\mathbf{q}, \nabla_w v)_K &= -(\nabla \cdot \mathbf{q}, v_0)_K + \langle \mathbf{q} \cdot \mathbf{n}, v_b \rangle_{\partial K}. \end{aligned}$$

Subtracting the above two equations yields

$$(\mathbf{q}, \nabla v_0)_K = (\mathbf{q}, \nabla_w v)_K + \langle \mathbf{q} \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial K}.$$

We notice the fact that $\nabla v_0|_K \in [P_{k-1}(K)]^2 \subset [P_{k+m-3}(K)]^2 \subseteq [P_{j_0}(K)]^2$ with $m \geq 3$ in 2D. In addition, by the definition of L^2 projection operator $\mathbb{Q}_h$ and letting $\mathbf{q} = \mathbb{Q}_h(A\nabla u)$, the above identity arrives at

$$\begin{aligned} &(A\nabla u, \nabla v_0)_K \\ &= (\mathbb{Q}_h(A\nabla u), \nabla v_0)_K \\ &= (\mathbb{Q}_h(A\nabla u), \nabla_w v)_K + \langle \mathbb{Q}_h(A\nabla u) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial K} \\ &= (A\nabla_w Q_h u, \nabla_w v)_K + (\mathbb{Q}_h(A\nabla u) - A\nabla_w Q_h u, \nabla_w v)_K + \langle \mathbb{Q}_h(A\nabla u) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial K}. \end{aligned}$$

Summing over all domain Ω, the proof of the lemma is completed. □

Lemma 3. *Let $\boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^2$ and $u \in H^1(\Omega)$. Then for all $v \in V_h$, there is*

$$(\boldsymbol{\beta} u, \nabla v_0) = (\boldsymbol{\beta} u, \nabla_w v) + \langle \mathbb{Q}_h(\boldsymbol{\beta} u) \cdot \mathbf{n}, v_0 - v_b \rangle.$$

Proof. Using integration by parts and definition of the discrete weak gradient, we obtain

$$\begin{aligned} &(\boldsymbol{\beta} u, \nabla v_0)_K \\ &\quad = (\mathbb{Q}_h(\boldsymbol{\beta} u), \nabla v_0)_K \end{aligned}$$

$$
\begin{aligned}
&= -(v_0, \nabla\cdot(\mathbb{Q}_h(\boldsymbol{\beta}\, u)))_K + \langle v_0, \mathbb{Q}_h(\boldsymbol{\beta}\, u)\cdot\mathbf{n}\rangle_{\partial K}\\
&= -(v_0, \nabla\cdot(\mathbb{Q}_h(\boldsymbol{\beta}\, u)))_K + \langle v_b, \mathbb{Q}_h(\boldsymbol{\beta}\, u)\cdot\mathbf{n}\rangle_{\partial K} + \langle v_0 - v_b, \mathbb{Q}_h(\boldsymbol{\beta}\, u)\cdot\mathbf{n}\rangle_{\partial K}\\
&= (\mathbb{Q}_h(\boldsymbol{\beta}\, u), \nabla_w v)_K + \langle \mathbb{Q}_h(\boldsymbol{\beta}\, u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial K}\\
&= (\boldsymbol{\beta}\, u, \nabla_w v)_K + \langle \mathbb{Q}_h(\boldsymbol{\beta}\, u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial K}. \qquad \square
\end{aligned}
$$

Summing over all domain Ω, the proof of the lemma is completed.

The L^2 projection operators Q_h and $\mathbb{Q}_h$ are known to comply with the optimal approximation properties as validated by references such as [33] and Lemma 2.5 in [2]. Moreover, it is established that the following inequality, referenced in Lemma 9 from [3], is satisfied.

$$\sum_{K\in\mathcal{T}_h} \|\mathbb{Q}_h(\nabla u) - \nabla_w Q_h u\|_K^2 \le C h^{2k} \|u\|_{k+1}^2. \tag{7}$$

For any $K \in \mathcal{T}_h$, edge $e \subset \partial K$ and function $\phi \in H^1(K)$, the following trace inequality holds true (cf. [34] Lemma A.3)

$$\|\phi\|_e^2 \le C\left(h_K^{-1}\|\phi\|_K^2 + h_K\|\nabla\phi\|_K^2\right). \tag{8}$$

The discrete Poincare inequality is established in Section 4.2 Remark 4.15 as noted in [12] and Lemma 1 in [3]. Given $k > 0$, there exists a constant $\sigma > 0$, which is independent of the mesh parameter h, asserting that for any $v \in V_h$, the following inequality holds:

$$\|v_0\|^2 \leqslant \sigma\left(\sum_{K\in\mathcal{T}_h}\|\nabla v_0\|_K^2 + \sum_{e\in\mathcal{E}_h}\frac{1}{h_e}\|[v_0]\|_e^2\right),$$

where the jump for a weak function $v \in V_h$ on e, a common edge of elements K_1 and K_2, is defined as

$$[v]_e = \begin{cases} v|_{\partial K_1}\mathbf{n}_1 + v|_{\partial K_2}\mathbf{n}_2, & e \in \mathcal{E}_h^0,\\ v\,\mathbf{n}, & e \in \partial\Omega. \end{cases}$$

The $\mathbf{n}_1$ and $\mathbf{n}_2$ represent the unit outward normal vectors on the common edge e, corresponding to the elements K_1 and K_2, respectively. Furthermore, Lemma 2 in [3] asserts that if $\mathcal{T}_h$ meets shape regularity conditions, then a constant $C > 0$ exists such that for each $K \in \mathcal{T}_h$:

$$\sum_{e\in\mathcal{E}_h}\frac{1}{h_e}\|[v_0]\|_e^2 \le C\sum_{K\in\mathcal{T}_h}\frac{1}{h_K}\|v_0 - v_b\|_{\partial K}^2.$$

These points lead to the establishment of the following lemma.

Lemma 4. *By the discrete Poincare inequality and the equivalence between norms $\|\cdot\|_{1,h}$ and $|||\cdot|||$, there holds*

$$\|v_0\|^2 \le C_\sigma\|v\|_{1,h}^2 \le (C_\sigma^*)^2\,|||v|||^2,$$

where $C_\sigma^* = \sqrt{\frac{C_\sigma}{C_1}}$ and C_σ are constants about σ.

To derive optimal L^2 error estimations, we define a non-standard Ritz projection operator, called weak Galerkin Ritz projection, $R_h = \{R_0, R_b\} : H^1(\Omega) \to V_h$ satisfying

$$
\begin{aligned}
&-\left((3C_2 n_d\,\nabla_w R_h\varphi + \mu\,\nabla_w R_h\varphi + \mathbf{C_3})\,n - 2C_2\,\nabla_w R_h\varphi\, n^2, \nabla_w v\right)\\
&\quad + \left((\tau\theta + \tau\mu^2\,\nabla_w R_h\varphi\otimes\nabla_w R_h\varphi)\,\nabla_w R_h n, \nabla_w v\right)\\
&\quad = \left(\nabla\cdot\left((3C_2 n_d\,\nabla\varphi + \mu\nabla\varphi + \mathbf{C_3})\,n - 2C_2\,\nabla\varphi\, n^2\right), v_0\right)\\
&\quad - \left(\nabla\cdot\left(\left(\tau\theta + \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\right)\,\nabla n\right), v_0\right), \qquad \forall\, v \in V_h^0, \qquad (9)\\
&\frac{\varepsilon}{q}(\nabla_w R_h\varphi, \nabla_w w) + (R_0 n, w_0) = (-\frac{\varepsilon}{q}\,\Delta\varphi + n, w_0), \qquad \forall\, w \in V_h^0. \qquad (10)
\end{aligned}
$$

Lemma 5. *Assume that the exact solutions to the HF model referred to in (3)–(4) are regular and adhere to the conditions $n, \varphi \in H^1(0,T; H^{k+1}(\Omega) \cap W^{1,\infty}(\Omega))$, there exists a constant C independent of mesh size h such that*

$$\|Q_0 s - R_0 s\| + h \,|||Q_h s - R_h s||| \le C h^{k+1}(\|n\|_{k+1} + \|\varphi\|_{k+1}), \qquad s = n, \varphi.$$

Proof. From the definition of weak Galerkin Ritz projection (9)–(10) and HF model (3)–(4), we get

$$\begin{aligned}
&-\left((3C_2 n_d \nabla_w R_h\varphi + \mu \nabla_w R_h\varphi + \mathbf{C_3})\, n - 2C_2 \nabla_w R_h\varphi\, n^2, \nabla_w v\right) \\
&\qquad + \left((\tau\theta + \tau\mu^2 \nabla_w R_h\varphi \otimes \nabla_w R_h\varphi)\, \nabla_w R_h n, \nabla_w v\right) = -\left(n_t, v_0\right), \qquad \forall\, v \in V_h^0, \qquad (11)\\
&\frac{\varepsilon}{q}(\nabla_w(R_h\varphi), \nabla_w w) + (R_0 n, w_0) = (n_d, w_0), \qquad \forall\, w \in V_h^0. \qquad (12)
\end{aligned}$$

Testing HF model (3)-(4) by $v, w \in V_h^0$ and using integration by parts it yields

$$\begin{aligned}
&(n_t, v_0) - \left((3C_2 n_d \nabla\varphi + \mu\nabla\varphi + \mathbf{C_3})\, n - 2C_2 \nabla\varphi\, n^2, \nabla v_0\right) + \left((\tau\theta + \tau\mu^2 \nabla\varphi \otimes \nabla\varphi)\, \nabla n, \nabla v_0\right)\\
&\quad + \left\langle \left((3C_2 n_d \nabla\varphi + \mu\nabla\varphi + \mathbf{C_3})\, n - 2C_2 \nabla\varphi\, n^2\right)\cdot \mathbf{n}, v_0\right\rangle\\
&- \left\langle (\tau\theta + \tau\mu^2 \nabla\varphi \otimes \nabla\varphi)\, \nabla n \cdot \mathbf{n}, v_0\right\rangle = 0, \qquad (13)\\
&\frac{\varepsilon}{q}(\nabla\varphi, \nabla w_0) - \frac{\varepsilon}{q}\left\langle \nabla\varphi \cdot \mathbf{n}, w_0\right\rangle + (Q_0 n, w_0) = (n_d, w_0), \qquad (14)
\end{aligned}$$

where the fact $(n, w_0) = (Q_0 n, w_0)$ for any $w \in V_h^0$ is utilized. Subtracting system (11)-(12) from (13)-(14) arrives at

$$\begin{aligned}
&-\left((3C_2 n_d \nabla\varphi + \mu\nabla\varphi + \mathbf{C_3})\, n - 2C_2 \nabla\varphi\, n^2, \nabla v_0\right)\\
&\quad + \left((3C_2 n_d \nabla_w R_h\varphi + \mu \nabla_w R_h\varphi + \mathbf{C_3})\, n - 2C_2 \nabla_w R_h\varphi\, n^2, \nabla_w v\right)\\
&\quad + \left\langle \left((3C_2 n_d \nabla\varphi + \mu\nabla\varphi + \mathbf{C_3})\, n - 2C_2 \nabla\varphi\, n^2\right)\cdot \mathbf{n}, v_0\right\rangle\\
&\quad + \left((\tau\theta + \tau\mu^2 \nabla\varphi \otimes \nabla\varphi)\, \nabla n, \nabla v_0\right) - \left((\tau\theta + \tau\mu^2 \nabla_w R_h\varphi \otimes \nabla_w R_h\varphi)\, \nabla_w R_h n, \nabla_w v\right)\\
&\quad - \left\langle (\tau\theta + \tau\mu^2 \nabla\varphi \otimes \nabla\varphi)\, \nabla n \cdot \mathbf{n}, v_0\right\rangle = 0, \qquad (15)\\
&\frac{\varepsilon}{q}\left[(\nabla\varphi, \nabla w_0) - (\nabla_w(R_h\varphi), \nabla_w w)\right] - \frac{\varepsilon}{q}\left\langle \nabla\varphi \cdot \mathbf{n}, w_0\right\rangle + (Q_0 n - R_0 n, w_0) = 0. \qquad (16)
\end{aligned}$$

By Lemma 2, there is

$$(\nabla\varphi, \nabla w_0) = (\nabla_w Q_h\varphi, \nabla_w w) + \langle \mathbb{Q}_h \nabla\varphi \cdot \mathbf{n}, w_0 - w_b\rangle + \left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi, \nabla_w w\right).$$

Denote $s - R_h s = (s - Q_h s) + (Q_h s - R_h s) := \rho_s + \eta_s$ with $s = n, \varphi$. Substituting the above identity into (16) leads to

$$\begin{aligned}
&\frac{\varepsilon}{q}(\nabla_w \eta_\varphi, \nabla_w w) + \frac{\varepsilon}{q}\langle (\mathbb{Q}_h\nabla\varphi - \nabla\varphi)\cdot \mathbf{n}, w_0 - w_b\rangle\\
&+ \frac{\varepsilon}{q}\left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi, \nabla_w w\right) + (\eta_{n0}, w_0) = 0, \qquad (17)
\end{aligned}$$

in which the fact $\langle \nabla\varphi \cdot \mathbf{n}, w_b\rangle = 0$ is applied.

First of all, we bound the energy norm between L^2 projection Q_h and Ritz projection R_h, i.e., $|||\eta_\varphi|||$. We let $w = \eta_\varphi$ in (17) and get

$$\begin{aligned}
&\frac{\varepsilon}{q}|||\eta_\varphi|||^2 \\
&= -\frac{\varepsilon}{q}\langle(\mathbb{Q}_h\nabla\varphi - \nabla\varphi)\cdot\mathbf{n}, \eta_{\varphi 0} - \eta_{\varphi b}\rangle - \frac{\varepsilon}{q}\left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi, \nabla_w\eta_\varphi\right) - (\eta_{n0}, \eta_{\varphi 0}) \\
&\le \frac{\varepsilon}{q}\left(\sum_{K\in\mathcal{T}_h} h_K \left\|\mathbb{Q}_h\nabla\varphi - \nabla\varphi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|\eta_{\varphi 0} - \eta_{\varphi b}\right\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\quad + \frac{\varepsilon}{q}\left\|\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\right\| |||\eta_\varphi||| + \|\eta_{n0}\| \|\eta_{\varphi 0}\| \\
&\le \left(Ch^k \|\varphi\|_{k+1} + C_\sigma^* \|\eta_{n0}\|\right)|||\eta_\varphi|||,
\end{aligned}$$

which implies

$$\frac{\varepsilon}{q}|||\eta_\varphi||| \le Ch^k \|\varphi\|_{k+1} + C_\sigma^* \|\eta_{n0}\| \le Ch^k \|\varphi\|_{k+1} + (C_\sigma^*)^2|||\eta_n|||, \tag{18}$$

where the trace inequality (8), approximation properties of L^2 projection operators, inequality (7) and Lemma 4 are used.

Next, we estimate the term $|||\eta_n|||$. It follows from Lemma 2, Lemma 3 and definition of the discrete weak gradient that the equation (15) can be transformed as:

$$\begin{aligned}
&-\left(\left(3C_2 n_d + \mu\right)\nabla\varphi\, n, \nabla v_0\right) \\
&\quad = -\left(\left(3C_2 n_d + \mu\right) n\, \nabla_w Q_h\varphi, \nabla_w v\right) - \langle\mathbb{Q}_h\left(\left(3C_2 n_d + \mu\right) n\, \nabla\varphi\right)\cdot\mathbf{n}, v_0 - v_b\rangle \\
&\qquad - \left(\mathbb{Q}_h\left(\left(3C_2 n_d + \mu\right) n\, \nabla\varphi\right) - \left(3C_2 n_d + \mu\right) n\, \nabla_w Q_h\varphi, \nabla_w v\right), \\
&-\left(\mathbf{C_3}\, n, \nabla v_0\right) \\
&\quad = -(\boldsymbol{C_3}\, n, \nabla_w v) - \langle\mathbb{Q}_h(\boldsymbol{C_3}\, n)\cdot\mathbf{n}, v_0 - v_b\rangle, \\
&\left(2C_2\nabla\varphi\, n^2, \nabla v_0\right) \\
&\quad = \left(2C_2 n^2\nabla_w Q_h\varphi, \nabla_w v\right) + \langle\mathbb{Q}_h(2C_2 n^2\nabla\varphi)\cdot\mathbf{n}, v_0 - v_b\rangle + \left(\mathbb{Q}_h(2C_2 n^2\nabla\varphi) - 2C_2 n^2\nabla_w Q_h\varphi, \nabla_w v\right), \\
&\left(\tau\theta\,\nabla n, \nabla v_0\right) \\
&\quad = \left(\tau\theta\,\nabla_w Q_h n, \nabla_w v\right) + \langle\mathbb{Q}_h(\tau\theta\,\nabla n)\cdot\mathbf{n}, v_0 - v_b\rangle + \left(\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla_w Q_h n, \nabla_w v\right), \\
&\left(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n, \nabla v_0\right) \\
&\quad = \left(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h n, \nabla_w v\right) + \langle\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n)\cdot\mathbf{n}, v_0 - v_b\rangle \\
&\qquad + \left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h n, \nabla_w v\right).
\end{aligned}$$

Substituting the above identities into equation (15), it yields

$$\begin{aligned}
&-\left(\left(3C_2 n_d + \mu\right) n\,\nabla_w\eta_\varphi, \nabla_w v\right) - \langle\left(\mathbb{Q}_h\left(\left(3C_2 n_d + \mu\right) n\,\nabla\varphi\right) - \left(3C_2 n_d + \mu\right) n\,\nabla\varphi\right)\cdot\mathbf{n}, v_0 - v_b\rangle \\
&\quad - \left(\mathbb{Q}_h\left(\left(3C_2 n_d + \mu\right) n\,\nabla\varphi\right) - \left(3C_2 n_d + \mu\right) n\,\nabla_w Q_h\varphi, \nabla_w v\right) \\
&\quad - \langle\left(\mathbb{Q}_h(\boldsymbol{C_3}\, n) - \boldsymbol{C_3}\, n\right)\cdot\mathbf{n}, v_0 - v_b\rangle + \left(2C_2 n^2\,\nabla_w\eta_\varphi, \nabla_w v\right) \\
&\quad + \langle\left(\mathbb{Q}_h(2C_2 n^2\,\nabla\varphi) - 2C_2 n^2\,\nabla\varphi\right)\cdot\mathbf{n}, v_0 - v_b\rangle + \left(\mathbb{Q}_h(2C_2 n^2\,\nabla\varphi) - 2C_2 n^2\,\nabla_w Q_h\varphi, \nabla_w v\right) \\
&\quad + \left(\tau\theta\,\nabla_w\eta_n, \nabla_w v\right) + \langle\left(\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla n\right)\cdot\mathbf{n}, v_0 - v_b\rangle \\
&\quad + \left(\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla_w Q_h n, \nabla_w v\right) + \left(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w\eta_n, \nabla_w v\right) \\
&\quad + \left(\tau\mu^2\,\left(\left(\nabla\varphi - \mathbb{Q}_h\nabla\varphi\right) + \left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\right) + \nabla_w\eta_\varphi\right)\otimes\nabla\varphi\,\nabla_w R_h n, \nabla_w v\right) \\
&\quad + \left(\tau\mu^2\,\nabla_w R_h\varphi\otimes\left(\left(\nabla\varphi - \mathbb{Q}_h\nabla\varphi\right) + \left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\right) + \nabla_w\eta_\varphi\right)\,\nabla_w R_h n, \nabla_w v\right) \\
&\quad + \langle\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n\right)\cdot\mathbf{n}, v_0 - v_b\rangle
\end{aligned}$$

$$+\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n)-\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h n,\nabla_w v\right)=0, \tag{19}$$

where the facts $\langle(3C_2n_d+\mu)\,n\,\nabla\varphi\cdot\mathbf{n},v_b\rangle=0$, $\langle\boldsymbol{C_3}\,n\cdot\mathbf{n},v_b\rangle=0$, $\langle 2C_2n^2\,\nabla\varphi\cdot\mathbf{n},v_b\rangle=0$, $\langle\tau\theta\,\nabla n\cdot\mathbf{n},v_b\rangle=0$ and $\langle\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n\cdot\mathbf{n},v_b\rangle=0$ are used.

We take $v=\eta_n$ in (19). Because, matrix $\nabla\varphi\otimes\nabla\varphi$ is symmetric semi-positive definite and $\tau\geq 0$, the term $\left(\tau\mu^2\nabla\varphi\otimes\nabla\varphi\nabla_w\eta_n,\nabla_w\eta_n\right)\geq 0$. It can be shown that

$$\begin{aligned}
&\tau\theta\,|||\eta_n|||^2\\
\leq&\tau\theta\,|||\eta_n|||^2+(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w\eta_n,\nabla_w\eta_n)\\
=&((3C_2n_d+\mu)\,n\,\nabla_w\eta_\varphi,\nabla_w\eta_n)+\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\varphi)-(3C_2n_d+\mu)\,n\,\nabla\varphi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&+(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\varphi)-(3C_2n_d+\mu)\,n\,\nabla_w Q_h\varphi,\nabla_w\eta_n)\\
&+\langle(\mathbb{Q}_h(\boldsymbol{C_3}\,n)-\boldsymbol{C_3}\,n)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle-\left(2C_2n^2\,\nabla_w\eta_\varphi,\nabla_w\eta_n\right)\\
&-\langle\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi)-2C_2n^2\,\nabla\varphi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle-\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi)-2C_2n^2\,\nabla_w Q_h\varphi,\nabla_w\eta_n\right)\\
&-\langle(\mathbb{Q}_h(\tau\theta\,\nabla n)-\tau\theta\,\nabla n)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle-(\mathbb{Q}_h(\tau\theta\,\nabla n)-\tau\theta\,\nabla_w Q_h n,\nabla_w\eta_n)\\
&-\left(\tau\mu^2\,((\nabla\varphi-\mathbb{Q}_h\nabla\varphi)+(\mathbb{Q}_h\nabla\varphi-\nabla_w Q_h\varphi)+\nabla_w\eta_\varphi)\otimes\nabla\varphi\,\nabla_w R_h n,\nabla_w\eta_n\right)\\
&-\left(\tau\mu^2\,\nabla_w R_h\varphi\otimes((\nabla\varphi-\mathbb{Q}_h\nabla\varphi)+(\mathbb{Q}_h\nabla\varphi-\nabla_w Q_h\varphi)+\nabla_w\eta_\varphi)\,\nabla_w R_h n,\nabla_w\eta_n\right)\\
&-\langle\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n)-\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&-\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n)-\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h n,\nabla_w\eta_n\right)\\
=&\sum_{i=1}^{17}\mathcal{I}_i.
\end{aligned}\tag{20}$$

Next, we bound the right-hand terms of the equation (20) one by one. With the help of Hölder inequality, we obtain

$$\begin{aligned}
|\mathcal{I}_1|&=|((3C_2n_d+\mu)\,n\,\nabla_w\eta_\varphi,\nabla_w\eta_n)|\\
&\leq(3\,\|C_2\|_\infty\,\|n_d\|_\infty+\|\mu\|_\infty)\,\|n\|_\infty\,|||\eta_\varphi|||\cdot|||\eta_n|||.
\end{aligned}\tag{21}$$

It follows from the trace inequality (8) the approximation property of projection operator $\mathbb{Q}_h$, the definition of norm $\|\cdot\|_{1,h}$ and Lemma (4) that

$$\begin{aligned}
|\mathcal{I}_2|&=|\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\varphi)-(3C_2n_d+\mu)\,n\,\nabla\varphi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle|\\
&\leq(3\,\|C_2\|_\infty\,\|n_d\|_\infty+\|\mu\|_\infty)\,\|n\|_\infty\left(\sum_{K\in\mathcal{T}_h}h_K\,\|\mathbb{Q}_h\nabla\varphi-\nabla\varphi\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\times\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\,\|\eta_{n0}-\eta_{nb}\|_{\partial K}^2\right)^{\frac{1}{2}}\leq Ch^k\,\|\varphi\|_{k+1}\,|||\eta_n|||,
\end{aligned}\tag{22}$$

$$\begin{aligned}
|\mathcal{I}_3|&=|(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\varphi)-(3C_2n_d+\mu)\,n\,\nabla_w Q_h\varphi,\nabla_w\eta_n)|\\
&\leq(3\,\|C_2\|_\infty\,\|n_d\|_\infty+\|\mu\|_\infty)\,\|n\|_\infty\,\|\mathbb{Q}_h\nabla\varphi-\nabla\varphi\|\,|||\eta_n|||\\
&\leq Ch^k\,\|\varphi\|_{k+1}\,|||\eta_n|||,
\end{aligned}\tag{23}$$

$$\begin{aligned}
|\mathcal{I}_4|&=|\langle(\mathbb{Q}_h(\boldsymbol{C_3}\,n)-\boldsymbol{C_3}\,n)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle|\\
&\leq\|\boldsymbol{C_3}\|_\infty\left(\sum_{K\in\mathcal{T}_h}h_K\,\|\mathbb{Q}_h n-n\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\,\|\eta_{n0}-\eta_{nb}\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\leq Ch^{k+1}\,\|n\|_{k+1}\,|||\eta_n|||.
\end{aligned}\tag{24}$$

An application of Hölder inequality implies that

$$\begin{aligned}|\mathcal{I}_5| &= \left|\left(2C_2n^2\,\nabla_w\eta_\varphi, \nabla_w\eta_n\right)\right| \\ &\le 2\left\|C_2\right\|_\infty \|n\|_\infty^2 \;|||\eta_\varphi||| \,|||\eta_n|||. \end{aligned} \tag{25}$$

With the help of approximation property and inequality (7), it can be shown that

$$\begin{aligned}|\mathcal{I}_6| &= \left|\langle\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi) - 2C_2n^2\,\nabla\varphi\right)\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle\right| \\ &\le 2\left\|C_2\right\|_\infty \|n\|_\infty^2 \;\left(\sum_{K\in\mathcal{T}_h} h_K\left\|\mathbb{Q}_h(\nabla\varphi)-\nabla\varphi\right\|_{\partial K}^2\right)^{\frac{1}{2}} \\ &\quad\times\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|\eta_{n0}-\eta_{nb}\right\|_{\partial K}^2\right)^{\frac{1}{2}} \le Ch^k\left\|\varphi\right\|_{k+1}|||\eta_n|||, \end{aligned} \tag{26}$$

$$\begin{aligned}|\mathcal{I}_7| &= \left|\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi) - 2C_2n^2\,\nabla_w Q_h\varphi, \nabla_w\eta_n\right)\right| \\ &\le 2\left\|C_2\right\|_\infty \|n\|_\infty^2 \;\left\|\mathbb{Q}_h(\nabla\varphi)-\nabla_w Q_h\varphi\right\| \,|||\eta_n||| \\ &\le Ch^k\left\|\varphi\right\|_{k+1}|||\eta_n|||, \end{aligned} \tag{27}$$

$$\begin{aligned}|\mathcal{I}_8| &= \left|\langle\left(\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla n\right)\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle\right| \\ &\le \left(\sum_{K\in\mathcal{T}_h} h_K\left\|\mathbb{Q}_h(\tau\theta\,\nabla n)-\tau\theta\,\nabla n\right\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|\eta_{n0}-\eta_{nb}\right\|_{\partial K}^2\right)^{\frac{1}{2}} \\ &\le Ch^k\left\|n\right\|_{k+1}|||\eta_n|||, \end{aligned} \tag{28}$$

$$\begin{aligned}|\mathcal{I}_9| &= \left|\left(\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla_w Q_h n, \nabla_w\eta_n\right)\right| \\ &\le \left\|\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla_w Q_h n\right\| \,|||\eta_n||| \\ &\le Ch^k\left\|n\right\|_{k+1}|||\eta_n|||. \end{aligned} \tag{29}$$

We make a priori assumption

$$|||n - R_h n||| + |||\varphi - R_h\varphi||| \le h^{\frac{1}{2}}, \tag{30}$$

which implies $\left\|\nabla_w R_h s\right\|_\infty \le \tilde{C}$ for $s = n, \varphi$. We verify the priori assumption (30) at the end of the proof. Based on the priori assumption (30) and approximation property of L^2 projection, we have

$$\begin{aligned}|\mathcal{I}_{10}| &= \left|\left(\tau\mu^2\left(\nabla\varphi - \mathbb{Q}_h\nabla\varphi\right)\otimes\nabla\varphi\,\nabla_w R_h n, \nabla_w\eta_n\right)\right| \\ &\le \tau\left\|\mu\right\|_\infty^2\left\|\varphi\right\|_{1,\infty}\left\|\nabla_w R_h n\right\|_\infty \left\|\nabla\varphi - \mathbb{Q}_h\nabla\varphi\right\| \,|||\eta_n||| \\ &\le Ch^k\left\|\varphi\right\|_{k+1}|||\eta_n|||. \end{aligned} \tag{31}$$

Employing inequality (7) leads to

$$\begin{aligned}|\mathcal{I}_{11}| &= \left|\left(\tau\mu^2\left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\right)\otimes\nabla\varphi\,\nabla_w R_h n, \nabla_w\eta_n\right)\right| \\ &\le \tau\left\|\mu\right\|_\infty^2\left\|\varphi\right\|_{1,\infty}\left\|\nabla_w R_h n\right\|_\infty \left\|\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\right\| \,|||\eta_n||| \\ &\le Ch^k\left\|\varphi\right\|_{k+1}|||\eta_n|||. \end{aligned} \tag{32}$$

With Cauchy-Schwarz inequality, we can derive

$$\begin{aligned}|\mathcal{I}_{12}| &= \left|\left(\tau\mu^2\,\nabla_w\eta_\varphi\otimes\nabla\varphi\,\nabla_w R_h n, \nabla_w\eta_n\right)\right| \\ &\le \tau\left\|\mu\right\|_\infty^2\left\|\varphi\right\|_{1,\infty}\left\|\nabla_w R_h n\right\|_\infty |||\eta_\varphi|||\,|||\eta_n||| \\ &\le \tau\left\|\mu\right\|_\infty^2\tilde{C}\;\left\|\varphi\right\|_{1,\infty}|||\eta_\varphi|||\,|||\eta_n|||. \end{aligned} \tag{33}$$

It follows from priori assumption (30), approximation property and inequality (7) that

$$\begin{aligned}|\mathcal{I}_{13}| &= \left|(\tau\mu^2\,\nabla_w R_h\varphi \otimes (\nabla\varphi - \mathbb{Q}_h\nabla\varphi)\,\nabla_w R_h n, \nabla_w\eta_n)\right| \\ &\le \tau\,\|\mu\|_\infty^2\,\|\nabla_w R_h\varphi\|_\infty \|\nabla_w R_h n\|_\infty\,\|\nabla\varphi - \mathbb{Q}_h\nabla\varphi\|\,|||\eta_n|||\\ &\le Ch^k\,\|\varphi\|_{k+1}\,|||\eta_n|||, \end{aligned} \tag{34}$$

$$\begin{aligned}|\mathcal{I}_{14}| &= \left|(\tau\mu^2\,\nabla_w R_h\varphi \otimes (\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi)\,\nabla_w R_h n, \nabla_w\eta_n)\right| \\ &\le \tau\,\|\mu\|_\infty^2\,\|\nabla_w R_h\varphi\|_\infty \|\nabla_w R_h n\|_\infty\,\|\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\|\,|||\eta_n|||\\ &\le Ch^k\,\|\varphi\|_{k+1}\,|||\eta_n|||. \end{aligned} \tag{35}$$

By Hölder inequality, trace inequality (8) and Lemma 7, it can be obtained such that

$$\begin{aligned}|\mathcal{I}_{15}| &= \left|(\tau\mu^2\,\nabla_w R_h\varphi \otimes \nabla_w\eta_\varphi\,\nabla_w R_h n, \nabla_w\eta_n)\right| \\ &\le \tau\,\|\mu\|_\infty^2\,\|\nabla_w R_h\varphi\|_\infty \|\nabla_w R_h n\|_\infty\,|||\eta_\varphi|||\,|||\eta_n|||\\ &\le \tau\,\|\mu\|_\infty^2\,\tilde{C}^2\,|||\eta_\varphi|||\,|||\eta_n|||, \end{aligned} \tag{36}$$

$$\begin{aligned}|\mathcal{I}_{16}| &= \left|\langle\big(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n\big)\cdot\mathbf{n}, \eta_{n0} - \eta_{nb}\rangle\right| \\ &\le \tau\,\|\mu\|_\infty^2\,\|\varphi\|_{1,\infty}^2\left(\sum_{K\in\mathcal{T}_h} h_K\,\|\mathbb{Q}_h\nabla n - \nabla n\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\,\|\eta_{n0} - \eta_{nb}\|_{\partial K}^2\right)^{\frac{1}{2}} \\ &\le Ch^k\,\|n\|_{k+1}\,|||\eta_n|||, \end{aligned} \tag{37}$$

$$\begin{aligned}|\mathcal{I}_{17}| &= \left|\big(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h n, \nabla_w\eta_n\big)\right| \\ &\le \tau\,\|\mu\|_\infty^2\,\|\varphi\|_{1,\infty}^2\,\|\mathbb{Q}_h\nabla n - \nabla_w Q_h n\|\,|||\eta_n|||\\ &\le Ch^k\,\|n\|_{k+1}\,|||\eta_n|||. \end{aligned} \tag{38}$$

With the above results (21)-(29), (31)-(38) and estimate (18), identity (20) is reduced to

$$\begin{aligned}&\tau\theta\,|||\eta_n|||^2 \le Ch^k\left(\|n\|_{k+1} + \|\varphi\|_{k+1}\right)|||\eta_n|||\\ &\quad + \Big((3\,\|C_2\|_\infty\,\|n_d\|_\infty + \|\mu\|_\infty)\,\|n\|_\infty + 2\,\|C_2\|_\infty\,\|n\|_\infty^2\\ &\quad +\tau\,\|\mu\|_\infty^2\,\tilde{C}\left(\|\varphi\|_{1,\infty} + \tilde{C}\right)\Big)\,|||\eta_\varphi|||\,|||\eta_n|||\\ &\le Ch^k\left(\|n\|_{k+1} + \|\varphi\|_{k+1}\right)|||\eta_n||| + \frac{q}{\varepsilon}(C_\sigma^*)^2\,((3\,\|C_2\|_\infty\,\|n_d\|_\infty\\ &\quad + \|\mu\|_\infty)\,\|n\|_\infty + 2\,\|C_2\|_\infty\,\|n\|_\infty^2 + \tau\,\|\mu\|_\infty^2\,\tilde{C}\left(\|\varphi\|_{1,\infty} + \tilde{C}\right)\Big)\,|||\eta_n|||^2, \end{aligned} \tag{39}$$

where the inequality

$$\tfrac{q}{\varepsilon}(C_\sigma^*)^2\left((3\,\|C_2\|_\infty\,\|n_d\|_\infty + \|\mu\|_\infty)\,\|n\|_\infty + 2\,\|C_2\|_\infty\,\|n\|_\infty^2 + \tau\,\|\mu\|_\infty^2\,\tilde{C}\left(\|\varphi\|_{1,\infty} + \tilde{C}\right)\right) < \tau\theta,$$

should be satisfied and C is a constant independent of h. Therefore, the estimate (39) implies that

$$|||\eta_n||| \le Ch^k(\|n\|_{k+1} + \|\varphi\|_{k+1}). \tag{40}$$

Substituting (40) into (18), we can derive

$$|||\eta_\varphi||| \le Ch^k(\|n\|_{k+1} + \|\varphi\|_{k+1}). \tag{41}$$

The proof of estimations in energy norm is completed.

Then, we introduce a dual problem to demonstrate the L^2 estimations between projection operators Q_h and R_h: find $\chi, \psi \in H_0^1(\Omega)\cap H^2(\Omega)$ satisfying

$$\nabla\cdot\left((3C_2 n_d\,\nabla\psi + \mu\nabla\psi + \mathbf{C_3})\,\chi - 2C_2\,\nabla\psi\,\chi^2\right)$$

$$-\nabla\cdot\left(\left(\tau\theta+\tau\mu^2\nabla\psi\otimes\nabla\psi\right)\nabla\chi\right)=\eta_{n0},\qquad \mathbf{x}\in\Omega, \tag{42}$$

$$-\frac{\varepsilon}{q}\Delta\psi+\chi=\eta_{\varphi 0},\qquad \mathbf{x}\in\Omega, \tag{43}$$

and H^2-regularity, i.e., there exists positive constants $\tilde{C}_1,\tilde{C}_2$ such that

$$\|\chi\|_2\le\tilde{C}_1\|\eta_{n0}\|,\qquad \|\psi\|_2\le\tilde{C}_2\|\eta_{\varphi 0}\|.$$

Multiplying (43) by η_φ and noting $\frac{\varepsilon}{q}\langle\nabla\psi\cdot\mathbf{n},\eta_{\varphi b}\rangle=0$, then

$$\begin{aligned}\|\eta_{\varphi 0}\|^2&=(-\frac{\varepsilon}{q}\Delta\psi+\chi,\eta_{\varphi 0})\\&=\frac{\varepsilon}{q}(\nabla\psi,\nabla\eta_{\varphi 0})-\frac{\varepsilon}{q}\langle\nabla\psi\cdot\mathbf{n},\eta_{\varphi 0}-\eta_{\varphi b}\rangle+(\chi,\eta_{\varphi 0}).\end{aligned} \tag{44}$$

It follows from Lemma 2 that

$$(\nabla\psi,\nabla\eta_\varphi)=(\nabla_w(Q_h\psi),\nabla_w\eta_\varphi)+\langle\mathbb{Q}_h\nabla\psi\cdot\mathbf{n},\eta_{\varphi 0}-\eta_{\varphi b}\rangle+(\mathbb{Q}_h\nabla\psi-\nabla_wQ_h\psi,\nabla_w\eta_\varphi).$$

Substituting the above identity into (44) arrives at

$$\begin{aligned}\|\eta_{\varphi 0}\|^2=&\frac{\varepsilon}{q}(\nabla_w(Q_h\psi),\nabla_w\eta_\varphi)+\frac{\varepsilon}{q}\langle(\mathbb{Q}_h\nabla\psi-\nabla\psi)\cdot\mathbf{n},\eta_{\varphi 0}-\eta_{\varphi b}\rangle\\&+\frac{\varepsilon}{q}(\mathbb{Q}_h\nabla\psi-\nabla_wQ_h\psi,\nabla_w\eta_\varphi)+(\chi,\eta_{\varphi 0}).\end{aligned} \tag{45}$$

With the help of identity (17), the Eqn. (45) is transformed as:

$$\begin{aligned}\|\eta_{\varphi 0}\|^2=&-\frac{\varepsilon}{q}\langle(\mathbb{Q}_h\nabla\varphi-\nabla\varphi)\cdot\mathbf{n},Q_0\psi-Q_b\psi\rangle-\frac{\varepsilon}{q}(\mathbb{Q}_h\nabla\varphi-\nabla_wQ_h\varphi,\nabla_wQ_h\psi)-(\eta_{n0},Q_0\psi)\\&+\frac{\varepsilon}{q}\langle(\mathbb{Q}_h\nabla\psi-\nabla\psi)\cdot\mathbf{n},\eta_{\varphi 0}-\eta_{\varphi b}\rangle+\frac{\varepsilon}{q}(\mathbb{Q}_h\nabla\psi-\nabla_wQ_h\psi,\nabla_w\eta_\varphi)+(\chi,\eta_{\varphi 0})\\:=&\sum_{i=1}^{6}R_i.\end{aligned} \tag{46}$$

Next, the terms $R_i\,(i=1,2,\cdots,6)$ on the right-hand side of (46) will be estimated. The estimate of term R_1 follows from approximation property, the trace inequality (8), the definition of norm $\|\cdot\|_{1,h}$ and Lemma 4 that

$$\begin{aligned}|R_1|=&\frac{\varepsilon}{q}|\langle(\mathbb{Q}_h\nabla\varphi-\nabla\varphi)\cdot\mathbf{n},Q_0\psi-Q_b\psi\rangle|\\\le&\frac{\varepsilon}{q}\left(\sum_{K\in\mathcal{T}_h}\|\mathbb{Q}_h\nabla\varphi-\nabla\varphi\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}\|Q_0\psi-Q_b\psi\|_{\partial K}^2\right)^{\frac{1}{2}}\\\le&\frac{\varepsilon}{q}\left(\sum_{K\in\mathcal{T}_h}h_K\|\mathbb{Q}_h\nabla\varphi-\nabla\varphi\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0\psi-\psi\|_{\partial K}^2\right)^{\frac{1}{2}}\\\le&Ch^{k+1}\|\varphi\|_{k+1}\,\|\psi\|_2.\end{aligned} \tag{47}$$

To bound R_2, we denote L^2 element-wise projection onto $[P_1(K)]^2$ by $\tilde{\mathbb{Q}}_h$. Then, it follows from the definition of the discrete weak gradient that

$$\left(\nabla_w(\varphi-Q_h\varphi),\tilde{\mathbb{Q}}_h\nabla\psi\right)=-\left(\varphi-Q_0\varphi,\nabla\cdot(\tilde{\mathbb{Q}}_h\nabla\psi)\right)+\left\langle\varphi-Q_b\varphi,\tilde{\mathbb{Q}}_h\nabla\psi\cdot\mathbf{n}\right\rangle=0.$$

Utilizing the above equality and inequality (7) yields

$$|R_2|=\frac{\varepsilon}{q}|(\mathbb{Q}_h\nabla\varphi-\nabla_wQ_h\varphi,\nabla_wQ_h\psi)|$$

$$
\begin{aligned}
&= \frac{\varepsilon}{q}\left|\left(\nabla_w(\varphi - Q_h\varphi), \nabla_w Q_h\psi - \tilde{\mathbb{Q}}_h\nabla\psi\right)\right| \\
&\leq Ch^{k+1}\left\|\varphi\right\|_{k+1}\left\|\psi\right\|_2.
\end{aligned}
$$

With the Cauchy-Schwarz inequality and H^2-regularity, we have

$$
\begin{aligned}
|R_3| &= |(\eta_{n0}, Q_0\psi)| \\
&\leq \|\eta_{n0}\|\ (\|Q_0\psi - \psi\| + \|\psi\|) \\
&\leq \|\eta_{n0}\|\ (Ch^2\left\|\psi\right\|_2 + \|\psi\|_2), \\
|R_6| &= |(\chi, \eta_{\varphi 0})| \\
&\leq \|\chi\|_2\ \|\eta_{\varphi 0}\| \\
&\leq \tilde{C}_1\left\|\eta_{n0}\right\|\ \left\|\eta_{\varphi 0}\right\|.
\end{aligned}
$$

By the energy estimate (41) of term η_φ, the estimations for R_4 and R_5 are as follows.

$$
\begin{aligned}
|R_4| &= \frac{\varepsilon}{q}\left|\langle(\mathbb{Q}_h\nabla\psi - \nabla\psi)\cdot\mathbf{n}, \eta_{\varphi 0} - \eta_{\varphi b}\rangle\right| \\
&\leq \frac{\varepsilon}{q}\left(\sum_{K\in\mathcal{T}_h} h_K\left\|\mathbb{Q}_h\nabla\psi - \nabla\psi\right\|^2_{\partial K}\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|\eta_{\varphi 0} - \eta_{\varphi b}\right\|^2_{\partial K}\right)^{\frac{1}{2}} \\
&\leq Ch\left\|\psi\right\|_2\ |||\eta_\varphi||| \\
&\leq Ch^{k+1}(\left\|n\right\|_{k+1} + \left\|\varphi\right\|_{k+1})\ \left\|\psi\right\|_2, \\
|R_5| &= \frac{\varepsilon}{q}\left|(\mathbb{Q}_h\nabla\psi - \nabla_w Q_h\psi, \nabla_w\eta_\varphi)\right| \\
&\leq \frac{\varepsilon}{q}\left\|\mathbb{Q}_h\nabla\psi - \nabla_w Q_h\psi\right\|\ |||\eta_\varphi||| \\
&\leq Ch\left\|\psi\right\|_2\ |||\eta_\varphi||| \\
&\leq Ch^{k+1}(\left\|n\right\|_{k+1} + \left\|\varphi\right\|_{k+1})\ \left\|\psi\right\|_2.
\end{aligned}
$$

Combining (46) with the inequalities (47) to (**??**) and using H^2-regularity arrive at

$$
\begin{aligned}
\left\|\eta_{\varphi 0}\right\|^2 &\leq Ch^{k+1}(\left\|n\right\|_{k+1} + \left\|\varphi\right\|_{k+1})\ \left\|\psi\right\|_2 + \left\|\eta_{n0}\right\|\ \left\|\psi\right\|_2 + \tilde{C}_1\left\|\eta_{n0}\right\|\ \left\|\eta_{\varphi 0}\right\| \\
&\leq Ch^{k+1}(\left\|n\right\|_{k+1} + \left\|\varphi\right\|_{k+1})\,\tilde{C}_2\left\|\eta_{\varphi 0}\right\| + (\tilde{C}_2 + \tilde{C}_1)\left\|\eta_{n0}\right\|\ \left\|\eta_{\varphi 0}\right\|,
\end{aligned}
$$

i.e.,

$$
\left\|\eta_{\varphi 0}\right\| \leq Ch^{k+1}(\left\|n\right\|_{k+1} + \left\|\varphi\right\|_{k+1}) + (\tilde{C}_1 + \tilde{C}_2)\left\|\eta_{n0}\right\|. \tag{48}
$$

Furthermore, to analyze the term $\|\eta_{n0}\|$, we multiply (42) by η_n and integrate over domain Ω deriving

$$
\begin{aligned}
\left\|\eta_{n0}\right\|^2 &= \left(\nabla\cdot\left((3C_2 n_d\,\nabla\psi + \mu\nabla\psi + \mathbf{C_3})\,\chi - 2C_2\,\nabla\psi\,\chi^2\right), \eta_{n0}\right) \\
&- \left(\nabla\cdot((\tau\theta + \tau\mu^2\,\nabla\psi\otimes\nabla\psi)\,\nabla\chi), \eta_{n0}\right) \\
&= -\left((3C_2 n_d + \mu)\,\nabla\psi\,\chi - 2C_2\,\nabla\psi\,\chi^2, \nabla\eta_{n0}\right) + (\nabla\cdot(\mathbf{C_3}\,\chi), \eta_{n0}) \\
&+ \langle\left((3C_2 n_d + \mu)\nabla\psi\,\chi - 2C_2\,\nabla\psi\,\chi^2\right)\cdot\mathbf{n}, \eta_{n0} - \eta_{nb}\rangle \\
&+ \left((\tau\theta + \tau\mu^2\,\nabla\psi\otimes\nabla\psi)\,\nabla\chi, \nabla\eta_{n0}\right) - \langle\left((\tau\theta + \tau\mu^2\,\nabla\psi\otimes\nabla\psi)\,\nabla\chi\right)\cdot\mathbf{n}, \eta_{n0} - \eta_{nb}\rangle,
\end{aligned} \tag{49}
$$

where the facts $\langle\left((3C_2 n_d\,\nabla\psi + \mu\nabla\psi + \mathbf{C_3})\,\chi - 2C_2\,\nabla\psi\,\chi^2\right)\cdot\mathbf{n}, \eta_{nb}\rangle = 0$ and $\langle(\tau\theta + \tau\mu^2\,\nabla\psi\otimes\nabla\psi)\,\nabla\chi\cdot\mathbf{n}, \eta_{nb}\rangle = 0$ are utilized. Similarly, it follows from Lemma 2 and integration by parts

that

$$
\begin{aligned}
&-\left((3C_2n_d+\mu)\,\nabla\psi\,\chi,\nabla\eta_{n0}\right)\\
&=-\left((3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)-\langle\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&-\left(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi)-(3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi,\nabla_w\eta_n\right),\\
&\left(2C_2\,\nabla\psi\,\chi^2,\nabla\eta_{n0}\right)\\
&=\left(2C_2\chi^2\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)+\langle\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&+\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi)-2C_2\chi^2\,\nabla_w Q_h\psi,\nabla_w\eta_n\right),\\
&(\tau\theta\,\nabla\chi,\nabla\eta_{n0})\\
&=(\tau\theta\,\nabla_w Q_h\chi,\nabla_w\eta_n)+\langle\mathbb{Q}_h(\tau\theta\,\nabla\chi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle+\left(\mathbb{Q}_h(\tau\theta\,\nabla\chi)-\tau\theta\,\nabla_w Q_h\chi,\nabla_w\eta_n\right),\\
&(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi,\nabla\eta_{n0})\\
&=(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n)+\langle\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&+\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n\right).
\end{aligned}
$$

Substituting the above identities into (49) derives

$$
\begin{aligned}
&\|\eta_{n0}\|^2\\
=&-\left((3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)-\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi)-(3C_2n_d+\mu)\,\chi\,\nabla\psi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&-\left(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi)-(3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)+(\nabla\cdot(\boldsymbol{C_3}\,\chi),\eta_{n0})\\
&+\left(2C_2\chi^2\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)+\langle\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi)-2C_2\chi^2\,\nabla\psi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&+\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi)-2C_2\chi^2\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)+(\tau\theta\,\nabla_w Q_h\chi,\nabla_w\eta_n)\\
&+\langle(\mathbb{Q}_h(\tau\theta\,\nabla\chi)-\tau\theta\,\nabla\chi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle+\left(\mathbb{Q}_h(\tau\theta\,\nabla\chi)-\tau\theta\,\nabla_w Q_h\chi,\nabla_w\eta_n\right)\\
&+(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n)+\langle\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&+\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n\right).
\end{aligned}
$$

With the identity (19), the above equation can be transformed as:

$$
\begin{aligned}
&\|\eta_{n0}\|^2\\
=&-\left((3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)-\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi)-(3C_2n_d+\mu)\,\chi\,\nabla\psi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&-\left(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi)-(3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)+(\nabla\cdot(\boldsymbol{C_3}\,\chi),\eta_{n0})\\
&+\left(2C_2\chi^2\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)+\langle\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi)-2C_2\chi^2\,\nabla\psi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&+\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi)-2C_2\chi^2\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)\\
&+\langle(\mathbb{Q}_h(\tau\theta\,\nabla\chi)-\tau\theta\,\nabla\chi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle+\left(\mathbb{Q}_h(\tau\theta\,\nabla\chi)-\tau\theta\,\nabla_w Q_h\chi,\nabla_w\eta_n\right)\\
&+(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n)+\langle\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\\
&+\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n\right)\\
&+\left((3C_2n_d+\mu)\,n\,\nabla_w\eta_\varphi,\nabla_w Q_h\chi\right)+\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,n\nabla\varphi)-(3C_2n_d+\mu)\,n\nabla\varphi)\cdot\mathbf{n},Q_0\chi-Q_b\chi\rangle\\
&+\left(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\varphi)-(3C_2n_d+\mu)\,n\,\nabla_w Q_h\varphi,\nabla_w Q_h\chi\right)\\
&+\langle(\mathbb{Q}_h(\boldsymbol{C_3}\,n)-\boldsymbol{C_3}\,n)\cdot\mathbf{n},Q_0\chi-Q_b\chi\rangle-\left(2C_2n^2\,\nabla_w\eta_\varphi,\nabla_w Q_h\chi\right)\\
&-\langle\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi)-2C_2n^2\,\nabla\varphi\right)\cdot\mathbf{n},Q_0\chi-Q_b\chi\rangle-\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi)-2C_2n^2\,\nabla_w Q_h\varphi,\nabla_w Q_h\chi\right)\\
&-\langle(\mathbb{Q}_h(\tau\theta\,\nabla n)-\tau\theta\,\nabla n)\cdot\mathbf{n},Q_0\chi-Q_b\chi\rangle-\left(\mathbb{Q}_h(\tau\theta\,\nabla n)-\tau\theta\,\nabla_w Q_h n,\nabla_w Q_h\chi\right)\\
&-(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w\eta_n,\nabla_w Q_h\chi)\\
&-(\tau\mu^2\,\left((\nabla\varphi-\mathbb{Q}_h\nabla\varphi)+(\mathbb{Q}_h\nabla\varphi-\nabla_w Q_h\varphi)+\nabla_w\eta_\varphi\right)\otimes\nabla\varphi\,\nabla_w R_h n,\nabla_w Q_h\chi)\\
&-(\tau\mu^2\,\nabla_w R_h\varphi\otimes\left((\nabla\varphi-\mathbb{Q}_h\nabla\varphi)+(\mathbb{Q}_h\nabla\varphi-\nabla_w Q_h\varphi)+\nabla_w\eta_\varphi\right)\,\nabla_w R_h n,\nabla_w Q_h\chi)
\end{aligned}
$$

$$
\begin{aligned}
&- \langle \big(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n\big)\cdot\mathbf{n}, Q_0\chi - Q_b\chi\rangle \\
&- \big(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h n, \nabla_w Q_h\chi\big) \\
:=&\sum_{i=1}^{30} I_i. \qquad (50)
\end{aligned}
$$

We estimate the $I_i\,(i=1,2,\cdots,30)$ in (50) individually. For instantce, we analyse I_2 and I_3 before analysing I_1. It follows from trace inequality (8) and approximation property of L^2 projection that

$$
\begin{aligned}
|I_2| &= |\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi) - (3C_2n_d+\mu)\,\chi\,\nabla\psi)\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle| \\
&\le (3\,\|C_2\|_\infty\,\|n_d\|_\infty + \|\mu\|_\infty)\,\|\chi\|_\infty \left(\sum_{K\in\mathcal{T}_h} h_K\,\|\mathbb{Q}_h\nabla\psi-\nabla\psi\|^2_{\partial K}\right)^{\frac{1}{2}} \\
&\times \left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\,\|\eta_{n0}-\eta_{nb}\|^2_{\partial K}\right)^{\frac{1}{2}} \le Ch\;\|\psi\|_2\;|||\eta_n||| \\
&\le Ch^{k+1}\left(\|n\|_{k+1}+\|\varphi\|_{k+1}\right)\|\psi\|_2\,. \qquad (51)
\end{aligned}
$$

With Lemma 7, the estimate of term I_3 is reduced to

$$
\begin{aligned}
|I_3| &= |(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi) - (3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi, \nabla_w\eta_n)| \\
&\le (3\,\|C_2\|_\infty\,\|n_d\|_\infty + \|\mu\|_\infty)\,\|\chi\|_\infty\,\|\mathbb{Q}_h\nabla\psi - \nabla_w Q_h\psi\|\;|||\eta_n||| \\
&\le Ch\;\|\psi\|_2\;|||\eta_n||| \\
&\le Ch^{k+1}\left(\|n\|_{k+1}+\|\varphi\|_{k+1}\right)\|\psi\|_2\,. \qquad (52)
\end{aligned}
$$

By Lemma 2, integration by parts and the above results (51)-(52), it can be shown that

$$
\begin{aligned}
|I_1| &= |((3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi, \nabla_w\eta_n)| \\
&\le |((3C_2n_d+\mu)\,\chi\,\nabla\psi, \nabla\eta_{n0}) - \langle\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi)\;\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle| \\
&+ |(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi) - (3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi, \nabla_w\eta_n)| \\
&\le |-\left(\nabla\cdot((3C_2n_d+\mu)\,\chi\nabla\psi)\,, \eta_{n0}\right) - \langle(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\nabla\psi) \\
&- (3C_2n_d+\mu)\,\chi\nabla\psi)\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle| \\
&+ |(\mathbb{Q}_h((3C_2n_d+\mu)\,\chi\,\nabla\psi) - (3C_2n_d+\mu)\,\chi\,\nabla_w Q_h\psi, \nabla_w\eta_n)| \\
&\le \left(3\,\|C_2\|_{1,\infty}\,\|n_d\|_{1,\infty} + \|\mu\|_{1,\infty}\right)\|\chi\|_{1,\infty}\,\|\psi\|_2\,\|\eta_{n0}\| \\
&+ Ch^{k+1}\left(\|n\|_{k+1}+\|\varphi\|_{k+1}\right)\|\psi\|_2\,. \qquad (53)
\end{aligned}
$$

It follows from the Hölder inequality that

$$
\begin{aligned}
|I_4| &= |(\nabla\cdot(\boldsymbol{C_3}\,\chi), \eta_{n0})| \\
&\le \|\boldsymbol{C_3}\|_{1,\infty}\,\|\chi\|_2\;\|\eta_{n0}\|\,.
\end{aligned}
$$

Using similar techniques in the derivation of (51)-(53), we get

$$
\begin{aligned}
|I_5| &= \left|\left(2C_2\chi^2\,\nabla_w Q_h\psi, \nabla_w\eta_n\right)\right| \\
&= \left|\left(2C_2\chi^2\nabla\psi, \nabla\eta_{n0}\right) - \langle\mathbb{Q}_h(2C_2\chi^2\nabla\psi)\;\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle\right. \\
&\left.+ \left(\mathbb{Q}_h(2C_2\chi^2\nabla\psi) - 2C_2\chi^2\nabla_w Q_h\psi, \nabla_w\eta_n\right)\right| \\
&\le \left|-\left(\nabla\cdot\left(2C_2\chi^2\,\nabla\psi\right), \eta_{n0}\right) - \langle\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi) - 2C_2\chi^2\,\nabla\psi\right)\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle\right| \\
&+ \left|\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi) - 2C_2\chi^2\,\nabla_w Q_h\psi, \nabla_w\eta_n\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\left\|C_2\right\|_{1,\infty}\left\|\chi\right\|_{1,\infty}^2\left\|\psi\right\|_2\left\|\eta_{n0}\right\| + Ch^{k+1}\left(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1}\right)\left\|\psi\right\|_2,\\
|I_6| &= \left|\langle\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi) - 2C_2\chi^2\,\nabla\psi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\right|\\
&\leq 2\left\|C_2\right\|_{\infty}\left\|\chi\right\|_{\infty}^2\left(\sum_{K\in\mathcal{T}_h}h_K\left\|\mathbb{Q}_h\nabla\psi-\nabla\psi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|\eta_{n0}-\eta_{nb}\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\leq Ch\ \left\|\psi\right\|_2\ |||\eta_n|||\\
&\leq Ch^{k+1}\left(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1}\right)\left\|\psi\right\|_2,\\
|I_7| &= \left|\left(\mathbb{Q}_h(2C_2\chi^2\,\nabla\psi) - 2C_2\chi^2\,\nabla_w Q_h\psi,\nabla_w\eta_n\right)\right|\\
&\leq 2\left\|C_2\right\|_{\infty}\left\|\chi\right\|_{\infty}^2\left\|\mathbb{Q}_h\nabla\psi-\nabla_w Q_h\psi\right\|\ |||\eta_n|||\\
&\leq Ch\ \left\|\psi\right\|_2\ |||\eta_n|||\\
&\leq Ch^{k+1}\left(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1}\right)\left\|\psi\right\|_2.
\end{aligned}
$$

An application of approximation property and the inequality (7) indicates that

$$
\begin{aligned}
|I_8| &= |\langle(\mathbb{Q}_h(\tau\theta\,\nabla\chi)-\tau\theta\,\nabla\chi)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle|\\
&\leq \tau\theta\left(\sum_{K\in\mathcal{T}_h}h_K\left\|\mathbb{Q}_h\nabla\chi-\nabla\chi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|\eta_{n0}-\eta_{nb}\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\leq Ch\left\|\chi\right\|_2\ |||\eta_n|||\\
&\leq Ch^{k+1}(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1})\ \left\|\chi\right\|_2,\\
|I_9| &= |(\mathbb{Q}_h(\tau\theta\,\nabla\chi)-\tau\theta\,\nabla_w Q_h\chi,\nabla_w\eta_n)|\\
&\leq \tau\theta\ \left\|\mathbb{Q}_h\nabla\chi-\nabla_w Q_h\chi\right\|\ |||\eta_n|||\\
&\leq Ch^{k+1}(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1})\ \left\|\chi\right\|_2.
\end{aligned}
$$

Using similar procedures in analyzing terms I_1, I_2 and I_3, we can derive

$$
\begin{aligned}
|I_{10}| &= \left|(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n)\right|\\
&\leq \left|\left(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi,\nabla\eta_{n0}\right)-\langle\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)\ \cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\right|\\
&+\left|\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n\right)\right|\\
&\leq \left|-\left(\nabla\cdot\left(\tau\mu^2\nabla\psi\otimes\nabla\psi\nabla\chi\right),\eta_{n0}\right)-\langle\left(\mathbb{Q}_h(\tau\mu^2\nabla\psi\otimes\nabla\psi\nabla\chi)\right.\right.\\
&\left.\left.-\tau\mu^2\nabla\psi\otimes\nabla\psi\nabla\chi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\right|\\
&+\left|\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n\right)\right|\\
&\leq \tau\left\|\mu\right\|_{1,\infty}^2\left\|\psi\right\|_{2,\infty}^2\left\|\chi\right\|_2\left\|\eta_{n0}\right\| + Ch^{k+1}\left(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1}\right)\left\|\chi\right\|_2,\\
|I_{11}| &= \left|\langle\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi\right)\cdot\mathbf{n},\eta_{n0}-\eta_{nb}\rangle\right|\\
&\leq \tau\left\|\mu\right\|_{\infty}^2\left\|\psi\right\|_{1,\infty}^2\left(\sum_{K\in\mathcal{T}_h}h_K\left\|\mathbb{Q}_h\nabla\chi-\nabla\chi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|\eta_{n0}-\eta_{nb}\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\leq Ch\left\|\chi\right\|_2\ |||\eta_n|||\\
&\leq Ch^{k+1}(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1})\ \left\|\chi\right\|_2,\\
|I_{12}| &= \left|\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla\chi)-\tau\mu^2\,\nabla\psi\otimes\nabla\psi\,\nabla_w Q_h\chi,\nabla_w\eta_n\right)\right|\\
&\leq \tau\left\|\mu\right\|_{\infty}^2\left\|\psi\right\|_{1,\infty}^2\left\|\mathbb{Q}_h\nabla\chi-\nabla_w Q_h\chi\right\|\ |||\eta_n|||\\
&\leq Ch^{k+1}(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1})\ \left\|\chi\right\|_2,
\end{aligned}
$$

$$
\begin{aligned}
|I_{13}| &= |((3C_2n_d+\mu)\,n\,\nabla_w\eta_\varphi,\nabla_w Q_h\chi)|\\
&\le |((3C_2n_d+\mu)\,n\,\nabla\chi,\nabla\eta_{\varphi 0})-\langle\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\chi)\ \cdot\mathbf{n},\eta_{\varphi 0}-\eta_{\varphi b}\rangle|\\
&+|(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\chi)-(3C_2n_d+\mu)\,n\,\nabla_w Q_h\chi,\nabla_w\eta_\varphi)|\\
&\le |-\left(\nabla\cdot((3C_2n_d+\mu)\,n\nabla\chi\right),\eta_{\varphi 0})-\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,n\nabla\chi)\\
&-(3C_2n_d+\mu)\,n\nabla\chi)\cdot\mathbf{n},\eta_{\varphi 0}-\eta_{\varphi b}\rangle|\\
&+|(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\chi)-(3C_2n_d+\mu)\,n\,\nabla_w Q_h\chi,\nabla_w\eta_\varphi)|\\
&\le \left(3\left\|C_2\right\|_{1,\infty}\left\|n_d\right\|_{1,\infty}+\left\|\mu\right\|_{1,\infty}\right)\left\|n\right\|_{1,\infty}\left\|\chi\right\|_2\left\|\eta_{\varphi 0}\right\|\\
&+Ch^{k+1}\left(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1}\right)\left\|\chi\right\|_2,\\
|I_{14}| &= |\langle(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\varphi)-(3C_2n_d+\mu)\,n\,\nabla\varphi)\cdot\mathbf{n},Q_0\chi-Q_b\chi\rangle|\\
&\le (3\left\|C_2\right\|_\infty\left\|n_d\right\|_\infty+\left\|\mu\right\|_\infty)\left\|n\right\|_\infty\left(\sum_{K\in\mathcal{T}_h}\left\|\mathbb{Q}_h\nabla\varphi-\nabla\varphi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\times\left(\sum_{K\in\mathcal{T}_h}\left\|Q_0\chi-Q_b\chi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\le (3\left\|C_2\right\|_\infty\left\|n_d\right\|_\infty+\left\|\mu\right\|_\infty)\left\|n\right\|_\infty\left(\sum_{K\in\mathcal{T}_h}h_K\left\|\mathbb{Q}_h\nabla\varphi-\nabla\varphi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\times\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|Q_0\chi-\chi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\le Ch^{k+1}\left\|\varphi\right\|_{k+1}\left\|\chi\right\|_2,\\
|I_{15}| &= |(\mathbb{Q}_h((3C_2n_d+\mu)\,n\,\nabla\varphi)-(3C_2n_d+\mu)\,n\,\nabla_w Q_h\varphi,\nabla_w Q_h\chi)|\\
&\le (3\left\|C_2\right\|_\infty\left\|n_d\right\|_\infty+\left\|\mu\right\|_\infty)\left\|n\right\|_\infty\left\|\mathbb{Q}_h\nabla\varphi-\nabla_w Q_h\varphi\right\|\ \left|\!\left|\!\left|\eta_n\right|\!\right|\!\right|\\
&\le Ch\ \left\|\varphi\right\|_2\ \left|\!\left|\!\left|\eta_n\right|\!\right|\!\right|\\
&\le Ch^{k+1}\left(\left\|n\right\|_{k+1}+\left\|\varphi\right\|_{k+1}\right)\left\|\psi\right\|_2.
\end{aligned}
$$

With the help of Hölder inequality and approximation property of operator $\mathbb{Q}_h$, it can be derived that

$$
\begin{aligned}
|I_{16}| &= |\langle(\mathbb{Q}_h(\boldsymbol{C_3}\,n)-\boldsymbol{C_3}\,n)\cdot\mathbf{n},Q_0\chi-Q_b\chi\rangle|\\
&\le\left(\sum_{K\in\mathcal{T}_h}\left\|\mathbb{Q}_h(\boldsymbol{C_3}\,n)-\boldsymbol{C_3}\,n\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\times\left(\sum_{K\in\mathcal{T}_h}\left\|Q_0\chi-Q_b\chi\right\|_{\partial K}^2\right)^{\frac{1}{2}}\\
&\le\left(\sum_{K\in\mathcal{T}_h}h_K\left\|\mathbb{Q}_h(\boldsymbol{C_3}\,n)-\boldsymbol{C_3}\,n\right\|_{\partial K}^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\times \left(\sum_{K\in\mathcal{T}_h} h_K^{-1} \|Q_0\chi - \chi\|_{\partial K}^2 \right)^{\frac{1}{2}}$$
$$\leq Ch^{k+2} \|\varphi\|_{k+1} \|\chi\|_2 .$$

An application of Lemma 2 and integration by parts implies that

$$\begin{aligned}
|I_{17}| &= \left|\left(2C_2n^2\,\nabla_w\eta_\varphi, \nabla_w Q_h\chi\right)\right| \\
&= \left|\left(2C_2n^2\nabla\chi, \nabla\eta_{\varphi 0}\right) - \langle \mathbb{Q}_h(2C_2n^2\nabla\chi)\ \cdot\mathbf{n}, \eta_{\varphi 0} - \eta_{\varphi b}\rangle\right. \\
&\quad \left. + \left(\mathbb{Q}_h(2C_2n^2\nabla\chi) - 2C_2n^2\nabla_w Q_h\chi, \nabla_w\eta_\varphi\right)\right| \\
&\leq \left|-\left(\nabla\cdot\left(2C_2n^2\,\nabla\chi\right), \eta_{\varphi 0}\right) - \langle\left(\mathbb{Q}_h(2C_2n^2\,\nabla\chi) - 2C_2n^2\,\nabla\chi\right)\cdot\mathbf{n}, \eta_{\varphi 0} - \eta_{\varphi b}\rangle\right| \\
&\quad + \left|\left(\mathbb{Q}_h(2C_2n^2\,\nabla\chi) - 2C_2n^2\,\nabla_w Q_h\chi, \nabla_w\eta_\varphi\right)\right| \\
&\leq 2\|C_2\|_{1,\infty}\|n\|_{1,\infty}^2\|\chi\|_2\|\eta_{\varphi 0}\| + Ch^{k+1}\left(\|n\|_{k+1} + \|\varphi\|_{k+1}\right)\|\chi\|_2 , \\
|I_{18}| &= \left|\langle\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi) - 2C_2n^2\,\nabla\varphi\right)\cdot\mathbf{n}, Q_0\chi - Q_b\chi\rangle\right| \\
&\leq 2\|C_2\|_\infty\|n\|_\infty^2\left(\sum_{K\in\mathcal{T}_h}\|\mathbb{Q}_h\nabla\varphi - \nabla\varphi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\quad \times\left(\sum_{K\in\mathcal{T}_h}\|Q_0\chi - Q_b\chi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\leq 2\|C_2\|_\infty\|n\|_\infty^2\left(\sum_{K\in\mathcal{T}_h} h_K\|\mathbb{Q}_h\nabla\varphi - \nabla\varphi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\quad \left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|Q_0\chi - \chi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\leq Ch^{k+1}\|\varphi\|_{k+1}\|\chi\|_2 , \\
|I_{19}| &= \left|\left(\mathbb{Q}_h(2C_2n^2\,\nabla\varphi) - 2C_2n^2\,\nabla_w Q_h\varphi, \nabla_w Q_h\chi\right)\right| \\
&\leq 2\|C_2\|_\infty\|n\|_\infty^2\left|\left(\nabla_w(\varphi - Q_h\varphi), \nabla_w Q_h\chi - \tilde{\mathbb{Q}}_h\nabla\chi\right)\right| \\
&\leq Ch^{k+1}\|\varphi\|_{k+1}\|\chi\|_2 .
\end{aligned}$$

By the Cauchy-Schwarz inequality and trace inequality (8), we arrive at

$$\begin{aligned}
|I_{20}| &= |\langle(\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla n)\cdot\mathbf{n}, Q_0\chi - Q_b\chi\rangle| \\
&\leq \tau\theta\left(\sum_{K\in\mathcal{T}_h}\|\mathbb{Q}_h\nabla n - \nabla n\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}\|Q_0\chi - Q_b\chi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\leq \tau\theta\left(\sum_{K\in\mathcal{T}_h} h_K\|\mathbb{Q}_h\nabla n - \nabla n\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|Q_0\chi - \chi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\leq Ch^{k+1}\|n\|_{k+1}\|\chi\|_2 .
\end{aligned}$$

To bound term I_{21}, we apply a similar technique in estimating R_2 and obtain

$$|I_{21}| = |(\mathbb{Q}_h(\tau\theta\,\nabla n) - \tau\theta\,\nabla_w Q_h n, \nabla_w Q_h\chi)|$$

$$\begin{aligned}
&\leq \tau\theta \left|\Big(\nabla_w(n-Q_hn), \nabla_w Q_h\chi - \tilde{\mathbb{Q}}_h\nabla\chi\Big)\right| \\
&\leq Ch^{k+1}\left\|n\right\|_{k+1}\left\|\chi\right\|_2.
\end{aligned}$$

By Lemma 2 and integration by parts, we obtain

$$\begin{aligned}
|I_{22}| &= \left|(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w\eta_n, \nabla_w Q_h\chi)\right| \\
&\leq \left|\left(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla\chi, \nabla\eta_{n0}\right) - \langle\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla\chi)\ \cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle\right| \\
&+ \left|\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla\chi) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h\chi, \nabla_w\eta_n\right)\right| \\
&\leq \left|-\left(\nabla\cdot\left(\tau\mu^2\nabla\varphi\otimes\nabla\varphi\nabla\chi\right), \eta_{n0}\right) - \langle\left(\mathbb{Q}_h(\tau\mu^2\nabla\varphi\otimes\nabla\varphi\nabla\chi) - \tau\mu^2\nabla\varphi\otimes\nabla\varphi\nabla\chi\right)\cdot\mathbf{n}, \eta_{n0}-\eta_{nb}\rangle\right| \\
&+ \left|\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla\chi) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h\chi, \nabla_w\eta_n\right)\right| \\
&\leq \tau\left\|\mu\right\|_{1,\infty}^2\left\|\varphi\right\|_{2,\infty}^2\left\|\chi\right\|_2\left\|\eta_{n0}\right\| + Ch^{k+1}\left(\left\|n\right\|_{k+1} + \left\|\varphi\right\|_{k+1}\right)\left\|\chi\right\|_2.
\end{aligned}$$

With the definition of L^2 projection operator $\mathbb{Q}_h$ and the inequality 7, there is

$$\begin{aligned}
|I_{23}| &= \left|(\tau\mu^2\left(\nabla\varphi - \mathbb{Q}_h\nabla\varphi\right)\otimes\nabla\varphi\,\nabla_w R_h n, \nabla_w Q_h\chi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\left\|\varphi\right\|_{1,\infty}\left\|\nabla_w R_h n\right\|_\infty\left|(\nabla\varphi - \mathbb{Q}_h\nabla\varphi, \nabla_w Q_h\chi)\right| \\
&= \tau\left\|\mu\right\|_\infty^2\tilde{C}\left\|\varphi\right\|_{1,\infty}\left|(\nabla\varphi - \mathbb{Q}_h\nabla\varphi, \nabla_w Q_h\chi - \mathbb{Q}_h\nabla\chi)\right| \\
&\leq Ch^{k+1}\left\|\varphi\right\|_{k+1}\left\|\chi\right\|_2.
\end{aligned}$$

With the priori assumption (30) and the definition of L^2 projection $\tilde{\mathbb{Q}}_h$, we get

$$\begin{aligned}
|I_{24}| &= \left|(\tau\mu^2\left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\right)\otimes\nabla\varphi\,\nabla_w R_h n, \nabla_w Q_h\chi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\left\|\varphi\right\|_{1,\infty}\left\|\nabla_w R_h n\right\|_\infty\left|\Big(\nabla_w(\varphi - Q_h\varphi), \nabla_w Q_h\chi - \tilde{\mathbb{Q}}_h\nabla\chi\Big)\right| \\
&\leq Ch^{k+1}\left\|\varphi\right\|_{k+1}\left\|\chi\right\|_2.
\end{aligned}$$

Applying Lemma 2 and integration by parts, it yields

$$\begin{aligned}
|I_{25}| &= \left|(\tau\mu^2\,\nabla_w\eta_\varphi\otimes\nabla\varphi\,\nabla_w R_h n, \nabla_w Q_h\chi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\left\|\varphi\right\|_{1,\infty}\left\|\nabla_w R_h n\right\|_\infty\left|(\nabla_w\eta_\varphi, \nabla_w Q_h\chi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\tilde{C}\left\|\varphi\right\|_{1,\infty}\left|(\nabla\chi, \nabla\eta_{\varphi 0}) - \langle\mathbb{Q}_h\nabla\chi\cdot\mathbf{n}, \eta_{\varphi 0}-\eta_{\varphi b}\rangle - (\mathbb{Q}_h\nabla\chi - \nabla_w Q_h\chi, \nabla_w\eta_\varphi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\tilde{C}\left\|\varphi\right\|_{1,\infty}\left|-(\nabla\cdot\nabla\chi, \eta_{\varphi 0}) - \langle(\mathbb{Q}_h\nabla\chi - \nabla\chi)\cdot\mathbf{n}, \eta_{\varphi 0}-\eta_{\varphi b}\rangle\right| \\
&+ \tau\left\|\mu\right\|_\infty^2\tilde{C}\left\|\varphi\right\|_{1,\infty}\left|(\mathbb{Q}_h\nabla\chi - \nabla_w Q_h\chi, \nabla_w\eta_\varphi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\tilde{C}\left\|\varphi\right\|_{1,\infty}\left\|\chi\right\|_2\left\|\eta_{\varphi 0}\right\| + Ch^{k+1}\left(\left\|n\right\|_{k+1} + \left\|\varphi\right\|_{k+1}\right)\left\|\chi\right\|_2.
\end{aligned}$$

With definition of L^2 projection operator $\mathbb{Q}_h$, we can derive

$$\begin{aligned}
|I_{26}| &= \left|(\tau\mu^2\,\nabla_w R_h\varphi\otimes\left(\nabla\varphi - \mathbb{Q}_h\nabla\varphi\right)\nabla_w R_h n, \nabla_w Q_h\chi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\left\|\nabla_w R_h\varphi\right\|_\infty\left\|\nabla_w R_h n\right\|_\infty\left|(\nabla\varphi - \mathbb{Q}_h\nabla\varphi, \nabla_w Q_h\chi)\right| \\
&= \tau\left\|\mu\right\|_\infty^2\tilde{C}^2\left\|\nabla_w R_h n\right\|_\infty\left|(\nabla\varphi - \mathbb{Q}_h\nabla\varphi, \nabla_w Q_h\chi - \mathbb{Q}_h\nabla\chi)\right| \\
&\leq Ch^{k+1}\left\|\varphi\right\|_{k+1}\left\|\chi\right\|_2.
\end{aligned}$$

Utilizing similar procedures as analyzing term I_{24} and Lemma 2 arrives at

$$\begin{aligned}
|I_{27}| &= \left|(\tau\mu^2\,\nabla_w R_h\varphi\otimes\left(\mathbb{Q}_h\nabla\varphi - \nabla_w Q_h\varphi\right)\nabla_w R_h n, \nabla_w Q_h\chi)\right| \\
&\leq \tau\left\|\mu\right\|_\infty^2\left\|\nabla_w R_h\varphi\right\|_\infty\left\|\nabla_w R_h n\right\|_\infty\left|\Big(\nabla_w(\varphi - Q_h\varphi), \nabla_w Q_h\chi - \tilde{\mathbb{Q}}_h\nabla\chi\Big)\right| \\
&\leq Ch^{k+1}\left\|\varphi\right\|_{k+1}\left\|\chi\right\|_2.
\end{aligned}$$

It follows from similar technique for term I_{25} that

$$\begin{aligned}
|I_{28}| &= \left|(\tau\mu^2\,\nabla_w R_h\varphi \otimes \nabla_w\eta_\varphi\,\nabla_w R_h n, \nabla_w Q_h\chi)\right| \\
&\le \tau\,\|\mu\|_\infty^2\,\|\nabla_w R_h\varphi\|_\infty\,\|\nabla_w R_h n\|_\infty\,|(\nabla_w\eta_\varphi, \nabla_w Q_h\chi)| \\
&\le \tau\,\|\mu\|_\infty^2\,\tilde{C}^2\,|(\nabla\chi, \nabla\eta_{\varphi 0}) - \langle \mathbb{Q}_h\nabla\chi\cdot\mathbf{n}, \eta_{\varphi 0}-\eta_{\varphi b}\rangle - (\mathbb{Q}_h\nabla\chi - \nabla_w Q_h\chi, \nabla_w\eta_\varphi)| \\
&\le \tau\,\|\mu\|_\infty^2\,\tilde{C}^2\,|-(\nabla\cdot\nabla\chi, \eta_{\varphi 0}) - \langle(\mathbb{Q}_h\nabla\chi - \nabla\chi)\cdot\mathbf{n}, \eta_{\varphi 0}-\eta_{\varphi b}\rangle| \\
&+ \tau\,\|\mu\|_\infty^2\,\tilde{C}^2\,|(\mathbb{Q}_h\nabla\chi - \nabla_w Q_h\chi, \nabla_w\eta_\varphi)| \\
&\le \tau\,\|\mu\|_\infty^2\,\tilde{C}^2\,\|\chi\|_2\,\|\eta_{\varphi 0}\| + Ch^{k+1}\left(\|n\|_{k+1} + \|\varphi\|_{k+1}\right)\|\chi\|_2\,.
\end{aligned}$$

An application of trace inequality, approximation property and the inequality (7) implies that

$$\begin{aligned}
|I_{29}| &= \left|\langle\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n\right)\cdot\mathbf{n}, Q_0\chi - Q_b\chi\rangle\right| \\
&\le \tau\,\|\mu\|_\infty^2\,\|\varphi\|_{1,\infty}^2\left(\sum_{K\in\mathcal{T}_h}\|\mathbb{Q}_h\nabla n - \nabla n\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}\|Q_0\chi - Q_b\chi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\le \tau\,\|\mu\|_\infty^2\,\|\varphi\|_{1,\infty}^2\left(\sum_{K\in\mathcal{T}_h}h_K\,\|\mathbb{Q}_h\nabla n - \nabla n\|_{\partial K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\,\|Q_0\chi - \chi\|_{\partial K}^2\right)^{\frac{1}{2}} \\
&\le Ch^{k+1}\,\|n\|_{k+1}\,\|\chi\|_2\,,
\end{aligned}$$

$$\begin{aligned}
|I_{30}| &= \left|\left(\mathbb{Q}_h(\tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla n) - \tau\mu^2\,\nabla\varphi\otimes\nabla\varphi\,\nabla_w Q_h n, \nabla_w Q_h\chi\right)\right| \\
&\le \tau\,\|\mu\|_\infty^2\,\|\varphi\|_{1,\infty}^2\,\|\mathbb{Q}_h\nabla n - \nabla_w Q_h n\|\,(|||Q_h\chi - \chi||| + |||\chi|||) \\
&\le Ch^{k+1}\,\|n\|_{k+1}\,(Ch\,\|\chi\|_2 + \|\chi\|_2). \qquad (54)
\end{aligned}$$

Combining (50) with inequalities (51) to (54) yields

$$\begin{aligned}
&\|\eta_{n0}\|^2 \\
&\le Ch^{k+1}\left(\|n\|_{k+1} + \|\varphi\|_{k+1}\right)\left(\|\chi\|_2 + \|\psi\|_2\right) \\
&+ \left(\|\boldsymbol{C_3}\|_{1,\infty} + \tau\,\|\mu\|_{1,\infty}^2\left(\|\psi\|_{2,\infty}^2 + \|\varphi\|_{2,\infty}^2\right)\right)\|\eta_{n0}\|\,\|\chi\|_2 \\
&+ \left(3\,\|C_2\|_{1,\infty}\,\|n_d\|_{1,\infty} + \|\mu\|_{1,\infty} + 2\,\|C_2\|_{1,\infty}\,\|\chi\|_{1,\infty}\right)\|\chi\|_{1,\infty}\,\|\eta_{n0}\|\,\|\psi\|_2 \\
&+ \left(\left(3\,\|C_2\|_{1,\infty}\,\|n_d\|_{1,\infty} + \|\mu\|_{1,\infty} + 2\,\|C_2\|_{1,\infty}\,\|n\|_{1,\infty}\right)\|n\|_{1,\infty} + \tau\,\|\mu\|_\infty^2\,\tilde{C}\left(\tilde{C} + \|\varphi\|_{1,\infty}\right)\right)\|\eta_{\varphi 0}\|\,\|\chi\|_2\,.
\end{aligned}$$

For simplicity, we let

$$\begin{aligned}
\alpha_1 &= \|\boldsymbol{C_3}\|_{1,\infty} + \tau\,\|\mu\|_{1,\infty}^2\left(\|\psi\|_{2,\infty}^2 + \|\varphi\|_{2,\infty}^2\right), \\
\alpha_2 &= \left(3\,\|C_2\|_{1,\infty}\,\|n_d\|_{1,\infty} + \|\mu\|_{1,\infty} + 2\,\|C_2\|_{1,\infty}\,\|\chi\|_{1,\infty}\right)\|\chi\|_{1,\infty}\,, \\
\alpha_3 &= \left(3\,\|C_2\|_{1,\infty}\,\|n_d\|_{1,\infty} + \|\mu\|_{1,\infty} + 2\,\|C_2\|_{1,\infty}\,\|n\|_{1,\infty}\right)\|n\|_{1,\infty} + \tau\,\|\mu\|_\infty^2\,\tilde{C}\left(\tilde{C} + \|\varphi\|_{1,\infty}\right).
\end{aligned}$$

By H^2-regularity, estimate (48) and ϵ-inequality that, the above estimate can be reduced to

$$\begin{aligned}
&\|\eta_{n0}\|^2 \\
&\le Ch^{k+1}\left(\|n\|_{k+1} + \|\varphi\|_{k+1}\right)\left(\tilde{C}_1\,\|\eta_{n0}\| + \tilde{C}_2\,\|\eta_{\varphi 0}\|\right) \\
&+ \tilde{C}_1\alpha_1\,\|\eta_{n0}\|^2 + \left(\tilde{C}_2\,\alpha_2 + \tilde{C}_1\alpha_3\right)\|\eta_{n0}\|\,\|\eta_{\varphi 0}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \tfrac{1}{4\epsilon}(\tilde{C}_1^2+\tilde{C}_2^2)\,Ch^{2(k+1)}\left(\|n\|_{k+1}^2+\|\varphi\|_{k+1}^2\right)+(2\epsilon+\tilde{\epsilon})\,\|\eta_{\varphi 0}\|^2\\
&+\left[2\epsilon+\tilde{C}_1\alpha_1+\tfrac{1}{4\tilde{\epsilon}}\left(\tilde{C}_2\,\alpha_2+\tilde{C}_1\alpha_3\right)^2\right]\|\eta_{n0}\|^2\\
&\leq \left[\tfrac{1}{4\epsilon}(\tilde{C}_1^2+\tilde{C}_2^2)\,Ch^{2(k+1)}+2\epsilon+\tilde{\epsilon}\right]Ch^{2(k+1)}\left(\|n\|_{k+1}^2+\|\varphi\|_{k+1}^2\right)\\
&+\left[2\epsilon+\tilde{C}_1\alpha_1+\tfrac{1}{4\tilde{\epsilon}}\left(\tilde{C}_2\,\alpha_2+\tilde{C}_1\alpha_3\right)^2+2(2\epsilon+\tilde{\epsilon})(\tilde{C}_1+\tilde{C}_2)^2\right]\|\eta_{n0}\|^2.
\end{aligned}
$$

We use $2\epsilon+\tilde{C}_1\alpha_1+\frac{1}{4\tilde{\epsilon}}\left(\tilde{C}_2\,\alpha_2+\tilde{C}_1\alpha_3\right)^2+2(2\epsilon+\tilde{\epsilon})(\tilde{C}_1+\tilde{C}_2)^2<1$ with arbitrarily small constants $\epsilon,\tilde{\epsilon}$ in domain $(0,1)$ satisfied to obtain

$$\|\eta_{n0}\|^2\leq Ch^{2(k+1)}(\|n\|_{k+1}^2+\|\varphi\|_{k+1}^2).$$

Substituting the above estimate into (48) leads to

$$\|\eta_{\varphi 0}\|^2\leq Ch^{2(k+1)}(\|n\|_{k+1}^2+\|\varphi\|_{k+1}^2).$$

The proof of estimations in the L^2 norm is completed.

To complete the proof of the lemma, we verify the priori assumption (30). For $k\geq 1$, we consider h is small enough so that $Ch^k<h^{\frac{1}{2}}$. Our proof implies that

$$|||s-R_hs|||\leq |||s-Q_hs|||+|||\eta_s|||\leq Ch^k\,\|s\|_{k+1}+Ch^k(\|n\|_{k+1}+\|\varphi\|_{k+1})<h^{\frac{1}{2}}.$$

Hence, the assumption (30) holds. The proof of the lemma is completed. □

Remark 3. To derive the weak Galerkin Ritz projection error $\|\eta_{s0}\|$ with $s=n,\varphi$, the introduction of a dual problem (42)–(43) is a critical step in the proof.

2.3. **Error estimations.**

Theorem 1. *Let n, φ and n_h, φ_h be solutions of the problem (3)–(4) and semi-discrete SFWG finite element scheme (5)–(6), respectively. Assume that the exact solution is regular that $n,\varphi\in H^1(0,T;H^{k+1}(\Omega)\cap W^{1,\infty}(\Omega))$. Then, there exists a constant C independent of mesh size h such that*

$$
\begin{aligned}
\|n-n_{h0}\|^2+\|\varphi-\varphi_{h0}\|^2\leq\; & Ch^{2(k+1)}\left\{\|n\|_{k+1}^2+\|\varphi\|_{k+1}^2\right.\\
&\left.+\int_0^T\left(\|n\|_{k+1}^2+\|n_t\|_{k+1}^2+\|\varphi\|_{k+1}^2+\|\varphi_t\|_{k+1}^2\right)\,dt\right\},
\end{aligned}
\tag{55}
$$

$$|||n-n_h|||^2+|||\varphi-\varphi_h|||^2\leq Ch^{2k}(\|n\|_{k+1}^2+\|\varphi\|_{k+1}^2),\tag{56}$$

$$\int_0^T|||n-n_h|||^2\,dt\leq\; Ch^{2k}\int_0^T(\|n\|_{k+1}^2+\|\varphi\|_{k+1}^2)\,dt.\tag{57}$$

Proof. Subtracting the semi-discrete scheme from the system (11)–(12) derived from the Ritz projection (5)–(6) yields

$$
\begin{aligned}
&(n_t-(n_{h0})_t,v_0)-\left((3C_2n_d\,\nabla_wR_h\varphi+\mu\,\nabla_wR_h\varphi+\mathbf{C_3})\;n-2C_2\,\nabla_wR_h\varphi\,n^2,\nabla_wv\right)\\
&+\left((3C_2n_d\,\nabla_w\varphi_h+\mu\,\nabla_w\varphi_h+\mathbf{C_3})\,n_{h0}-2C_2\,\nabla_w\varphi_h\,n_{h0}^2,\nabla_wv\right)\\
&+\left((\tau\theta+\tau\mu^2\,\nabla_wR_h\varphi\otimes\nabla_wR_h\varphi)\,\nabla_wR_hn,\nabla_wv\right)\\
&-\left((\tau\theta+\tau\mu^2\,\nabla_w\varphi_h\otimes\nabla_w\varphi_h)\,\nabla_wn_h,\nabla_wv\right)=0,\quad\forall\,v\in V_h^0,
\end{aligned}
\tag{58}
$$

$$\frac{\varepsilon}{q}(\nabla_w(R_h\varphi-\varphi_h),\nabla_ww)+(R_0n-n_{h0},w_0)=0,\quad\forall\,w\in V_h^0.\tag{59}$$

Denoting

$$s - s_h = (s - Q_h s) + (Q_h s - R_h s) + (R_h s - s_h) := \rho_s + \eta_s + \xi_s, \quad s = n, \varphi.$$

Taking $w = \xi_\varphi$ in (59) and utilizing Lemma 4 arrive at

$$\frac{\varepsilon}{q}|||\xi_\varphi|||^2 = -(\xi_{n0}, \xi_{\varphi 0}) \le \|\xi_{n0}\| \, \|\xi_{\varphi 0}\| \le C_\sigma^* \|\xi_{n0}\| \, |||\xi_\varphi|||,$$

which implies

$$\frac{\varepsilon}{q}|||\xi_\varphi||| \le C_\sigma^* \|\xi_{n0}\|. \tag{60}$$

Since

$$\begin{aligned}
&-((3C_2 n_d + \mu)\,(\nabla_w R_h\varphi\, n - \nabla_w \varphi_h\, n_{h0}), \nabla_w v)\\
&\quad = -((3C_2 n_d + \mu)\,(\nabla_w R_h\varphi\,(n - n_{h0}) + \nabla_w (R_h\varphi - \varphi_h)\, n_{h0}), \nabla_w v)\\
&\quad = -((3C_2 n_d + \mu)\,(\nabla_w R_h\varphi\,(\rho_{n0} + \eta_{n0} + \xi_{n0}) + \nabla_w \xi_\varphi\, n_{h0}), \nabla_w v),\\
&-(\mathbf{C_3}(n - n_{h0}), \nabla_w v)\\
&\quad = -(\mathbf{C_3}(\rho_{n0} + \eta_{n0} + \xi_{n0}), \nabla_w v),\\
&\big(2C_2\,\nabla_w R_h\varphi\, n^2 - 2C_2\,\nabla_w \varphi_h\, n_{h0}^2, \nabla_w v\big)\\
&\quad = \big(2C_2\,\nabla_w R_h\varphi\,(n^2 - n_{h0}^2) - 2C_2\,\nabla_w (R_h\varphi - \varphi_h)\, n_{h0}^2, \nabla_w v\big)\\
&\quad = \big(2C_2\,\nabla_w R_h\varphi\,(n^2 - n_{h0}^2) - 2C_2\,\nabla_w \xi_\varphi\, n_{h0}^2, \nabla_w v\big),\\
&(\tau\mu^2\,\nabla_w R_h\varphi \otimes \nabla_w R_h\varphi\,\nabla_w R_h n, \nabla_w v) - (\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \varphi_h\,\nabla_w n_h, \nabla_w v)\\
&\quad = \big(\tau\mu^2\,\nabla_w (R_h\varphi - \varphi_h) \otimes \nabla_w R_h\varphi\,\nabla_w R_h n, \nabla_w v\big)\\
&\qquad + \big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w (R_h\varphi - \varphi_h)\,\nabla_w R_h n, \nabla_w v\big)\\
&\qquad + \big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \varphi_h\,\nabla_w (R_h n - n_h), \nabla_w v\big)\\
&\quad = \big(\tau\mu^2\,\nabla_w \xi_\varphi \otimes \nabla_w R_h\varphi\,\nabla_w R_h n, \nabla_w v\big) + \big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \xi_\varphi\,\nabla_w R_h n, \nabla_w v\big)\\
&\qquad + \big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \varphi_h\,\nabla_w \xi_n, \nabla_w v\big),
\end{aligned}$$

the error equation (58) can be transformed as:

$$\begin{aligned}
&((\eta_{n0})_t + (\xi_{n0})_t, v_0) - ((3C_2 n_d + \mu)\,(\nabla_w R_h\varphi\,(\rho_{n0} + \eta_{n0} + \xi_{n0}) + \nabla_w \xi_\varphi\, n_{h0}), \nabla_w v)\\
&- (\mathbf{C_3}(\rho_{n0} + \eta_{n0} + \xi_{n0}), \nabla_w v) + \big(2C_2\,\nabla_w R_h\varphi\,(n^2 - n_{h0}^2) - 2C_2\,\nabla_w \xi_\varphi\, n_{h0}^2, \nabla_w v\big)\\
&+ (\tau\theta\,\nabla_w \xi_n, \nabla_w v) + \big(\tau\mu^2\,\nabla_w \xi_\varphi \otimes \nabla_w R_h\varphi\,\nabla_w R_h n, \nabla_w v\big) + \big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \xi_\varphi\,\nabla_w R_h n, \nabla_w v\big)\\
&\quad + \big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \varphi_h\,\nabla_w \xi_n, \nabla_w v\big) = 0,
\end{aligned} \tag{61}$$

where the definition of Q_h is utilized.

Then, we take $v = \xi_n$ in Eqn. (61). Because $\tau \ge 0$ and matrix $\nabla_w \varphi_h \otimes \nabla_w \varphi_h$ is semi-positive, the term $\big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \varphi_h\,\nabla_w \xi_n, \nabla_w \xi_n\big) \ge 0$. It follows from inequality $a^2 - b^2 \le 2a(a-b)$ and ϵ-inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$ that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|\xi_{n0}\|^2 + \tau\theta\,|||\xi_n|||^2\\
\le&\tfrac{1}{2}\tfrac{d}{dt}\|\xi_{n0}\|^2 + \tau\theta\,|||\xi_n|||^2 + \big(\tau\mu^2\,\nabla_w \varphi_h \otimes \nabla_w \varphi_h\,\nabla_w \xi_n, \nabla_w \xi_n\big)\\
=&-((\eta_{n0})_t, \xi_{n0}) + ((3C_2 n_d + \mu)\,(\nabla_w R_h\varphi\,(\rho_{n0} + \eta_{n0} + \xi_{n0}) + \nabla_w \xi_\varphi\, n_{h0}), \nabla_w \xi_n)\\
&+ (\mathbf{C_3}(\rho_{n0} + \eta_{n0} + \xi_{n0}), \nabla_w \xi_n) - \big(2C_2\,\nabla_w R_h\varphi\,(n^2 - n_{h0}^2) - 2C_2\,\nabla_w \xi_\varphi\, n_{h0}^2, \nabla_w \xi_n\big)
\end{aligned}$$

$$
\begin{aligned}
&- \left(\tau\mu^2\,\nabla_w\xi_\varphi\otimes\nabla_w R_h\varphi\,\nabla_w R_h n,\nabla_w\xi_n\right) - \left(\tau\mu^2\,\nabla_w\varphi_h\otimes\nabla_w\xi_\varphi\,\nabla_w R_h n,\nabla_w\xi_n\right)\\
\le&\,\|(\eta_{n0})_t\|\,\|\xi_{n0}\| + \|\mathbf{C_3}\|_\infty\left(\|\rho_{n0}\|+\|\eta_{n0}\|+\|\xi_{n0}\|\right)|||\xi_n|||\\
&+\left(3\,\|C_2\|_\infty\,\|n_d\|_\infty+\|\mu\|_\infty\right)\left(\|\nabla_w R_h\varphi\|_\infty\left(\|\rho_{n0}\|+\|\eta_{n0}\|+\|\xi_{n0}\|\right)+\|n_{h0}\|_\infty\,|||\xi_\varphi|||\right)|||\xi_n|||\\
&+2\,\|C_2\|_\infty\left(2\,\|\nabla_w R_h\varphi\|_\infty\,\|n\|_\infty\left(\|\rho_{n0}\|+\|\eta_{n0}\|+\|\xi_{n0}\|\right)+\|n_{h0}\|_\infty^2\,|||\xi_\varphi|||\right)|||\xi_n|||\\
&+\tau\,\|\mu\|_\infty^2\,\|\nabla_w R_h n\|_\infty\left(\|\nabla_w R_h\varphi\|_\infty+\|\nabla_w\varphi_h\|_\infty\right)|||\xi_\varphi|||\,|||\xi_n|||\\
\le&\,\|(\eta_{n0})_t\|\,\|\xi_{n0}\| + C\left(1+\tilde{C}(1+2\,\|n\|_\infty)\right)\left(\|\rho_{n0}\|+\|\eta_{n0}\|+\|\xi_{n0}\|\right)|||\xi_n|||\\
&+C\left(\|n_{h0}\|_\infty+\|n_{h0}\|_\infty^2+\tilde{C}(\tilde{C}+\|\nabla_w\varphi_h\|_\infty)\right)|||\xi_\varphi|||\,|||\xi_n|||,
\end{aligned}
$$

where C is a constant of $\tau, \|\mu\|_\infty, \|C_2\|_\infty, \|\mathbf{C_3}\|_\infty$ and $\|n_d\|_\infty$.

We make a priori assumption

$$\|n-n_{h0}\|\le h^{\frac{3}{2}}, \tag{62}$$

which implies that $\|n_{h0}\|_\infty\le C$ and $\|\nabla_w\varphi_h\|_\infty\le C$. We will justify the priori assumption (62) later.

The above identity implies that

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\,\|\xi_{n0}\|^2+\tau\theta\,|||\xi_n|||^2\\
\le&\,\|(\eta_{n0})_t\|\,\|\xi_{n0}\|+C(\|\rho_{n0}\|+\|\eta_{n0}\|+\|\xi_{n0}\|)\,|||\xi_n|||+C|||\xi_\varphi|||\,|||\xi_n|||\\
\le&\,\tfrac{1}{2}\left(\|(\eta_{n0})_t^2\|+\|\xi_{n0}\|^2\right)+\tfrac{4}{8}\tau\theta\,|||\xi_n|||^2+C\tfrac{2}{\tau\theta}\left(\|\rho_{n0}\|^2+\|\eta_{n0}\|^2+\|\xi_{n0}\|^2+|||\xi_\varphi|||^2\right),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\,\|\xi_{n0}\|^2+\tfrac{1}{2}\tau\theta\,|||\xi_n|||^2\\
\le&\,\tfrac{1}{2}\left(\|(\eta_{n0})_t^2\|+\|\xi_{n0}\|^2\right)+C\tfrac{2}{\tau\theta}\left(\|\rho_{n0}\|^2+\|\eta_{n0}\|^2+\|\xi_{n0}\|^2+|||\xi_\varphi|||^2\right)\\
\le&\,\tfrac{1}{2}\|(\eta_{n0})_t^2\|+C\tfrac{2}{\tau\theta}\left(\|\rho_{n0}\|^2+\|\eta_{n0}\|^2\right)+\left(\tfrac{1}{2}+C\tfrac{2}{\tau\theta}\left(1+(\tfrac{q}{\varepsilon}C_\sigma^*)^2\right)\right)\|\xi_{n0}\|^2,
\end{aligned}\tag{63}
$$

where the estimate (60) is employed.

Integrating (63) from 0 to T for t, setting $n_h(x,0)=R_h n(x,0)$, using approximation properties of L^2 projection and Ritz projection Lemma 4, and the Gronwall inequality, we can derive

$$\|\xi_{n0}\|^2+|||\xi_n|||^2+\int_0^T|||\xi_n|||^2\,dt\le Ch^{2(k+1)}\int_0^T\left(\|n\|_{k+1}^2+\|n_t\|_{k+1}^2+\|\varphi\|_{k+1}^2+\|\varphi_t\|_{k+1}^2\right)dt. \tag{64}$$

Combining (60) with (64), it follows from Lemma 4 that

$$\|\xi_{\varphi 0}\|^2+|||\xi_\varphi|||^2\le Ch^{2(k+1)}\int_0^T\left(\|n\|_{k+1}^2+\|n_t\|_{k+1}^2+\|\varphi\|_{k+1}^2+\|\varphi_t\|_{k+1}^2\right)dt. \tag{65}$$

By the approximation properties of projection operators Q_h and R_h, and Lemma 5, the error estimations (55), (56) and (57) are completed.

Finally, to complete the proof, we verify the assumption (62). For $k\ge 1$, we consider h is small enough so that $Ch^{k+1}<h^{\frac{3}{2}}$, where C is the constant in (62) determined by the final time T. Then, if $t^*=sup\left\{t:\|n(t)-n_{h0}(t)\|\le h^{\frac{3}{2}}\right\}$, we should have $\|n(t^*)-n_{h0}(t^*)\|=h^{\frac{3}{2}}$ by continuity if t^* is finite. On the other hand, our proof implies that (55) holds for $t\le t^*$, in particular $\|n(t^*)-n_{h0}(t^*)\|\le Ch^{k+1}<h^{\frac{3}{2}}$. This is a contradiction if $t^*<T$. Thus, $t^*\ge T$ and the assumption (62) is correct. □

Remark 4. Without employing the weak Galerkin Ritz projection, one observes in numerical experiments that the errors $|||Q_h s - s_h|||$, i.e., L^2 projection error in energy norm, only reach a rate of $O(k)$. This behavior leads to sub-optimal L^2 convergence rates for electrical potential and concentration functions. To address this issue, the introduction of the Ritz projection operator is crucial. This adjustment attains superconvergence for the energy estimations of $|||\xi_n|||$ and $|||\xi_\varphi|||$, leading to optimal L^2 convergence rates as detailed in the referenced inequalities (64) and (65) (cf. [14, 15]).

3. Fully discrete SFWG finite element method

The objective of this section is to establish a fully discrete SFWG finite element scheme and derive the error estimations for the numerical solutions.

3.1. **Fully discrete scheme.** Let Δt represent the size of the time step, and define $t_m = m\Delta t\,(m = 0, 1, \cdots, M)$. The linearized method is employed for time discretization to develop a fully discrete SFWG finite element scheme. The backward difference quotient $\bar{\partial}_t n_h^m = (n_h^m - n_h^{m-1})/\Delta t$ is used as an approximation $(n_h)_t$ at time t_m. Consequently, the fully discrete SFWG finite element scheme can be articulated as follows: $(n_h^m, \varphi_h^m) \in V_h^n \times V_h^\varphi, m = 1, 2, \cdots, M$ such that

$$\begin{aligned}(\bar{\partial}_t n_{h0}^m, v_0) - \left((3C_2 n_d\,\nabla_w\varphi_h^{m-1} + \mu\,\nabla_w\varphi_h^{m-1} + \mathbf{C_3})\,n_{h0}^m - 2C_2\,\nabla_w\varphi_h^m\,(n_{h0}^{m-1})^2, \nabla_w v\right)\\ + \left((\tau\theta + \tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\varphi_h^{m-1})\,\nabla_w n_h^m, \nabla_w v\right) = 0, \quad \forall\, v \in V_{h0}, \qquad (66)\\ \frac{\varepsilon}{q}(\nabla_w\varphi_h^m, \nabla_w w) + (n_{h0}^m, w_0) = (n_d, w_0), \qquad \forall\, w \in V_{h0}. \qquad (67)\end{aligned}$$

3.2. **Error estimations.**

Theorem 2. *Let n, φ and n_h^m, φ_h^m be solutions of the problem (3)-(4) and fully discrete SFWG finite element scheme (66)-(67), respectively. Assume that the exact solution is regular such that $n, \varphi \in H^1(0, T; H^{k+1}(\Omega) \cap W^{1,\infty}(\Omega))$, then*

$$\begin{aligned}&\left\|n^M - n_{h0}^M\right\|^2 + \|\varphi^M - \varphi_{h0}^M\|^2 \le C\left\{\Delta t^2\left(1 + \int_0^T \|n_{tt}\|^2\,dt\right)\right.\\ &\left. + h^{2(k+1)}\int_0^T (\|n_t\|_{k+1}^2 + \|\varphi_t\|_{k+1}^2)\,dt + h^{2(k+1)}\max_{0\leqslant i\leqslant M}\left(\|n^i\|_{k+1}^2 + \|\varphi^i\|_{k+1}^2\right)\right\}, \qquad (68)\\ &|||n^M - n_h^M|||^2 + |||\varphi^M - \varphi_h^M|||^2\\ &\le C\left\{\Delta t^2\left(1 + \int_0^T \|n_{tt}\|^2\,dt\right) + h^{2k}\left(\|n^M\|_{k+1}^2 + \|\varphi^M\|_{k+1}^2\right)\right\}, \qquad (69)\\ &\Delta t\sum_{i=1}^M |||n^i - n_h^i|||^2 \le C\left\{\Delta t^2\left(1 + \int_0^T \|n_{tt}\|^2\,dt\right) + h^{2k}\max_{1\leqslant i\leqslant M}\left(\|n^i\|_{k+1}^2 + \|\varphi^i\|_{k+1}^2\right)\right\}, \qquad (70)\end{aligned}$$

where C is a positive constant independent of Δt and h.

Proof. Subtracting a fully discrete scheme (66)-(67) from the system (11)-(12), the following error equations can be obtained as follows.

$$\begin{aligned}&(n_t^m - \bar{\partial}_t n_{h0}^m, v_0) - \left((3C_2 n_d\,\nabla_w R_h\varphi^m + \mu\,\nabla_w R_h\varphi^m + \mathbf{C_3})\,n^m - 2C_2\,\nabla_w R_h\varphi^m\,(n^m)^2, \nabla_w v\right)\\ &+ \left((3C_2 n_d\,\nabla_w\varphi_h^{m-1} + \mu\,\nabla_w\varphi_h^{m-1} + \mathbf{C_3})\,n_{h0}^m - 2C_2\,\nabla_w\varphi_h^m\,(n_{h0}^{m-1})^2, \nabla_w v\right)\end{aligned}$$

$$+\left((\tau\theta+\tau\mu^2\,\nabla_w R_h\varphi^m\otimes\nabla_w R_h\varphi^m)\,\nabla_w R_h n^m,\nabla_w v\right)$$
$$-\left((\tau\theta+\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\varphi_h^{m-1})\,\nabla_w n_h^m,\nabla_w v\right)=0,\forall\ v\in V_h^0, \tag{71}$$
$$\frac{\varepsilon}{q}(\nabla_w(R_h\varphi^m-\varphi_h^m),\nabla_w w)+(R_0 n^m-n_{h0}^m,w_0)=0,\qquad\forall\, w\in V_h^0. \tag{72}$$

Denoting

$$\begin{aligned}&n_t^m-\bar\partial_t n_h^m\\&=(n_t^m-\bar\partial_t n^m)+\bar\partial_t(n^m-Q_h n^m)+\bar\partial_t(Q_h n^m-R_h n^m)+\bar\partial_t(R_h n^m-n_h^m)\\&:=(n_t^m-\bar\partial_t n^m)+\bar\partial_t\rho_n^m+\bar\partial_t\eta_n^m+\bar\partial_t\xi_n^m.\end{aligned}$$

Because of the definition of L^2 projetion Q_h, there holds $(\bar\partial_t\rho_n^m,v_0)=0$. By employing a similar approach in semi-discrete procedures in equation (61), there hold

$$\begin{aligned}&-\left((3C_2n_d+\mu)\,\left(\nabla_w R_h\varphi^m\,n^m-\nabla_w\varphi_h^{m-1}\,n_{h0}^m\right),\nabla_w v\right)\\&\quad=-\left((3C_2n_d+\mu)\,\left(\nabla_w R_h(\varphi^m-\varphi^{m-1})\,n^m+\nabla_w\xi_\varphi^{m-1}\,n^m\right),\nabla_w v\right)\\&\qquad-\left((3C_2n_d+\mu)\,\nabla_w\varphi_h^{m-1}\,(\rho_{n0}^m+\eta_{n0}^m+\xi_{n0}^m),\nabla_w v\right),\\&\left(2C_2\,\nabla_w R_h\varphi^m\,(n^m)^2-2C_2\,\nabla_w\varphi_h^m\,(n_{h0}^{m-1})^2,\nabla_w v\right)\\&\quad=\left(2C_2\,\left\{\nabla_w R_h\varphi^m\,[(n^m)^2-(n^{m-1})^2+(n^{m-1})^2-(n_{h0}^{m-1})^2]+\nabla_w\xi_\varphi^m\,(n_{h0}^{m-1})^2\right\},\nabla_w v\right),\\&(\tau\mu^2\,\nabla_w R_h\varphi^m\otimes\nabla_w R_h\varphi^m\,\nabla_w R_h n^m,\nabla_w v)-(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\varphi_h^{m-1}\,\nabla_w n_h^m,\nabla_w v)\\&\quad=\left(\tau\mu^2\,\nabla_w R_h(\varphi^m-\varphi^{m-1})\otimes\nabla_w R_h\varphi^m\,\nabla_w R_h n^m,\nabla_w v\right)\\&\qquad+\left(\tau\mu^2\,\nabla_w\xi_\varphi^{m-1}\otimes\nabla_w R_h\varphi^m\,\nabla_w R_h n^m,\nabla_w v\right)\\&\qquad+\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w R_h(\varphi^m-\varphi^{m-1})\,\nabla_w R_h n^m,\nabla_w v\right)\\&\qquad+\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\xi_\varphi^{m-1}\,\nabla_w R_h n^m,\nabla_w v\right)\\&\qquad+\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\varphi_h^{m-1}\,\nabla_w\xi_n^m,\nabla_w v\right).\end{aligned}$$

Therefore, Eqn. (71) can be transformed into

$$\begin{aligned}&\left((n_t^m-\bar\partial_t n^m)+\bar\partial_t\eta_{n0}^m+\bar\partial_t\xi_{n0}^m,v_0\right)+(\tau\theta\,\nabla_w\xi_n^m,\nabla_w v)-(\mathbf{C_3}(\rho_{n0}^m+\eta_{n0}^m+\xi_{n0}^m),\nabla_w v)\\&-\left((3C_2n_d+\mu)\,\left(\nabla_w R_h(\varphi^m-\varphi^{m-1})\,n^m+\nabla_w\xi_\varphi^{m-1}\,n^m\right),\nabla_w v\right)\\&-\left((3C_2n_d+\mu)\,\nabla_w\varphi_h^{m-1}\,(\rho_{n0}^m+\eta_{n0}^m+\xi_{n0}^m),\nabla_w v\right)\\&+\left(2C_2\,\left\{\nabla_w R_h\varphi^m\,[(n^m)^2-(n^{m-1})^2+(n^{m-1})^2-(n_{h0}^{m-1})^2]+\nabla_w\xi_\varphi^m(n_{h0}^{m-1})^2\right\},\nabla_w v\right)\\&\quad+\left(\tau\mu^2\,\nabla_w R_h(\varphi^m-\varphi^{m-1})\otimes\nabla_w R_h\varphi^m\,\nabla_w R_h n^m,\nabla_w v\right)\\&\quad+\left(\tau\mu^2\,\nabla_w\xi_\varphi^{m-1}\otimes\nabla_w R_h\varphi^m\,\nabla_w R_h n^m,\nabla_w v\right)\\&\quad+\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w R_h(\varphi^m-\varphi^{m-1})\,\nabla_w R_h n^m,\nabla_w v\right)\\&\quad+\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\xi_\varphi^{m-1}\,\nabla_w R_h n^m,\nabla_w v\right)\\&\quad+\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\varphi_h^{m-1}\,\nabla_w\xi_n^m,\nabla_w v\right)=0.\end{aligned}\tag{73}$$

Taking $w=\xi_\varphi^m$ in (72) and utilizing Lemma 4, we arrive at

$$\frac{\varepsilon}{q}\left|\!\left|\!\left|\xi_\varphi^m\right|\!\right|\!\right|\le C_\sigma^*\left\|\xi_{n0}^m\right\|. \tag{74}$$

Then, we analyze the term $\|\xi_{n0}^m\|$ by letting $v=\xi_n^m$ in (73) to obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{1}{\Delta t}\left(\|\xi_{n0}^m\|^2-\left\|\xi_{n0}^{m-1}\right\|^2\right)+\tau\theta\left|\!\left|\!\left|\xi_n^m\right|\!\right|\!\right|^2\\&\le(\bar\partial_t\xi_{n0}^m,\xi_{n0}^m)+\tau\theta\left|\!\left|\!\left|\xi_n^m\right|\!\right|\!\right|^2+\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\varphi_h^{m-1}\,\nabla_w\xi_n^m,\nabla_w\xi_n^m\right)\\&=-\left((n_t^m-\bar\partial_t n^m)+\bar\partial_t\eta_{n0}^m,\xi_{n0}^m\right)+(\mathbf{C_3}(\rho_{n0}^m+\eta_{n0}^m+\xi_{n0}^m),\nabla_w\xi_n^m)\end{aligned}$$

$$
\begin{aligned}
&+\left((3C_2n_d+\mu)\,\left(\nabla_w R_h(\varphi^m-\varphi^{m-1})\,n^m+\nabla_w\xi_\varphi^{m-1}\,n^m\right),\nabla_w\xi_n^m\right)\\
&+\left((3C_2n_d+\mu)\,\nabla_w\varphi_h^{m-1}\,(\rho_{n0}^m+\eta_{n0}^m+\xi_{n0}^m),\nabla_w\xi_n^m\right)\\
&-\left(2C_2\,\left\{\nabla_w R_h\varphi^m\,[(n^m)^2-(n^{m-1})^2+(n^{m-1})^2-(n_{h0}^{m-1})^2]+\nabla_w\xi_\varphi^m(n_{h0}^{m-1})^2\right\},\nabla_w\xi_n^m\right)\\
&-\left(\tau\mu^2\,\nabla_w R_h(\varphi^m-\varphi^{m-1})\otimes\nabla_w R_h\varphi^m\,\nabla_w R_h n^m,\nabla_w\xi_n^m\right)\\
&-\left(\tau\mu^2\,\nabla_w\xi_\varphi^{m-1}\otimes\nabla_w R_h\varphi^m\,\nabla_w R_h n^m,\nabla_w\xi_n^m\right)\\
&-\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w R_h(\varphi^m-\varphi^{m-1})\,\nabla_w R_h n^m,\nabla_w\xi_n^m\right)\\
&-\left(\tau\mu^2\,\nabla_w\varphi_h^{m-1}\otimes\nabla_w\xi_\varphi^{m-1}\,\nabla_w R_h n^m,\nabla_w\xi_n^m\right)\\
\le\;&\left(\left\|n_t^m-\bar\partial_t n^m\right\|+\left\|\bar\partial_t\eta_{n0}^m\right\|\right)\,\|\xi_{n0}^m\|+C(1+\left\|\nabla_w\varphi_h^{m-1}\right\|_\infty)\,(\|\rho_{n0}^m\|+\|\eta_{n0}^m\|+\|\xi_{n0}^m\|)\,|||\xi_n^m|||\\
&+C\left(\left[\tilde C\,\|n^m\|_\infty\,(1+\|n_t^m\|_\infty)+\tilde C^2(\tilde C^2+\|\nabla_w\varphi_h^{m-1}\|_\infty)\right]\,(\Delta t+o(\Delta t))+\|n^m\|_\infty\,|||\xi_\varphi^{m-1}|||\right)|||\xi_n^m|||\\
&+C\left(2\tilde C\left\|n^{m-1}\right\|_\infty\,(\left\|\rho_{n0}^{m-1}\right\|+\left\|\eta_{n0}^{m-1}\right\|+\left\|\xi_{n0}^{m-1}\right\|)+\left\|n_{h0}^{m-1}\right\|_\infty^2\,|||\xi_\varphi^m|||\right)|||\xi_n^m|||\\
&+C\,\left(\|n^m\|_\infty+\tilde C(\tilde C+\|\nabla_w\varphi_h^{m-1}\|_\infty)\right)\,|||\xi_\varphi^{m-1}|||\,|||\xi_n^m|||,
\end{aligned}
\tag{75}
$$

where C is a constant of $\tau,\|\mu\|_\infty,\|C_2\|_\infty,\|\mathbf{C_3}\|_\infty$ and $\|n_d\|_\infty$.

A priori assumption is taken for m,

$$
\left\|n^j-n_{h0}^j\right\|\le h^{\frac{3}{2}},\quad\text{for } 0\le j\le M-1,
\tag{76}
$$

which implies that $\max\limits_{0\le j\le M-1}\left\|n_{h0}^j\right\|_\infty\le C$ and $\max\limits_{0\le j\le M-1}\left\|\nabla_w\varphi_h^j\right\|_\infty\le C$. We will justify the priori assumption (76) at the end of the proof.

The estimate (75) can be transformed as:

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{1}{\Delta t}\left(\|\xi_{n0}^m\|^2-\left\|\xi_{n0}^{m-1}\right\|^2\right)+\tau\theta\,|||\xi_n^m|||^2\\
\le\;&\left(\left\|n_t^m-\bar\partial_t n^m\right\|+\left\|\bar\partial_t\eta_{n0}^m\right\|\right)\,\|\xi_{n0}^m\|+C\,(\Delta t+o(\Delta t))|||\xi_n^m|||\\
&+C(\|\rho_{n0}^m\|+\|\eta_{n0}^m\|+\|\xi_{n0}^m\|+\left|||\xi_\varphi^m|||\right|^2+\left\|\rho_{n0}^{m-1}\right\|+\left\|\eta_{n0}^{m-1}\right\|+\left\|\xi_{n0}^{m-1}\right\|+\left|||\xi_\varphi^{m-1}|||\right|)\,|||\xi_n^m|||\\
\le\;&\tfrac{1}{2}\left(\left\|n_t^m-\bar\partial_t n^m\right\|^2+\left\|\bar\partial_t\eta_{n0}^m\right\|^2\right)+\|\xi_{n0}^m\|^2+C\,(\Delta t^2+o(\Delta t^2))+\tfrac{9}{10}\tau\theta\,|||\xi_n^m|||^2\\
&+C\tfrac{5}{2\tau\theta}\,\left(\|\rho_{n0}^m\|^2+\|\eta_{n0}^m\|^2+\|\xi_{n0}^m\|^2+\left|||\xi_\varphi^m|||\right|^2+\left\|\rho_{n0}^{m-1}\right\|+\left\|\eta_{n0}^{m-1}\right\|+\left\|\xi_{n0}^{m-1}\right\|+\left|||\xi_\varphi^{m-1}|||\right|^2\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{1}{\Delta t}\left(\|\xi_{n0}^m\|^2-\left\|\xi_{n0}^{m-1}\right\|^2\right)+\tfrac{1}{10}\tau\theta\,|||\xi_n^m|||^2\\
\le\;&\tfrac{1}{2}\left(\left\|n_t^m-\bar\partial_t n^m\right\|^2+\left\|\bar\partial_t\eta_{n0}^m\right\|^2\right)+C\,\Delta t^2+\left(1+C\tfrac{5}{2\tau\theta}\,\left[1+(\tfrac{q}{\varepsilon}C_\sigma^*)^2\right]\right)\,\|\xi_{n0}^m\|^2,\\
&+C\tfrac{5}{2\tau\theta}\,\left(\|\rho_{n0}^m\|^2+\|\eta_{n0}^m\|^2+\left\|\rho_{n0}^{m-1}\right\|^2+\left\|\eta_{n0}^{m-1}\right\|^2\right)+C\tfrac{5}{2\tau\theta}\,\left(1+(\tfrac{q}{\varepsilon}C_\sigma^*)^2\right)\,\left\|\xi_{n0}^{m-1}\right\|^2,
\end{aligned}
\tag{77}
$$

where the estimate (74) is used.

Next, summing the inequality (77) over the time step m, letting $n_h^0=R_hn_0$ and utilizing the discrete Gronwall inequality and priori assumption (76), we get

$$
\begin{aligned}
&\left\|\xi_{n0}^M\right\|^2+\tau\theta\,\Delta t\sum_{i=1}^M\left|||\xi_n^i|||\right|^2\\
&\le\Delta t\sum_{i=1}^M\left(\left\|n_t^i-\bar\partial_t n^i\right\|^2+\left\|\bar\partial_t\eta_{n0}^i\right\|^2\right)+C\,\Delta t^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C\tfrac{5}{2\tau\theta}\,\Delta t \sum_{i=0}^{M}\left(\left\|\rho_{n0}^{i}\right\|^{2}+\left\|\eta_{n0}^{i}\right\|^{2}\right)\\
&\leq \Delta t\sum_{i=1}^{M}\left(\left\|n_{t}^{i}-\bar{\partial}_{t}n^{i}\right\|^{2}+\left\|\bar{\partial}_{t}\eta_{n0}^{i}\right\|^{2}\right)+C\,\Delta t^{2}\\
&+ C\tfrac{5}{2\tau\theta}\Delta t\,h^{2(k+1)}\sum_{i=0}^{M}\left(\left\|n^{i}\right\|_{k+1}^{2}+\left\|\varphi^{i}\right\|_{k+1}^{2}\right). \qquad (78)
\end{aligned}
$$

By Taylor expansion, there holds

$$
n_{t}^{i}-\bar{\partial}_{t}n^{i}=\frac{1}{\Delta t}\int_{t_{i-1}}^{t_{i}}(t-t_{i-1})\,n_{tt}\,dt,
$$

$$
\bar{\partial}_{t}\eta_{n}^{i}=\bar{\partial}_{t}\left(Q_{h}n(t_{i})-R_{h}n(t_{i})\right)=\frac{1}{\Delta t}\int_{t_{i-1}}^{t_{i}}\left(Q_{h}-R_{h}\right)n_{t}\,dt.
$$

With Hölder inequality, the approximation properties, and Lemma 5, we obtain

$$
\sum_{i=1}^{M}\left\|n_{t}^{i}-\bar{\partial}_{t}n^{i}\right\|^{2}\leq\sum_{i=1}^{M}\frac{1}{\Delta t^{2}}\left(\int_{t_{i-1}}^{t_{i}}(t-t_{i-1})^{2}dt\right)\cdot\left(\int_{t_{i-1}}^{t_{i}}\left\|n_{tt}\right\|^{2}dt\right)=\frac{\Delta t}{3}\int_{0}^{T}\left\|n_{tt}\right\|^{2}\,dt, \qquad (79)
$$

$$
\begin{aligned}
&\sum_{i=1}^{M}\left\|\bar{\partial}_{t}\eta_{n0}^{i}\right\|^{2}=\frac{1}{\Delta t}\sum_{i=1}^{M}\int_{t_{i-1}}^{t_{i}}\left\|\left(Q_{0}-R_{0}\right)n_{t}\right\|^{2}\,dt\\
&\leq\Delta t^{-1}\,Ch^{2(k+1)}\int_{0}^{T}(\|n_{t}\|_{k+1}^{2}+\|\varphi_{t}\|_{k+1}^{2})dt. \qquad (80)
\end{aligned}
$$

It is easy to verify that

$$
\Delta t\sum_{i=0}^{M}\|s^{i}\|_{k+1}^{2}\leq M\Delta t\max_{0\leqslant i\leqslant M}\|s^{i}\|_{k+1}^{2}\leq T\max_{0\leqslant i\leqslant M}\|s^{i}\|_{k+1}^{2},\quad s=n,\varphi. \qquad (81)
$$

Combining (78) with inequalities (79) to (81) and utilizing Lemma 4 lead to

$$
\begin{aligned}
&\left\|\xi_{n0}^{M}\right\|^{2}+\left|\left|\left|\xi_{n}^{M}\right|\right|\right|^{2}+\Delta t\sum_{i=1}^{M}\left|\left|\left|\xi_{n}^{i}\right|\right|\right|^{2}\\
&\leq C\left\{\Delta t^{2}\left(1+\int_{0}^{T}\left\|n_{tt}\right\|^{2}\,dt\right)+h^{2(k+1)}\int_{0}^{T}(\|n_{t}\|_{k+1}^{2}+\|\varphi_{t}\|_{k+1}^{2})\,dt\right.\\
&\left.+h^{2(k+1)}\max_{0\leqslant i\leqslant M}\left(\|n^{i}\|_{k+1}^{2}+\|\varphi^{i}\|_{k+1}^{2}\right)\right\}, \qquad (82)
\end{aligned}
$$

where C is a constant of $\tau,\theta,\|\mu\|_{\infty}$, $\max\limits_{1\leq j\leq M-1}\left\|n_{h0}^{j}\right\|_{\infty}$ and $\max\limits_{1\leq j\leq M-1}\left\|\nabla_{w}\varphi_{h}^{j}\right\|_{\infty}$. By substituting estimate (82) into (74), it follows from Lemma 4 that

$$
\|\xi_{\varphi0}^{M}\|^{2}+\left|\left|\left|\xi_{\varphi}^{M}\right|\right|\right|^{2}\leq C\left\{\Delta t^{2}\left(1+\int_{0}^{T}\left\|n_{tt}\right\|^{2}\,dt\right)\right.
$$

$$+ h^{2(k+1)} \int_0^T (\|n_t\|_{k+1}^2 + \|\varphi_t\|_{k+1}^2)\, dt$$

$$+h^{2(k+1)} \max_{0\leqslant i\leqslant M} \left(\|n^i\|_{k+1}^2 + \|\varphi^i\|_{k+1}^2\right)\Bigg\}. \tag{83}$$

Because of the approximation properties of operators Q_h and R_h, and Lemma 5, the demonstration of estimations (68), (69) and (70) is completed.

Finally, to complete the proof, we verify the assumption (76). For $m = 0$, due to we choose $n_h^0 = R_h n_0$, obviously the assumption (76) holds. If (76) holds for $m = 1, 2, \cdots, M-1$, then for $m = M$, we can get (68), i.e., (76) also holds true for $m = M$ with $k \geq 1$. □

4. Numerical experiments

In this section, numerical examples are presented with the objective of validating theoretical results. For temporal and nonlinear discretization, the linearized method is employed. Uniform triangular meshes, constructed by dividing the domain uniformly into $N_1 \times N_2$ sub-intervals and bisecting the rectangles into two right triangles using a backward slash, are utilized for the numerical simulations. The weak finite element space with the degree of freedom $j = k + 1$ is employed for computing the numerical solutions.

4.1. **Convergence order test.** For the domain $\Omega = (0, 2\pi) \times (0, 2\pi)$ and time $T = 1$, a nonlinear coupled equations are considered as follows.

$$\begin{aligned} n_t + \nabla \cdot \left((4\nabla\varphi + 1)n - 2\nabla\varphi\, n^2\right) - \nabla \cdot \left((1 + \nabla\varphi \otimes \nabla\varphi)\, \nabla n\right) &= f(x, y, t), \\ -\Delta\varphi + n &= g(x, y, t). \end{aligned}$$

The accurate functions are $n = e^{-t} \sin x \sin y$ and $\varphi = e^{-t} \cos x \cos y$. Errors and associated convergence orders for electron concentration and electric potential for $1 \leq k \leq 2$, $\Delta t = h^{k+1}$ at $T = 1$ are recorded in the referenced tables, Tables 1 and 2.

Table 1. Error profiles and convergence rates of Example 1 with $k = 1$ and $\tau = h^2$.

N	$k=1$											
	$\|\|\rho_{n0}\|\|$	Rate	$\|\|\|\rho_n\|\|\|$	Rate	$\|\|\eta_{n0}\|\|$	Rate	$\|\|\|\eta_n\|\|\|$	Rate	$\|\|\xi_{n0}\|\|$	Rate	$\|\|\|\xi_n\|\|\|$	Rate
8	5.72E-03	-	1.18E-01	-	5.37E-03	-	7.81E-02	-	2.89E-02	-	6.97E-02	-
16	1.44E-03	1.99	5.97E-02	0.99	1.36E-03	1.98	3.92E-02	1.00	9.16E-03	1.66	2.25E-02	1.63
32	3.59E-04	2.00	2.99E-02	1.00	3.42E-04	1.99	1.96E-02	1.00	2.28E-03	2.01	5.55E-03	2.02
64	8.99E-05	2.00	1.49E-02	1.00	8.56E-05	2.00	9.80E-03	1.00	5.72E-04	2.00	1.39E-03	2.00
	$\|\|\rho_{\varphi 0}\|\|$	Rate	$\|\|\|\rho_\varphi\|\|\|$	Rate	$\|\|\eta_{\varphi 0}\|\|$	Rate	$\|\|\|\eta_\varphi\|\|\|$	Rate	$\|\|\xi_{\varphi 0}\|\|$	Rate	$\|\|\|\xi_\varphi\|\|\|$	Rate
8	5.72E-03	-	1.18E-01	-	1.06E-02	-	7.90E-02	-	1.00E-02	-	1.56E-02	-
16	1.44E-03	1.99	5.97E-02	0.99	2.68E-03	1.98	3.95E-02	1.00	3.01E-03	1.73	4.77E-03	1.71
32	3.59E-04	2.00	2.99E-02	1.00	6.71E-04	1.99	1.97E-02	1.00	7.58E-04	1.99	1.20E-03	2.00
64	8.99E-05	2.00	1.49E-02	1.00	1.68E-04	2.00	9.87E-03	1.00	1.90E-04	1.99	3.00E-04	1.99

It is shows that the terms ρ_s and η_s with $s = n, \varphi$ in Tables 1 and 2 achieve $O(h^{k+1})$ in L^2 norm and $O(h^k)$ in energy norm, respectively. This behaviour is consistent with approximation property of L^2 projection operator Q_h and Lemma 5. In particular, the term ξ_n and ξ_φ reveal that the errors in both norms achieve a convergence rate of $O(h^{k+1})$, which is compatible with the estimations (82) and (83) in the proof of Theorem 2. The numerical solutions depicted validate the theoretical predictions, showcasing convergence orders of the SFWG-FEM are consistent with the theoretical analysis presented in Lemma 5 and Theorem 2.

TABLE 2. Error profiles and convergence rates of Example 1 with $k = 2$ and $\tau = h^3$.

N	$k=1$											
	$\|\|\rho_{n0}\|\|$	Rate	$\|\|\|\rho_n\|\|\|$	Rate	$\|\|\eta_{n0}\|\|$	Rate	$\|\|\|\eta_n\|\|\|$	Rate	$\|\|\xi_{n0}\|\|$	Rate	$\|\|\|\xi_n\|\|\|$	Rate
8	3.17E-04	-	8.99E-03	-	1.78E-04	-	5.83E-03	-	1.92E-02	-	4.74E-02	-
16	3.98E-05	2.99	2.26E-03	1.99	2.20E-05	3.01	1.45E-03	2.01	2.52E-03	2.93	6.12E-03	2.95
32	4.98E-06	3.00	5.66E-04	2.00	2.74E-06	3.01	3.62E-04	2.00	3.21E-04	2.97	7.79E-04	2.97
	$\|\|\rho_{\varphi 0}\|\|$	Rate	$\|\|\|\rho_\varphi\|\|\|$	Rate	$\|\|\eta_{\varphi 0}\|\|$	Rate	$\|\|\|\eta_\varphi\|\|\|$	Rate	$\|\|\xi_{\varphi 0}\|\|$	Rate	$\|\|\|\xi_\varphi\|\|\|$	Rate
8	3.17E-04	-	8.99E-03	-	1.96E-04	-	6.07E-03	-	6.21E-03	-	9.89E-03	-
16	3.98E-05	2.99	2.26E-03	1.99	2.32E-05	3.08	1.51E-03	2.01	8.41E-04	2.88	1.33E-03	2.90
32	4.98E-06	3.00	5.66E-04	2.00	2.82E-06	3.04	3.74E-04	2.01	1.07E-04	2.97	1.69E-04	2.97

4.2. **Semiconductor device simulation.** The high-field model simulation for a 2D MESFET device considers a device measuring $0.6 \times 0.2\,\mu m^2$. In this example, the source and drain are each $0.1\,\mu m$ in size, localized at $x \in [0, 0.1]$ and $y = 0.2$ for the source, and $x \in [0.5, 0.6]$ for the drain, with both at $y = 0.2$. The gate occupies $0.2\,\mu m$ across $x \in [0.2, 0.4]$ and is positioned at $y = 0.2$. The doping profile defined for the simulation sets $n_d = 3 \times 10^{17} cm^{-3}$ within the regions $[0, 0.1] \times [0.15, 0.2]$ and $[0.5, 0.6] \times [0.15, 0.2]$, and $n_d = 1 \times 10^{17} cm^{-3}$ in all other areas of the device, with abrupt transitions at the junctions.

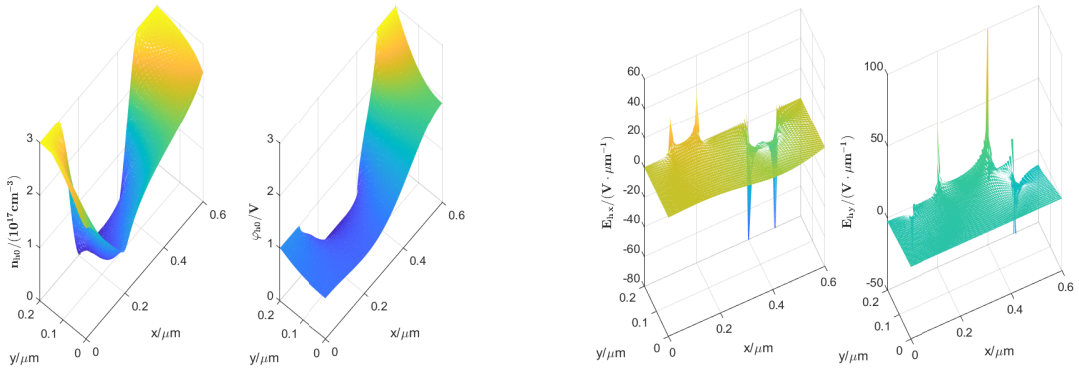


FIGURE 1. SFWG-FEM solution to simulate 2D MESFET HF model with a uniform 72×24 mesh with $k = 1$. Left: density $n\,(10^{12} cm^{-3})$; middle left: electric potential $\varphi\,(V)$; middle right: the horizontal electric field $E_{hx}\,(V/\mu m)$; right: the vertical electric field $E_{hy}\,(V/\mu m)$.

For the simulation, the lattice temperature is set at $T_0 = 300°K$. The constants involved are the Boltzmann constant $k = 0.138 \times 10^{-4}$, permittivity $\varepsilon = 11.7 \times 8.85418$, electron charge $q = 0.1602$, effective electron mass $m = 0.26 \times 0.9109 \times 10^{-31} kg$, and mobility $\mu = 0.75$. The initial electron concentration condition is set as $n(x, 0) = n_d(x)$. Detailed numerical boundary conditions have been listed as follows for both potential and concentration:

(1) At the source $(0 \le x \le 0.1, y = 0.2)$: $\varphi = \varphi_0$, $n = 3 \times 10^{17} cm^{-3}$, $\varphi_0 = \frac{kT}{q} \ln\left(\frac{n_d}{n_i}\right)$, $n_i = 1.4 \times 10^{10} cm^{-3}$.
(2) At the drain $(0.5 \le x \le 0.6, y = 0.2)$: $\varphi = \varphi_0 + 2$, $n = 3 \times 10^{17} cm^{-3}$.
(3) At the gate $(0.2 \le x \le 0.4, y = 0.2)$: $\varphi = \varphi_0 - 0.8$, $n = 3.9 \times 10^5 cm^{-3}$.
(4) For all other parts of the boundary, zero Neumann boundary conditions are applied to both unknown functions.

The simulation results using the SFWG-FEM solver with a parameter $k = 1$ on a 72×24 uniform mesh have been visualized, showcasing the surface plots for electron concentration n_{h0}, electric potential φ_{h0}, and the electric field components E_{hx} and E_{hy}. The final solution at

$T = 0.226$ reveals several peculiarities, such as singularities and sharp transitions which are likely due to the discontinuous nature of the doping profile n_d and the application of mixed Dirichlet-Neumann boundary conditions. The trend of the graphs, especially the results of electric potential and electric field, aligns with what the authors reported in [36] using local discontinuous Galerkin methods.

5. Conclusion

The linearized stabilizer-free weak Galerkin finite element method (SFWG-FEM) for two-dimensional unsteady-state nonlinear coupled high-field model is developed. The piecewise polynomial elements $P_k(K), P_k(e)$, $[P_j(K)]^2$ are associated with different parts of each finite element K and boundary $e \subset \partial K$, where $k \geq 1$ denotes the polynomial degree. The parameter satisfies $j \geq j_0$, and $j_0 = k + m - 2$, or alternatively $j_0 = k + m - 3$ under the condition that each edge of the element K is parallel to another edge of K, where m represents the number of edges of the element K. Incorporating a weak Galerkin Ritz projection, optimal order error estimations for both energy and L^2 norms are derived. A rigorous analysis of the non-standard Ritz projection error is given by introducing a dual problem. The superconvergence between Ritz projection of the exact functions and the numerical solutions in energy norm results in optimal L^2 error estimations.

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