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MODIFIED QUADRATURE METHOD WITH EXTRAPOLATION FOR SOLVING THE FIRST KIND BOUNDARY INTEGRAL EQUATIONS ON SMOOTH OPEN ARCS

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Abstract. To investigate the numerical solution of first kind boundary integral equations on smooth open arcs, a modified quadrature method was developed. The method is constructed based on the trapezoidal rule, and its convergence is established. Subsequently, we demonstrated the single-parameter odd power asymptotic expansion for the errors. Utilizing this expansion, we created an extrapolation algorithm achieving a numerical solution error accuracy of $O(h^5)$ with one extrapolation and $o(h^5)$ with another. The method's effectiveness was validated through a numerical example.

Keywords: Modified Quadrature Method, Extrapolation Algorithm, First Kind, Boundary Integral Equations, Smooth Open Arcs.

AMS Subject Classification: 65N38, 65R20.

1. Introduction

We consider the Dirichlet boundary value problem on an infinite plane as:

$$\begin{cases} \Delta u(x) = 0, x \in R^2 \backslash \Gamma, \\ u(x) = d(x), x \in \Gamma, \\ u(x) = O(\log(r)), r = |x| \longrightarrow \infty, \end{cases} \tag{1}$$

with Γ denoting a smooth open arc, which has significant practical importance and is applied in various fields such as fluid dynamics, hydrodynamics, electrostatics, and crack simulation. Using the single-layer potential theory, this problem is transformed into the following boundary integral equation given by:

$$d(x) = -\frac{1}{2\pi} \int_{\Gamma} \ln|x-y| v(y) ds_y, x \in \Gamma, \tag{2}$$

where $\ln|x-y|$ is the fundamental solution, and $v(y)$ is unknown function.

If the logarithmic capacity of Γ, denoted as C_Γ, is different from 1, Eqn. (1) has a unique solution [19]. Once the unknown function $v(y)$ has been determined from Eqn. (2), the numerical solution of Eqn. (1) can be computed by the following form:

$$u(x) = -\frac{1}{2\pi} \int_{\Gamma} \ln|x-y| v(y) ds_y, x \in R^2 \backslash \Gamma,$$

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Currently, several numerical methods exist for solving the first kind boundary integral equations on open arcs. The mechanical quadrature method and split extrapolation are employed to solve this equation, with splitting extrapolation enhancing accuracy in a single step [10]. Additionally, the mechanical quadrature method combined with extrapolation is utilized to solve two-dimensional scattering problems on a smooth arc, yielding high accurate numerical results [12]. The fast Fourier-Galerkin method is applied to solve the first kind boundary integral equation on smooth open arcs, and numerical examples confirm its efficiency [21]. Recently, various numerical methods have been proposed to solve Eqn. (2), including the Nyström method [2, 7, 8], the collocation method [1], the spectral Galerkin method [4, 22], the fast multiscale Galerkin method [5,9], the artificial neural network algorithm [14,16], the deep learning approach [15,20], etc.

In this work, we aim to solve Eqn. (2) using the modified quadrature method. In [6], the author initially introduced this method, which combines the trapezoidal rule to provide both simplicity and high accuracy. Then, the author of [17] applied this method to solve logarithmic kernel integral equations on general smooth closed curves. In [13], the authors used this method for axisymmetric problems, obtaining numerical solutions with convergence order $O(h^3)$. In [11], the authors developed an error asymptotic expansion for numerical solutions of logarithmic-kernel integral equations on smooth closed curves using this method and created an extrapolation algorithm to attain numerical solutions with an accuracy of $O(h^5)$. Building on our prior research, we are extending the modified quadrature method to solve the first-kind boundary integral equation on smooth open arcs. In contrast to smooth closed boundary curves, solutions to this equation on smooth open arcs exhibit endpoint singularities, which adversely affect numerical accuracy. To mitigate this, we employ the Sidi periodic transformation [18] to eliminate singularities and periodize the boundary curve, achieving an accuracy of $O(h^3)$ in numerical solutions, and we derive an asymptotic expansion of the error with an odd power of the mesh width h and develop an extrapolation algorithm based on it. The accuracy of the first extrapolation of the numerical solutions is $O(h^5)$, while the accuracy of the second extrapolation is $o(h^5)$. This significantly enhances the accuracy of the modified quadrature method, reaching the highest possible accuracy for any quadrature method. A numerical example confirms the effectiveness of this method.

The remaining work is organized into three sections. In Section 2, we analyze the singularity of the solution of the first-kind boundary integral equations on smooth open arcs and discretize the equation using the modified quadrature method. In Section 3, we prove that the errors of the numerical solutions have a single-parameter asymptotic expansion with odd powers. Using this expansion, we create an extrapolation algorithm. After the first extrapolation, the accuracy of the numerical solutions improves to $O(h^5)$, and after the second extrapolation, it reaches $o(h^5)$. In Section 4, we provide a numerical example to validate our theoretical analysis.

2. Singularity analysis and modified quadrature method

The smooth open arc Γ is parameterized by $y = x(\tau) = (x_1(\tau), x_2(\tau)) : [0,1] \to \Gamma$, where $|x'(\tau)|^2 = |x_1'(\tau)|^2 + |x_2'(\tau)|^2 > 0$. Here, $x_1(\tau)$ and $x_2(\tau)$ belong to $C^{2m}[0,1]$, denotes the space of $2m$-times continuously differentiable functions. So, Eqn. (2) can be transformed into the following integral equation

$$d(s) = -\frac{1}{2\pi} \int_0^1 \ln|x(s) - x(\tau)| v(\tau) x'(\tau) d\tau, s \in [0,1], \tag{3}$$

where $d(s) = d(x(s))$ and $v(\tau) = v(x(\tau))$. Since the function $v(\tau)$ exhibits singularities at $\tau = 0$ and $\tau = 1$, we apply the Sidi periodic transformation [18] to remove the endpoint singularities.

The transformation is defined as follows:

$$\tau = \chi_p(t) : [0,1] \to [0,1], p \in N,$$

where $\chi_p(t) = \psi_p(t)/\psi_p(1)$, $\psi_p(t) = \int_0^t (\sin(\pi\tau))^p d\tau$. Therefore, Eqn. (3) is equivalent to

$$d(s) = -\frac{1}{2\pi} \int_0^1 \ln|x(s) - x(t)| v(t) x'(t) \chi_p'(t) dt, s \in [0,1], \tag{4}$$

where $x(t) = x(\chi_p(t))$, $v(t) = v(x(\chi_p(t)))$, $x'(t) = x'(\chi_p(t))$, they are periodic functions with period 1, and $\chi_p'(t)$ has zeros of degree p at $t = 0$ and $t = 1$.

We analyze the singularity of the solution $v(\tau)$ of Eqn. (3). Let Q_1 and Q_2 denote the endpoints of Γ, corresponding to parameter values τ_λ for $\lambda = 1, 2$. It is well established that the solution $v(\tau)$ exhibits singularities at points τ_λ for $\lambda = 1, 2$ (see [19, 23]). The solution $v(\tau)$ can be represented as follows:

$$v(\tau) = \tau^{\kappa_{\lambda-1}} (1-\tau)^{\kappa_\lambda} \vartheta(t),$$

where $\vartheta(0) \neq 0$, $\vartheta(1) \neq 0$, and $\kappa_\lambda \geq -\frac{1}{2}$. Further, we get

$$v(t) = \mu_1 \vartheta(0) t^{(p+1)\kappa_{\lambda-1}+p} (1 + O(t^2)), t \to 0^+, \tag{5}$$

and

$$v(t) = \mu_2 \vartheta(1) (1-t)^{(p+1)\kappa_\lambda+p} (1 + O((1-t)^2)), t \to 1^-, \tag{6}$$

where μ_1 and μ_2 are constants. From Eqns. (5) and (6), it is clear that $v(t)$ is bounded on $[0,1]$, with zeros at $t = 0$ and $t = 1$. Consequently, $v(t)$ is a smooth function on $[0,1]$.

An integral operator on $[0,1]$ is defined as follows:

$$(Cv)(s) = \int_0^1 C(s,t) v(t) dt, \tag{7}$$

where $c(s,t) = -\frac{1}{2\pi} \ln|x(s) - x(t)|$ and $v(t) = v(t) x'(t) \chi_p'(t)$. Eqn. (4) is equivalent to

$$Cv = d. \tag{8}$$

Let $h = 1/n, n \in N$ be the mesh width, and $t_k = s_k = kh, (k = 0, 1, \cdots, n-1)$ be the grid nodes. The value of $(Cv)(t_k)$ is defined as

$$(Cv)(t_k) = v(t_k) \int_0^1 c(t_k, t) dt + \int_0^1 c(t_k, t)(v(t) - v(t_k)) dt. \tag{9}$$

Since the second integral of Eqn. (9) equals zero at $t = t_k$, applying the composite trapezoid formula yields

$$\int_0^1 c(t_k, t)(v(t) - v(t_k)) dt = h \sum_{j=0, j\neq k}^{n-1} c(t_k, t_j)(v(t_j) - v(t_k)). \tag{10}$$

The first integral of Eqn. (9) can be written as follows:

$$\int_0^1 c(t_k, t) dt = \int_0^1 c'(t_k, t) dt + \int_0^1 [c(t_k, t) - c'(t_k, t)] dt, \tag{11}$$

where $c'(t_k,t) = -1/(2\pi)\ln|x_p(t_k)-x_p(t)|$, and $x_p(t) = pe^{i2\pi t}$ is a circle with a radius of $p = e^{-1/2}$. The integration kernel of the second part of the Eqn. (11) is smooth when t is not equal to t_k. When t equals t_k, we have

$$\begin{aligned}\int_0^1 [c(t_k,t)-c'(t_k,t)]d\tau &= -\frac{1}{2\pi}\int_0^1 \ln\frac{|x(t_k)-x(t)|}{|x_p(t_k)-x_p(t)|}dt,\\ &= -\frac{1}{2\pi}\int_0^1 \ln\frac{|x'(t_k)|}{2\pi e^{-1/2}}dt, t\to t_k,\end{aligned}$$

that ensures smoothness.

For all $t_k, k = 0,1,\ldots,n-1$, since $-2\int_0^1 \ln|x_p(t_k)-x_p(t)|dt = 1$ as in [17], by applying the composite trapezoidal formula, we get

$$\int_0^1 c(t_k,t)dt = \lambda_k = \frac{1}{4\pi} - \frac{h}{2\pi}\ln\frac{|x'(t_k)|}{2\pi e^{-1/2}} + h\sum_{j=0,j\neq k}^{n-1}[c(t_k,t_j)-c'(t_k,t_j)]. \tag{12}$$

We derive an algebraic equation using Eqns. (9)-(12) as:

$$\lambda_k v(t_k) + h\sum_{j=0,j\neq k}^{n-1} c(t_k,t_j)(v(t_j)-v(t_k)) = d(t_k). \tag{13}$$

Eqn. (13) can be expressed in the following operator form

$$C^h v^h = d^h, \tag{14}$$

where $v^h = (v(t_0),\ldots,v(t_{n-1}))$, $d^h = (d(t_0),\ldots,d(t_{n-1}))$, and $C^h = (c^h_{kj})_{n\times n}$ is a matrix, with c^h_{kj} being defined as follows:

$$c^h_{kj} = hc(t_k,t_j), j\neq k,$$

and

$$\begin{aligned}c^h_{kk} &= \lambda_k - h\sum_{j=0,j\neq k}^{n-1} c(t_k,t_j) = \frac{1}{4\pi} - \frac{h}{2\pi}\ln\frac{|x'(s_k)|}{2\pi e^{-1/2}} - h\sum_{j=0,j\neq k}^{n-1} c'(t_k,t_j) \qquad (15)\\ &= \frac{h}{2\pi}\Big[-\ln\frac{|x'(s_k)|}{2\pi} + \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big].\end{aligned}$$

When the value v^h is provided by Eqn. (14), then $u^h(x)$ can be obtained by:

$$u^h(x) = h\sum_{j=0}^{n-1} c(x,t_j)v(t_j), x\in R^2\backslash\Gamma. \tag{16}$$

We will analyze the convergence of numerical solutions of Eqn. (14) in the following content. We will divide operator C into two components, as follows:

$$(Cv)(s) = (Sv)(s) + (Dv)(s),$$

where

$$(Sv)(s) = -\frac{1}{2\pi}\int_0^1 \ln|x_p(s)-x_p(t)|v(t)dt,$$

and

$$(Dv)(s) = \int_0^1 c(s,t)v(t)dt + \frac{1}{2\pi}\int_0^1 \ln|x_p(s) - x_p(t)|v(t)dt, \tag{17}$$

with $x_p(t) = (e^{-1/2}\cos(2\pi t), e^{-1/2}\sin(2\pi t))$. The operator D is smooth, whereas operator S is logarithmically weakly singular and invertible as shown in [11]. Therefore, Eqn. (8) is equivalent to

$$(I + S^{-1}D)v = S^{-1}d.$$

By giving $-2\int_0^1 \ln|x_p(s) - x_p(t)|dt = 1$, $(Sv)(s)$ can be expressed as

$$(Sv)(s) = \frac{1}{4\pi}v(s) - \frac{1}{2\pi}\int_0^1 \ln|x_p(s) - x_p(t)|(v(t) - v(s))dt.$$

Based on the trapezoidal rule, the approximation operator S^h can be obtained as following:

$$(S^h v)(t_k) = \frac{1}{4\pi}v(t_k) - \frac{h}{2\pi}\sum_{j=0, j\neq k}^{n-1} \ln|x_p(t_k) - x_p(t_j)|(v(t_j) - v(t_k)), k = 0, 1, \ldots, n-1,$$

further, we have

$$(S^h v)(t_k) = -\frac{h}{2\pi}\Big(\ln|\frac{e^{-1/2}}{2\pi}|v(t_k) - \sum_{j=1}^{n-1}\ln|2e^{-1/2}\sin(\pi jh)|v(t_j)\Big), k = 0, 1, \ldots, n-1.$$

Since the operator D has a smooth kernel, the approximation operator D^h is defined as

$$\begin{aligned}(D^h v)(t_k) = {} & h\sum_{j=0, j\neq k}^{n-1}\Big(c(t_k, t_j) + \frac{1}{2\pi}\ln|x_p(t_k) - x_p(t_j)|\Big)v(t_j) \\ & - \frac{h}{2\pi}\ln\frac{|x'(t)|}{2\pi e^{-1/2}}v(t_k), k = 0, 1, \ldots, n-1.\end{aligned}$$

The approximation operator C^h is defined as follows:

$$(C^h v)(s) = (S^h v)(s) + (D^h v)(s), \tag{18}$$

where S^h is a circulant matrix and given by:

$$S^h = -\frac{h}{2\pi}circ\Big(\ln|he^{-1/2}|, \ln|2e^{-1/2}\sin(\pi h)|, \ldots, \ln|2e^{-1/2}\sin(\pi(n-1)h)|\Big).$$

Since the operator S^h is invertible as in [11], Eqn. (14) is rewritten as

$$(I + (S^h)^{-1}D^h)v = (S^h)^{-1}d^h.$$

Corollary 1. *[11] If $G(s,t) = \varphi(t)\ln|s-t|$ is a periodic function with period 1, satisfying $\varphi(t), G(s,t) \in C^{2m}[0,1]$, the approximate integration formula for $\int_0^1 G(s,t)dt$ is given by:*

$$Q_n(G) = h\sum_{j=0, t_j\neq s}^{n-1} G(s,t_j)\varphi(t_j) + h\varphi(s)\Big[\ln\big(\frac{1}{2\pi}\big) - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big],$$

and the error is given by:

$$E_n(G) = \int_0^1 G(s,t)d\tau - Q_n(G) = 2\sum_{j=1}^{m-1}\frac{\zeta'(-2j)}{(2j)!}\varphi^{(2j)}(s)h^{2j+1} + O(h^{2m}),$$

where ζ' is the derivative of the Riemann Zeta function.

Theorem 1. *Let Γ be a smooth open arc with $C_\Gamma \neq 1$, $\{(S^h)^{-1}D^h\}$ is collectively compact operator sequence and convergent to $S^{-1}D$ in $C[0,1]$.*

Proof. According to $(S^h)^{-1}D^h = ((S^h)^{-1}S)(S^{-1}D^h)$, we obtain

$$\| (S^h)^{-1}D^h \| \leq \| (S^h)^{-1}S \| \| S^{-1}D^h \| .$$

Next, we prove $(S^h)^{-1}S \to I$. Let $\rho \in C[0,1]$, ρ^h be the solutions of the equations $S\rho = g$ and $S^h\rho^h = g^h$, respectively, where $\rho^h = (\rho_0, \ldots, \rho_{n-1})^T$ and $g^h = (g_0, \ldots, g_{n-1})^T$. By using Corollary 1, we get

$$(S\rho)(t_k) = -\frac{1}{2\pi}\int_0^1 \ln|x_p(t_k) - x_p(\tau)|\rho(\tau)d\tau = -\frac{h}{2\pi}\sum_{j=0,t_k\neq t_j}^{n-1} \ln|x_p(t_k) - x_p(t_j)|\rho(t_j)$$

$$-\frac{h}{2\pi}\rho(t_k)\Big[\ln\frac{|x_p'(t_k)|}{2\pi} - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big] + O(h^3).$$

Assume that $e = (e(t_0), \ldots, e(t_{n-1}))^T$ with $e(t_j) = \rho^h(t_j) - \rho(t_j)$, we obtain

$$S^h e = -\frac{h}{2\pi}\sum_{j=0,t_k\neq t_j}^{n-1} \ln|x_p(t_k) - x_p(t_j)|e(t_j) - \frac{h}{2\pi}e(t_k)\Big[\ln\frac{|x_p'(t_k)|}{2\pi} - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big]$$

$$= -\frac{h}{2\pi}\sum_{j=0,t_k\neq t_j}^{n-1} \ln|x_p(t_k) - x_p(t_j)|\rho^h(t_j) - \frac{h}{2\pi}\rho^h(t_k)\Big[\ln\frac{|x_p'(t_k)|}{2\pi} - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big]$$

$$+\frac{h}{2\pi}\sum_{j=0,t_k\neq t_j}^{n-1} \ln|x_p(t_k) - x_p(t_j)|\rho(t_j) + \frac{h}{2\pi}\rho(t_k)\Big[\ln\frac{|x_p'(t_k)|}{2\pi} - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big]$$

$$= g(t_k) + \frac{h}{2\pi}\sum_{j=0,t_k\neq t_j}^{n-1} \ln|x_p(t_k) - x_p(t_j)|\rho(t_j) + \frac{h}{2\pi}\rho(t_k)\Big[\ln\frac{|x_p'(t_k)|}{2\pi} - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big]$$

$$= -\frac{1}{2\pi}\int_0^1 \ln|x_p(t_k) - x_p(\tau)|\rho(\tau)d\tau + \frac{h}{2\pi}\sum_{j=0,t_k\neq t_j}^{n-1} \ln|x_p(t_k) - x_p(t_j)|\rho(t_j)$$

$$+\frac{h}{2\pi}\rho(t_k)\Big[\ln\frac{|x_p'(t_k)|}{2\pi} - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big] = O(h^3) = \epsilon,$$

where $\epsilon = (\epsilon_0, \ldots, \epsilon_{n-1})^T$. Further, we have

$$S^h e = \epsilon,$$

and since S^h is invertible, we get

$$e = (S^h)^{-1}\epsilon.$$

A restricted operator X^h is defined as follows:

$$X^h : C[0,1] \to R^n, X^h g = (g(t_0), \ldots, g(t_{n-1}))^T, \forall g \in C[0,1],$$

then, we have

$$\begin{aligned} \| e \| &= \| (S^h)^{-1}\epsilon \| \\ &= \| X^h(S)^{-1}g - (S^h)^{-1}X^h g \| \\ &= \| X^h\rho - (S^h)^{-1}X^h S\rho \| \to 0, h \to 0. \end{aligned}$$

So, we obtain $(S^h)^{-1}S \to I$.

Since the kernel function of the integral operator D is smooth, the operator sequence $\{S^{-1}D^h\}$ is collectively compact and converges to $S^{-1}D$ in $C[0,1]$. According to the compact operator theory given in [3], the operator sequence $\{(S^h)^{-1}D^h\}$ is also collectively compact and converges to $S^{-1}D$ in $C[0,1]$. □

3. Asymptotic expansion of error and Extrapolation Algorithm

Theorem 2. *$(Cv)(s) = \int_0^1 c(s,t)v(t)dt$ has the following approximate formula*

$$Q_n(C) = h\sum_{j=0,t_j\neq s}^{n-1} c(s,t_j)v(t_j) + \frac{h}{2\pi}v(s)\Big[-\ln\frac{|x'(s)|}{2\pi} + \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big],$$

and the error term is

$$E_n(C) = \int_0^1 c(s,t)v(t)dt - Q_n(C) = -2\sum_{j=1}^{m-1}\frac{\zeta'(-2j)}{(2j)!}v^{(2j)}(s)h^{2j+1} + O(h^{2m}).$$

Proof. Let $c(s,t) = \frac{1}{2\pi}[-\ln|x(s)-x(t)| + \ln|s-t| - \ln|s-t|]$, and divide it into two parts as follows:

$$c(t,\tau) = c_1(s,t) + c_2(s,t),$$

where $c_1(s,t) = \frac{1}{2\pi}[-\ln|x(s)-x(t)| + \ln|s-t|]$ and $c_2(s,t) = -\frac{1}{2\pi}\ln|s-t|$.

Obviously, $c_1(s,t)$ is smooth, by using the composite trapezoid formula, we have

$$\int_0^1 c_1(s,t)v(t)dt = h\sum_{j=0,t_j\neq s}^{n-1} c_1(s,t_j)v(t_j) - \frac{h}{2\pi}v(s)\ln|x'(s)| + O(h^{2m}). \tag{19}$$

According to Corollary 1, the approximate formula of $\int_0^1 c_2(s,t)v(t)dt$ is given by:

$$Q_n(c_2) = h\sum_{j=0,t_j\neq s}^{n-1} c_2(s,t_j)v(t_j) + \frac{h}{2\pi}v(s)\Big[-\ln\frac{1}{2\pi} + \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big], \tag{20}$$

and the error term is given by

$$E_n(c_2) = \int_0^1 c_2(s,t)v(t)dt - Q_n(c_2) = -2\sum_{j=1}^{m-1}\frac{\zeta'(-2j)}{(2j)!}v^{(2j)}(s)h^{2j+1} + O(h^{2m}). \tag{21}$$

By combining Eqns. (19), (20), and (21), the result has been successfully proven. □

Theorem 3. *Let $v(t) \in C^{2m}[0,1]$, the errors have the following asymptotic expansion*

$$v^h(t) - v(t) = \omega(t)h^3 + \varpi(t)h^5 + o(h^5), t \in \{t_k, k = 0,1,\ldots,n-1\} \tag{22}$$

where $\omega(t)$ and $\varpi(t)$ are two functions in $C[0,1]$ that are independent of the mesh width h.

Proof. By Theorem 2, the $k-$th element of $C^h(v^h - X^h v)$ is considered as:

$$\begin{aligned} C^h(v^h - X^h v)|_k = {} & d(t_k) - h\sum_{j=0,t_j\neq t_k}^{n-1} c(t_j,t_k)v(t_j) \\ & + \frac{h}{2\pi}v(t_k)\Big[\ln\frac{|x'(t_k)|}{2\pi} - \sum_{j=1}^{n-1}\ln(2|sin(\pi jh)|)\Big] \end{aligned}$$

$$= \zeta'(-2)v^{(2)}(t_k)h^3 + \frac{\zeta'(-4)}{12}v^{(4)}(t_k)h^5 + o(h^5),$$

where X^h is the restricted operator defined previously. So, we get

$$C^h(v^h(t) - v(t)) = \zeta'(-2)v^{(2)}(t)h^3 + \frac{\zeta'(-4)}{12}v^{(4)}(t)h^5 + o(h^5),$$

where $t \in \{t_k, k = 0, 1, \ldots, n-1\}$. By using Eqn. (18), we have

$$(S^h + D^h)(v^h(t) - v(t)) = \zeta'(-2)v^{(2)}(t)h^3 + \frac{\zeta'(-4)}{12}v^{(4)}(t)h^5 + o(h^5).$$

Due to the invertibility of the operator S^h, we obtain

$$\begin{aligned}(I + (S^h)^{-1}D^h)(v^h(t) - v(t)) &= \zeta'(-2)(S^h)^{-1}v^{(2)}(t)h^3 \\ &+ \frac{\zeta'(-4)}{12}(S^h)^{-1}v^{(4)}(\tau)h^5 + o(h^5).\end{aligned}$$

Based on Theorem 1, it is known that $\{(S^h)^{-1}D^h\}$ is collectively convergent to $S^{-1}D$. Consequently, $(I + (S^h)^{-1}D^h)^{-1}$ exists and is bounded as in [3]. Furthermore, we have

$$\begin{aligned}v^h(t) - v(t) &= \zeta'(-2)(I + (S^h)^{-1}D^h)^{-1}(S^h)^{-1}v^{(2)}(t)h^3 \\ &+ \frac{\zeta'(-4)}{12}(I + (S^h)^{-1}D^h)^{-1}(S^h)^{-1}v^{(4)}(t)h^5 + o(h^5).\end{aligned}$$

Let

$$\omega(t) = \zeta'(-2)(I + (S^h)^{-1}D^h)^{-1}(S^h)^{-1}v^{(2)}(t), \varpi(t) = \frac{\zeta'(-4)}{12}(I + (S^h)^{-1}D^h)^{-1}(S^h)^{-1}v^{(4)}(t),$$

Hence, the proof is completed. □

Using the error asymptotic expansion (22), the extrapolation algorithm is constructed as follows: The first step involves eliminating the h^3 term in the expansion,

$$v^h_{(1)}(t_k) = \frac{1}{7}(8v^{h/2}(t_k) - v^h(t_k)), k = 0, 1, \cdots, n-1, \tag{23}$$

where $v^h_{(1)}$ is the approximate solution of the first extrapolation, and v^h and $v^{h/2}$ are the approximate solutions with the mesh width h and $h/2$, respectively; the second step involves eliminating the h^5 term in the expansion,

$$v^h_{(2)}(t_k) = \frac{1}{31}(32v^{h/2}_{(1)}(t_k) - v^h_{(1)}(t_k)), k = 0, 1, \cdots, n-1, \tag{24}$$

where $v^h_{(2)}$ is the approximate solution of the second extrapolation, and $v^h_{(1)}$ and $v^{h/2}_{(1)}$ are the approximate solutions of the first extrapolation with the mesh width h and $h/2$, respectively. Consequently, after two rounds of extrapolation, the accuracy of the approximation error becomes $o(h^5)$.

An accuracy of order $o(h^5)$ can be achieved by solving the discrete equations in parallel. A detailed description of the algorithm is provided below.

Step 1. Set $h^{(0)} = h/2^0$, $h^{(1)} = h/2^1$, and $h^{(2)} = h/2^2$;

Step 2. Solve Eqn. (14) with $h = h^{(0)}$, $h^{(1)}$, and $h^{(2)}$, in parallel to obtain $v^{h^{(0)}}$, $v^{h^{(1)}}$, and $v^{h^{(2)}}$;

Step 3. Compute the first extrapolations $v^{h^{(0)}}_{(1)}$ and $v^{h^{(1)}}_{(1)}$, by using Eqn. (23);

Step 4. Compute the second extrapolation $v^{h^{(0)}}_{(2)}$, by using Eqn. (24);

Step 5. Compute $u^{h^{(0)}}(x)$, $u^{h^{(0)}}_{(1)}(x)$, and $u^{h^{(0)}}_{(2)}(x)(x \in R^2 \backslash \Gamma)$, by using Eqn. (16), $v^{h^{(0)}}$, $v^{h^{(0)}}_{(1)}$, and $v^{h^{(0)}}_{(2)}$.

The multiprocessor system of the algorithm comprises three processors: pro(0), pro(1), and pro(2). For simplicity, pro(0) serves as the host processor. The parallel algorithm, which is based on two extrapolations, can be outlined using the following flowchart.

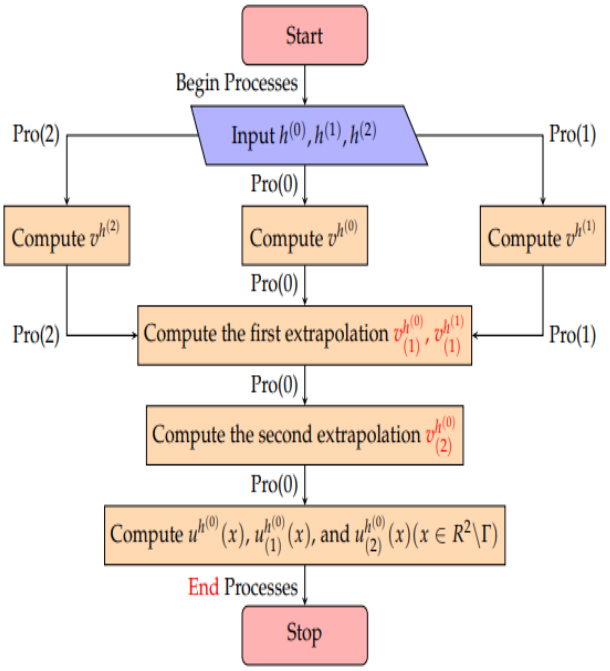


FIGURE 1. Flowchart of Parallel Algorithm.

4. NUMERICAL EXAMPLE

Here, we provide a numerical example to verify the effectiveness of the proposed method.

Example 1. (See [10,19,23]) We consider the Dirichlet boundary value problem on an infinite plane with $d(x) = -1/2\pi$ and the smooth open arcs $\Gamma = \{(y_1, 0)| - 1 \leq y_1 \leq 1\}$. Using the Sidi periodic transformation $\chi_4(t)$, we calculate the approximate values of $u(0, 2)$ and $u(1.3, 0)$, which are presented in Tables 1 and 2. The precise values of $u(0, 2)$ and $u(1.3, 0)$ are as follows:

$$u(0,2) = -\frac{1}{\ln(2)} \ln\left(\frac{2+\sqrt{2^2+1}}{2}\right) = -1.082725740891852$$

and

$$u(1.3,0) = -\frac{1}{\ln(2)} \ln\left(\frac{1.3+\sqrt{1.3^2-1}}{2}\right) = -0.091302009258539.$$

TABLE 1. Numerical results for $u(0, 2)$ with $\chi_4(t)$.

n	$Error$	Co_1	$Error_1^{Ea}$	Co_2	$Error_2^{Ea}$
8	1.4651E-3	-	-	-	-
16	1.6531E-4	3.1478	-	-	-
32	2.0090E-5	3.0406	6.5485E-7	4.9600	2.5537E-9
64	2.4931E-6	3.0105	2.0845E-8	4.9734	2.7023E-10

TABLE 2. Numerical results for $u(1.3, 0)$ with $\chi_4(t)$.

n	$Error$	Co_1	$Error_1^{Ea}$	Co_2	$Error_2^{Ea}$
128	7.5082E-8	-	-	-	-
256	9.3923E-9	2.9989	8.0073E-12	-	-
512	1.1743E-9	2.9997	2.5548E-13	4.9701	2.7352E-15
1024	1.4679E-10	2.9999	7.4385E-15	5.1020	7.7759E-16

In Tables 1 and 2, $Error = |u^h(x) - u(x)|$ is the errors of the approximate solutions, $Co_1 = \log_2 |(u^h(x) - u(x))/(u^{h/2}(x) - u(x))|$ and $Co_2 = \log_2 |(u^h_{(1)}(x) - u(x))/(u^{h/2}_{(1)}(x) - u(x))|$ are the convergence order of the errors, where $u^h_{(1)}(x)$ and $u^{h/2}_{(1)}(x)$ is first extrapolations of the approximate solutions, and $Error_1^{Ea}$ and $Error_2^{Ea}$ are the errors of the two extrapolations. From numerical results, we can see $Co_1 \approx 3$ and $Co_2 \approx 5$ which agree with Theorem 3 very well.

5. Conclusion

In this work, we investigate the modified quadrature method with two extrapolations to solve first-kind boundary integral equations on smooth open arcs. This method offers several advantages: (a) the entries of the discrete matrices can be computed easily without evaluating singular integrals; (b) the method has high accuracy, with errors exhibiting an asymptotic expansion characterized by an odd power of the mesh width h. Using one extrapolation, the accuracy of the numerical solutions reaches $O(h^5)$, and with another extrapolation, it achieves $o(h^5)$, representing the highest accuracy currently attainable through quadrature methods. Overall, the proposed method provides an efficient numerical algorithm for solving first-kind weakly singular boundary integral equations. Future applications will extend the proposed method to solve three-dimensional axisymmetric boundary integral equations.

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