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A NOVEL MULTI-ATTRIBUTE DECISION-MAKING METHOD VIA PICTURE FUZZY ACZÉL-ALSINA INTERACTIONAL AGGREGATION OPERATORS

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Abstract. This paper investigates the interactional operational laws based on Aczél-Alsina t-norms and t-conorms for picture fuzzy numbers (PFNs) and proposes an efficient picture fuzzy multi-attribute decision-making (MADM) method. To overcome the lack of closure of existing picture fuzzy operations, we first construct a new class of picture fuzzy interactional operations based on Aczél-Alsina t-norms and t-conorms, including addition, product, scalar multiplication, and power, which are shown to be closed on the set of PFNs. Based on these four operations, we introduce the picture fuzzy Aczél-Alsina interactional weighted average (PFAAIWA) operator and the picture fuzzy Aczél-Alsina interactional weighted geometric (PFAAIWG) operator and prove that they are monotonicity idempotent, bounded, shift-invariant, and homogeneous. Finally, we propose a novel MADM method under the picture fuzzy environment using these operators, and present an investment selection example for a financing organization to illustrate our method.

Keywords: Picture Fuzzy Set, Aggregation Operator, Aczél-Alsina t-norms, Multi-Attribute Decision Making.

AMS Subject Classification: 03E72.

1. Introduction

In 1965, Zadeh [57] introduced the concept of fuzzy set (FS) by using the membership degree to measure the importance of a fuzzy element in a universe of discourse, which is regarded as the basis for fuzzy decision-making [7, 8, 26]. As a generalization of Zadeh's FS, the author introduced intuitionistic fuzzy set (IFS) to address issues involving two types of responses: 'yes' and 'no' in [5, 6]. Recently, in [9, 10], the authors proposed the concept of picture fuzzy sets (PFSs) to handle situations with multiple selection options as follows: 'yes', 'no', 'abstain', and 'refusal', which are characterized by three membership functions: positive, neutral, and negative, meeting the requirement that the sum of these membership degrees for each point must be less than or equal to 1. In particular, every triple of positive, neutral, and negative membership degrees for PFSs, i.e., a triple whose sum is less than or equal to 1, is referred to as a picture fuzzy number (PFN) in [21]. Then, some basic operations for PFSs, including conjunctions, disjunctions, negations, and implications, were defined in [9, 11]. In the same papers, the authors also explored the concepts of picture fuzzy distance, picture fuzzy relations, and Zadeh's extension principle in the context of PFSs. In analogy to IFSs, considering the uncertainty of three membership degrees for PFSs, the author of the paper [9] introduced the

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concept of interval-valued picture fuzzy sets (IvPFSs). In [17], the authors defined spherical fuzzy sets (SFSs), which can be regard as a 2-rung version of PFSs. Mathematically, picture fuzzy set (PFS) and SFS are isomorphic.

Observing that a PFS reduces to an IFS when the neutral membership degree is equal to 0 (i.e., every IFS is a special form of PFS), many researchers have focused on MADM problems expressed using PFSs or PFNs and have developed various useful PF MADM methods by extending MADM approaches within the IFS framework, including PF aggregation operators [2,14,21,32,34,38,40,51,53], PF linguistic aggregation operators [28,54], PF similarity similarity measures [12,29,31,45,46,55,58], PF TOPSIS method [36,44], PF VIKOR method [3,22], PF TODIM method [3,48,52], and PF AHP method [16,37], and others. Based on well-known t-norm families, several PF aggregation operators were developed and successfully applied to MADM problems, including PF Heronian mean aggregation operators [27, 53], PF Hamacher aggregation operators [20, 51], PF Dombi aggregation operators [21, 33], PF Einstein aggregation operations [24], PF Frank aggregation operators [32,39], PF Aczél-Alsina average aggregation operators [40], and PF Schweizer-Sklar aggregation operators [47]. Using general t-norms and t-conorms with additive generators, the researchers developed weighted average, ordered weighted average, and hybrid average aggregation operators for handling MADM problems with PFSs [4, 13].

Among all t-norms, Aczél-Alsina t-norms are the unique solutions to the following functional equations:

$$T(x^p, y^p) = (T(x, y))^p \quad \text{and} \quad T(x^q, y^q) = (T(x, y))^q,$$

where $p, q \in (0, 1) \cup (0, +\infty)$ and the quotient $\frac{\ln p}{\ln q} \in \mathbb{R} \setminus \mathbb{Q}$ (i.e., it is an irrational number). These t-norms can handle power operations effectively and have been widely applied in PF MADM problems. In [35, 40], the researchers independently introduced Aczél-Alsina t-norms into the PF framework and developed several new operations for PFNs. Furthermore, combining the power average with the Bonferroni mean operators, the authors proposed new Aczél-Alsina power Bonferroni aggregation operators for PFSs in [30].

Due to the special three-dimensional structure of PFNs, the aggregation outcomes obtained by the PF Aczél-Alsina aggregation operators [30,35,40] may not always result in a PFN, i.e., the sum of the aggregation results can exceed 1 (see Remark 1). This undermines the fairness of the decision-making process because the aggregated results are no longer evaluated within a unified framework. Considering the importance of Aczél-Alsina t-norms, to overcome the lack of closure of the PF Aczél-Alsina aggregation operators [30, 35, 40], we aim to develop PF Aczel-Alsina aggregation operators that satisfy the closure property and an effective PF Aczel-Alsina-based MADM method. Specifically, our main contributions are as follows:

(1) We introduce four basic algebraic operations for PFNs based on Aczél-Alsina t-norms and t-conorms: Addition, product, scalar multiplication, and power, which consider the interactional operations that occur in the position of the neutral degrees. Furthermore, we prove that the four operations are closed in PFNs and establish some fundamental properties, including the commutative law, associative law, and distributive law. These four operations overcome the limitations of uncloseness of PF Aczél-Alsina aggregation operators [30, 35, 40].

(2) By applying above four basic algebraic operations, we define the picture fuzzy Aczél-Alsina interactional weighted average (PFAAIWA) operator and the picture fuzzy Aczél-Alsina interactional weighted geometric (PFAAIWG) operator to aggregate PF information, which are proved to be monotonicity, idempotent, bounded, shift-invariant, and homogeneous. In particular, this shows that PFAAIWA and PFAAIWG operators are aggregation operators on PFNs. It should be note that the monotonicity proofs for existing operators on PFNs are imprecise, because, according to our knowledge, there is no general methodology to rank any two PFNs until the recent work [56], which introduced a total order $\preceq_{\mathrm{W}}$ for PFNs. The monotonicity in this paper is proved in the sense of this total order.

(3) By applying the PFAAIWA and PFAAIWG operators, we build up a new approach for solving MADM problems expressed by PFNs. Additionally, we present an example involving investment choices for a financing organization to illustrate the application of our method.

The block diagram of this paper is stated in Fig.1.

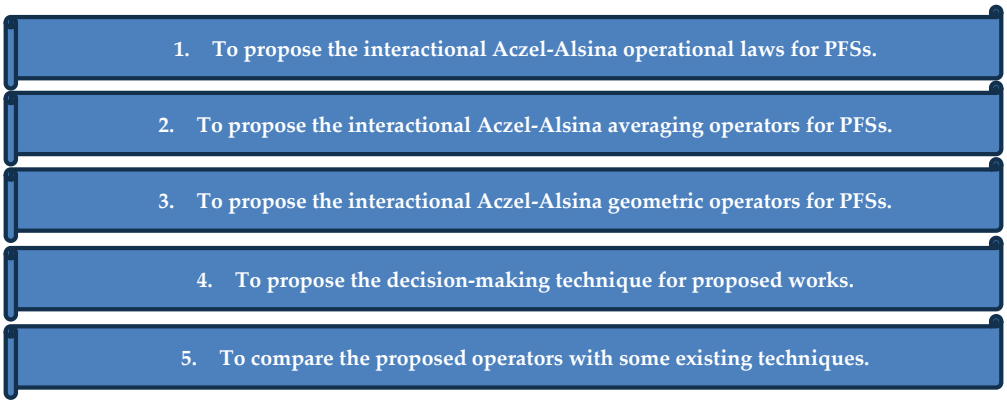


FIGURE 1. Block diagram of this paper.

2. PRELIMINARIES

2.1. Picture fuzzy set.

Definition 1 ([9, Definition 3.1]). Let X be the universe of discourse. A picture fuzzy set (PFS) P in X is defined as an object with the following form

$$P = \{\langle x, \mu_P(x), \eta_P(x), \nu_P(x)\rangle \mid x \in X\},$$

where $\mu_P(x)$, $\eta_P(x)$, $\nu_P(x) \in [0,1]$, $\mu_P(x) + \eta_P(x) + \nu_P(x) \leq 1$, and $\pi_P(x) = 1 - (\mu_P(x) + \eta_P(x) + \nu_P(x))$ for any $x \in X$. Here, $\mu_P(x)$, $\eta_P(x)$, and $\nu_P(x)$ denote the degree of positive membership, neutral membership, and negative membership of x in P, respectively. The value $\pi_P(x)$ is the degree of refusal membership of x in P. In particular, the triplet $\langle \mu, \eta, \nu\rangle$ satisfying that $\mu, \eta, \nu \in [0,1]$ and $\mu + \eta + \nu \leq 1$ is called a picture fuzzy number (PFN). For convenience, a PFN α is denoted by $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha\rangle$.

Let $\mathscr{P}$ denote the set of all PFNs, i.e., $\mathscr{P} = \{\langle \mu, \eta, \nu\rangle \mid \mu, \eta, \nu \in [0,1] \text{ and } \mu + \eta + \nu \leq 1\}$. If $\eta_P(x) = 0$, then the PFS reduces to Atanassov's IFS. Hence, the Atanassov's IFS are special form of PFSs.

The following inclusion relation was introduced in [9].

Definition 2. [9] For two PFNs α and β, $\alpha \subseteq \beta$ if and only if $\mu_\alpha \leq \mu_\beta$, $\eta_\alpha \leq \eta_\beta$, and $\nu_\alpha \geq \nu_\beta$.

It can be verified that '$\subseteq$' is a partial order on $\mathscr{P}$. However, the poset $(\mathscr{P}, \subseteq)$ does not form a lattice, since the two PFNs $\langle 1,0,0\rangle$ and $\langle 0,1,0\rangle$ does not have a supremum in $\mathscr{P}$.

With a view to comparing two PFNs, the authors introduced the following comparison laws in [49].

Definition 3 ([49, Definition 3.3]). Given a picture fuzzy number $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha\rangle$, let the score function $S(\alpha)$ and the accuracy function $H(\alpha)$ of α be defined as:

$$S(\alpha) = \mu_\alpha - \nu_\alpha, H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$$

In this case, for any two PFNs α and β,

- if $S(\alpha) < S(\beta)$, then $\alpha \prec \beta$;
- if $S(\alpha) = S(\beta)$, then
  - if $H(\alpha) = H(\beta)$, then $\alpha \sim \beta$;

– if $H(\alpha) < H(\beta)$, then $\alpha \prec \beta$.

If $\alpha \prec \beta$ or $\alpha \sim \beta$, we denote it by $\alpha \preceq \beta$.

However, the order introduced in [49] can not distinguish some pair of PFNs. Recently, the authors introduced the following linear order for PFNs, which can compare any pair of PFNs and preserves the natural inclusion relation '$\subseteq$' in [56].

Definition 4 ([56, Definition 11]). Given a picture fuzzy number $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$, the score function $S(\alpha)$, the first accuracy function $H_1(\alpha)$, and the second accuracy function $H_2(\alpha)$ of α are defined as follows:

$$S(\alpha) = \mu_\alpha - \nu_\alpha, H_1(\alpha) = \mu_\alpha + \nu_\alpha and H_2(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$$

In this case, for any two PFNs α and β,

- if $S(\alpha) < S(\beta)$, then $\alpha \prec_{\mathrm{W}} \beta$;
- if $S(\alpha) = S(\beta)$, then
  - if $H_1(\alpha) < H_1(\beta)$, then $\alpha \prec_{\mathrm{W}} \beta$;
  - if $H_1(\alpha) = H_1(\beta)$, then
    - if $H_2(\alpha) = H_2(\beta)$, then $\alpha = \beta$;
    - if $H_2(\alpha) < H_2(\beta)$, then $\alpha \prec_{\mathrm{W}} \beta$.

If $\alpha \prec_{\mathrm{W}} \beta$ or $\alpha = \beta$, we denote it by $\alpha \preceq_{\mathrm{W}} \beta$.

2.2. **Triangular norm.**

Definition 5 ([25, Definition 1.1]). A function $T : [0,1]^2 \longrightarrow [0,1]$ is called a triangular norm (t-norm) if, for any x_1, x_2, $x_3 \in [0,1]$, the following are satisfied:

(T1) $T(x_1, x_2) = T(x_2, x_1)$ (commutativity);
(T2) $T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3)$ (associativity);
(T3) $T(x_1, x_3) \leq T(x_2, x_3)$ for $x_1 \leq x_2$ (monotonicity);
(T4) $T(x_1, 1) = x_1$ (neutrality).

A function $S : [0,1]^2 \longrightarrow [0,1]$ is called a *t-conorm* if, for any x_1, x_2, $x_3 \in [0,1]$, it satisfies (Tl)–(T3) and (S4) described by

(S4) $S(x_1, 0) = x_1$ (neutrality).

The duality expression of t-norms and t-conorms is given in the following result.

Proposition 1 ([25, Proposition 1.15]). *A binary function $S : [0,1]^2 \longrightarrow [0,1]$ is a t-conorm if and only if there exists a t-norm T such that, for any $(x, y) \in [0,1]^2$,*

$$S(x, y) = 1 - T(1 - x, 1 - y). \tag{1}$$

The t-norm given by formula (1) is called the dual t-conorm of T, analogously, we can give the definition of the dual t-norm of a t-conorm S.

The associativity of t-norms $T : [0,1]^2 \longrightarrow [0,1]$ helps us to define their extensions to n-ary functions $T^{(n)} : [0,1]^n \longrightarrow [0,1]$ in the following way (see [25, Remark 1.10] and [15, Definition 3.23])

$$T^{(n)}(x_1, \ldots, x_{n-1}, x_n) \triangleq T(T^{(n-1)}(x_1, \ldots, x_{n-1}), x_n).$$

If, in particular, we have $x_1 = x_2 = \cdots = x_n = x$, we shall briefly write $x_T^{(n)} = T^{(n)}(x, x, \ldots, x)$. Finally we put, by convention, $x_T^{(0)} = 1$ and $x_T^{(1)} = x$ for any $x \in [0,1]$.

Definition 6 ([25, Definition 3.2]). Let $f : [a,b] \longrightarrow [c,d]$ be a monotone function, where $[a,b]$ and $[c,d]$ are two closed subintervals of $[-\infty, +\infty]$. The pseudo-inverse $f^{(-1)} : [c,d] \longrightarrow [a,b]$ of f is defined by:

$$f^{(-1)}(y) = \sup\{x \in [a,b] \mid (f(x) - y)(f(b) - f(a)) < 0\}.$$

Definition 7 ([25, Definition 3.25])**.** A strictly decreasing function $\tau : [0,1] \longrightarrow [0,+\infty]$ is called an additive generator (AG) of a t-norm T if it is right-continuous in 0, $\tau(1) = 0$, and, for any $(x,y) \in [0,1]^2$, we have $\tau(x)+\tau(y) \in \mathrm{Ran}(\tau) \cup [\tau(0),+\infty]$ and $T(x,y) = \tau^{(-1)}(\tau(x)+\tau(y))$.

The family of Aczél-Alsina t-norms was formally introduced by Aczél and Alsina [1] in the context of functional equations, which are the only t-norms satisfying that, whenever for $p,q \in (0,1) \cup (1,+\infty)$ with $\frac{\ln p}{\ln q} \in \mathbb{Q}$, for all $(x,y) \in [0,1]^2$, we have $T(x^p,y^p) = (T(x,y))^p$ and $T(x^q,y^q) = (T(x,y))^q$ (see [25, Proposition 5.17] or [1]).

The family $(T_\gamma^{\mathbf{AA}})_{\gamma\in[0,+\infty]}$ of Aczél-Alsina t-norms is given by:

$$T_\gamma^{\mathbf{AA}}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y), & \gamma = 0, \\ T_{\mathbf{M}}(x,y), & \gamma = +\infty, \\ e^{-((-\ln x)^\gamma+(-\ln y)^\gamma)^{1/\gamma}}, & \gamma \in (0,+\infty). \end{cases}$$

The family $(T_\gamma^{\mathbf{AA}})_{\gamma\in[0,+\infty]}$ of Aczél-Alsina t-conorms is given by:

$$S_\gamma^{\mathbf{AA}}(x,y) = \begin{cases} S_{\mathbf{D}}(x,y), & \gamma = 0, \\ S_{\mathbf{M}}(x,y), & \gamma = +\infty, \\ 1 - e^{-((-\ln(1-x))^\gamma+(-\ln(1-y))^\gamma)^{1/\gamma}}, & \gamma \in (0,+\infty). \end{cases}$$

For $\gamma \in (0,+\infty)$, the additive generators $\tau_\gamma^{\mathbf{AA}}$ and $\zeta_\gamma^{\mathbf{AA}}$ of $T_\gamma^{\mathbf{AA}}$ and $S_\gamma^{\mathbf{AA}}$ are given by, respectively,

$$\tau_\gamma^{\mathbf{AA}}(x) = (-\ln x)^\gamma, \tag{2}$$

and

$$\zeta_\gamma^{\mathbf{AA}}(x) = (-\ln(1-x))^\gamma. \tag{3}$$

3. Aczél-Alsina interactional operational laws for PFNs

Based on Aczél-Alsina t-norms and Aczél-Alsina t-conorms, the researchers independently introduced introduced the following Aczél-Alsina operations for PFNs in [35, 40].

Definition 8 ([35, Definition 13], [40, Definition 9])**.** Let $\gamma > 0$. For $\alpha_1 = \langle \mu_1, \eta_1, \nu_1 \rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2 \rangle$, $\alpha = \langle \mu, \eta, \nu \rangle \in \mathscr{P}$, define

(i) $\alpha_1 \oplus \alpha_2 = \big\langle 1 - e^{-((-\ln(1-\mu_1))^\gamma+(-\ln(1-\mu_2))^\gamma)^{1/\gamma}}, e^{-((-\ln \eta_1)^\gamma+(-\ln \eta_2)^\gamma)^{1/\gamma}}, e^{-((-\ln \nu_1)^\gamma+(-\ln \nu_2)^\gamma)^{1/\gamma}} \big\rangle$;

(ii) $\alpha_1 \otimes \alpha_2$
$= \big\langle e^{-((-\ln \mu_1)^\gamma+(-\ln \mu_2)^\gamma)^{1/\gamma}}, 1 - e^{-((-\ln(1-\eta_1))^\gamma+(-\ln(1-\eta_2))^\gamma)^{1/\gamma}},$
$1 - e^{-((-\ln(1-\nu_1))^\gamma+(-\ln(1-\nu_2))^\gamma)^{1/\gamma}} \big\rangle$;

(iii) $\lambda \cdot \alpha = \big\langle 1 - e^{-(\lambda(-\ln(1-\mu))^\gamma)^{1/\gamma}}, e^{-(\lambda(-\ln \eta)^\gamma)^{1/\gamma}}, e^{-(\lambda(-\ln \nu)^\gamma)^{1/\gamma}} \big\rangle$, $\lambda > 0$;

(iv) $\alpha^\lambda = \big\langle e^{-(\lambda(-\ln \mu)^\gamma)^{1/\gamma}}, 1 - e^{-(\lambda(-\ln(1-\eta))^\gamma)^{1/\gamma}}, 1 - e^{-(\lambda(-\ln(1-\nu))^\gamma)^{1/\gamma}} \big\rangle$, $\lambda > 0$.

Remark 1. By direct calculation, we have

(1) $\langle 0.2, 0.8, 0 \rangle \otimes \langle 0.2, 0, 0.8 \rangle$
$= \langle e^{-((-\ln 0.2)^\gamma+(-\ln 0.2)^\gamma)^{1/\gamma}}, 1-e^{-((-\ln(1-0.8))^\gamma+(-\ln(1-0))^\gamma)^{1/\gamma}}, 1-e^{-((-\ln(1-0))^\gamma+(-\ln(1-0.8))^\gamma)^{1/\gamma}} \rangle$
$= \langle 0.2^{\sqrt[\gamma]{2}}, 0.8, 0.8 \rangle \notin \mathscr{P}$ since $0.2^{\sqrt[\gamma]{2}} + 0.8 + 0.8 > 1$;

(2) $\lim\limits_{\lambda\to 0^+} \lambda \cdot \langle 0.3, 0.3, 0.3 \rangle$
$= \lim\limits_{\lambda\to 0^+} \langle 1-e^{-(\lambda(-\ln 0.7)^\gamma)^{1/\gamma}}, e^{-(\lambda(-\ln 0.3)^\gamma)^{1/\gamma}}, e^{-(\lambda(-\ln 0.3)^\gamma)^{1/\gamma}} \rangle = \lim\limits_{\lambda\to 0^+} \langle 1-0.7^{\sqrt[\gamma]{\lambda}}, 0.3^{\sqrt[\gamma]{\lambda}}, 0.3^{\sqrt[\gamma]{\lambda}} \rangle$
$= \langle 0, 1, 1 \rangle \notin \mathscr{P}$ since $0 + 1 + 1 > 1$.
This implies that there exists $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, $\lambda \cdot \langle 0.3, 0.3, 0.3 \rangle \notin \mathscr{P}$;

(3) $\lim\limits_{\lambda\to+\infty} \langle 0.3, 0.3, 0.3 \rangle^\lambda = \lim\limits_{\lambda\to+\infty} \langle 0.3^{\sqrt[\gamma]{\lambda}}, 1 - 0.7^{\sqrt[\gamma]{\lambda}}, 1 - 0.7^{\sqrt[\gamma]{\lambda}} \rangle =$
$\langle 0, 1, 1 \rangle \notin \mathscr{P}$ since $0 + 1 + 1 > 1$.

This implies that there exists $\tilde{\lambda}_0 > 0$ such that, for any $\lambda > \tilde{\lambda}_0$, $\langle 0.3, 0.3, 0.3\rangle^\lambda \notin \mathscr{P}$.
The above show that the operational laws $\otimes$, $\lambda \cdot \alpha$, and α^λ are not closed in $\mathscr{P}$.

Due to the uncloseness of Aczél-Alsina operations in Definition 8 (see Remark 1 or [40, Example 3]), and inspired by the interactional operations proposed in [23, Definition 4] and [49, Definition 3.1] , we introduce the following interaction operational laws for PFNs via Aczél-Alsina t-norms and Aczél-Alsina t-conorms, which will be proved to be closed in $\mathscr{P}$ (see Theorem 1).

Definition 9. Let $\gamma > 0$. For $\alpha_1 = \langle \mu_1, \eta_1, \nu_1\rangle$, $\alpha_2 = \langle \mu_2, \eta_2, \nu_2\rangle$, and $\alpha = \langle \mu, \eta, \nu\rangle \in \mathscr{P}$, define

(i) (Complement) $\alpha^{\complement} = \langle \nu, \eta, \mu\rangle$;

(ii) (Addition)

$$\begin{aligned}
\alpha_1 \oplus \alpha_2 =& \left\langle S_\gamma^{\mathbf{AA}}(\mu_1, \mu_2), S_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1, \mu_2 + \eta_2) - S_\gamma^{\mathbf{AA}}(\mu_1, \mu_2), T_\gamma^{\mathbf{AA}}(\nu_1, \nu_2)\right\rangle \\
=& \big\langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_1) + \zeta_\gamma^{\mathbf{AA}}(\mu_2)), \\
& (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \zeta_\gamma^{\mathbf{AA}}(\mu_2 + \eta_2)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_1) + \zeta_\gamma^{\mathbf{AA}}(\mu_2)), \\
& (\tau_\gamma^{\mathbf{AA}})^{-1}(\tau_\gamma^{\mathbf{AA}}(\nu_1) + \tau_\gamma^{\mathbf{AA}}(\nu_2))\big\rangle \\
=& \Big\langle 1 - e^{-((-\ln(1-\mu_1))^\gamma + (-\ln(1-\mu_2))^\gamma)^{1/\gamma}}, \\
& e^{-((-\ln(1-\mu_1))^\gamma + (-\ln(1-\mu_2))^\gamma)^{1/\gamma}} - e^{-((-\ln(1-\mu_1-\eta_1))^\gamma + (-\ln(1-\mu_2-\eta_2))^\gamma)^{1/\gamma}}, \\
& e^{-((-\ln \nu_1)^\gamma + (-\ln \nu_2)^\gamma)^{1/\gamma}}\Big\rangle;
\end{aligned}$$

(iii) (Product)

$$\begin{aligned}
\alpha_1 \otimes \alpha_2 =& \left\langle T_\gamma^{\mathbf{AA}}(\mu_1, \mu_2), S_\gamma^{\mathbf{AA}}(\eta_1 + \nu_1, \eta_2 + \nu_2) - S_\gamma^{\mathbf{AA}}(\nu_1, \nu_2), S_\gamma^{\mathbf{AA}}(\nu_1, \nu_2)\right\rangle \\
=& \big\langle (\tau_\gamma^{\mathbf{AA}})^{-1}(\tau_\gamma^{\mathbf{AA}}(\mu_1) + \tau_\gamma^{\mathbf{AA}}(\mu_2)), \\
& (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\eta_1 + \nu_1) + \zeta_\gamma^{\mathbf{AA}}(\eta_2 + \nu_2)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\nu_1) + \zeta_\gamma^{\mathbf{AA}}(\nu_2)), \\
& (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\nu_1) + \zeta_\gamma^{\mathbf{AA}}(\nu_2))\big\rangle \\
=& \Big\langle e^{-((-\ln \mu_1)^\gamma + (-\ln \mu_2)^\gamma)^{1/\gamma}}, \\
& e^{-((-\ln(1-\nu_1))^\gamma + (-\ln(1-\nu_2))^\gamma)^{1/\gamma}} - e^{-((-\ln(1-\eta_1-\nu_1))^\gamma + (-\ln(1-\eta_2-\nu_2))^\gamma)^{1/\gamma}}, \\
& 1 - e^{-((-\ln(1-\nu_1))^\gamma + (-\ln(1-\nu_2))^\gamma)^{1/\gamma}}\Big\rangle;
\end{aligned}$$

(iv) (Scalar multiplication) For $\lambda > 0$,

$$\begin{aligned}
\lambda \cdot \alpha =& \langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu + \eta)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu)), \\
& (\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \tau_\gamma^{\mathbf{AA}}(\mu))\rangle \\
=& \Big\langle 1 - e^{-(\lambda(-\ln(1-\mu))^\gamma)^{1/\gamma}}, e^{-(\lambda(-\ln(1-\mu))^\gamma)^{1/\gamma}} - e^{-(\lambda(-\ln(1-\mu-\eta))^\gamma)^{1/\gamma}}, \\
& e^{-(\lambda(-\ln \nu)^\gamma)^{1/\gamma}}\Big\rangle \\
=& \left\langle 1 - (1-\mu)^{\sqrt[\gamma]{\lambda}}, (1-\mu)^{\sqrt[\gamma]{\lambda}} - (1-\mu-\eta)^{\sqrt[\gamma]{\lambda}}, \nu^{\sqrt[\gamma]{\lambda}}\right\rangle;
\end{aligned}$$

(v) (Power) For $\lambda > 0$,

$$\begin{aligned}
\alpha^\lambda =& \big\langle (\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\eta + \nu)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\nu)), \\
& (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \tau_\gamma^{\mathbf{AA}}(\nu))\big\rangle \\
=& \Big\langle e^{-(\lambda(-\ln \mu)^\gamma)^{1/\gamma}}, e^{-(\lambda(-\ln(1-\nu))^\gamma)^{1/\gamma}} - e^{-(\lambda(-\ln(1-\eta-\nu))^\gamma)^{1/\gamma}}, \\
& 1 - e^{-(\lambda(-\ln(1-\nu))^\gamma)^{1/\gamma}}\Big\rangle
\end{aligned}$$

$$= \left\langle \mu^{\sqrt[\gamma]{\lambda}}, (1-\nu)^{\sqrt[\gamma]{\lambda}} - (1-\eta-\nu)^{\sqrt[\gamma]{\lambda}}, 1-(1-\nu)^{\sqrt[\gamma]{\lambda}} \right\rangle;$$

where $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

The following theorem demonstrates that all operational laws defined in Definition 9 are closed in $\mathscr{P}$.

Theorem 1. *Let α_1, α_2, $\alpha \in \mathscr{P}$. Then, for any $\lambda > 0$, $\alpha_1 \oplus \alpha_2$, $\alpha_1 \otimes \alpha_2$, $\lambda \cdot \alpha$, and α^λ are PFNs.*

Proof. (i) Since $T_\gamma^{\mathbf{AA}}$ is increasing, by $\nu_1 \le 1-\mu_1-\eta_1$ and $\nu_2 \le 1-\mu_2-\eta_2$, we have

$$T_\gamma^{\mathbf{AA}}(\nu_1,\nu_2) \le T_\gamma^{\mathbf{AA}}(1-\mu_1-\eta_1, 1-\mu_2-\eta_2).$$

This, together with

$\alpha_1 \oplus \alpha_2 = \langle S_\gamma^{\mathbf{AA}}(\mu_1,\mu_2), S_\gamma^{\mathbf{AA}}(\mu_1+\eta_1,\mu_2+\eta_2) - S_\gamma^{\mathbf{AA}}(\mu_1,\mu_2), T_\gamma^{\mathbf{AA}}(\nu_1,\nu_2)\rangle$,

implies that

$S_\gamma^{\mathbf{AA}}(\mu_1,\mu_2) + (S_\gamma^{\mathbf{AA}}(\mu_1+\eta_1,\mu_2+\eta_2) - S_\gamma^{\mathbf{AA}}(\mu_1,\mu_2)) + T_\gamma^{\mathbf{AA}}(\nu_1,\nu_2)$
$= S_\gamma^{\mathbf{AA}}(\mu_1+\eta_1,\mu_2+\eta_2) + T_\gamma^{\mathbf{AA}}(\nu_1,\nu_2) \le S_\gamma^{\mathbf{AA}}(\mu_1+\eta_1,\mu_2+\eta_2)$
$+T_\gamma^{\mathbf{AA}}(1-\mu_1-\eta_1, 1-\mu_2-\eta_2) = 1$,

and thus $\alpha_1 \oplus_T \alpha_2 \in \mathscr{P}$. Similarly, we can prove that $\alpha_1 \otimes \alpha_2 \in \mathscr{P}$.

(ii) Fix $\lambda > 0$. By $\nu \le 1-\mu-\eta$, we have $\nu^{\sqrt[\gamma]{\lambda}} \le (1-\mu-\eta)^{\sqrt[\gamma]{\lambda}}$. This, together with

$\lambda \cdot \alpha = \langle 1-(1-\mu)^{\sqrt[\gamma]{\lambda}}, (1-\mu)^{\sqrt[\gamma]{\lambda}} - (1-\mu-\eta)^{\sqrt[\gamma]{\lambda}}, \nu^{\sqrt[\gamma]{\lambda}}\rangle$,

implies that

$1-(1-\mu)^{\sqrt[\gamma]{\lambda}} + [(1-\mu)^{\sqrt[\gamma]{\lambda}} - (1-\mu-\eta)^{\sqrt[\gamma]{\lambda}}] + \nu^{\sqrt[\gamma]{\lambda}} = 1-(1-\mu-\eta)^{\sqrt[\gamma]{\lambda}} + \nu^{\sqrt[\gamma]{\lambda}} \le 1$,

and thus $\lambda \cdot \alpha \in \mathscr{P}$. Similarly, we can prove that $\alpha^\lambda \in \mathscr{P}$. □

Theorem 2. *Let α, β, $\vartheta \in \mathscr{P}$. Then, for any λ, $\xi > 0$, we have*

(i) $\alpha \oplus \beta = \beta \oplus \alpha$;
(ii) $\alpha \otimes \beta = \beta \otimes \alpha$;
(iii) $(\alpha \oplus \beta) \oplus \vartheta = \alpha \oplus (\beta \oplus \vartheta)$;
(iv) $(\alpha \otimes \beta) \otimes \vartheta = \alpha \otimes (\beta \otimes \vartheta)$;
(v) $(\xi\alpha) \oplus (\lambda\alpha) = (\xi+\lambda)\alpha$;
(vi) $(\alpha^\xi) \otimes (\alpha^\lambda) = \alpha^{(\xi+\lambda)}$;
(vii) $\lambda(\alpha \oplus \beta) = (\lambda\alpha) \oplus (\lambda\beta)$;
(viii) $(\alpha \otimes \beta)^\lambda = \alpha^\lambda \otimes \beta^\lambda$;
(ix) $\xi(\lambda\alpha) = (\xi \cdot \lambda)\alpha$;
(x) $(\alpha^\lambda)^\xi = \alpha^{(\lambda \cdot \xi)}$.

Proof. The statements (i)–(ii) follow directly from the commutativity of $T_\gamma^{\mathbf{AA}}$ and $S_\gamma^{\mathbf{AA}}$.

By direct calculation, we have

(iii) $(\alpha \oplus \beta) \oplus \vartheta = \langle S_\gamma^{\mathbf{AA}}(\mu_\alpha,\mu_\beta), S_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha,\mu_\beta+\eta_\beta) - S_\gamma^{\mathbf{AA}}(\mu_\alpha,\mu_\beta), T_\gamma^{\mathbf{AA}}(\nu_\alpha,\nu_\beta)\rangle \oplus \vartheta$
$= \langle S_\gamma^{\mathbf{AA}}(S_\gamma^{\mathbf{AA}}(\mu_\alpha,\mu_\beta),\mu_\vartheta), S_\gamma^{\mathbf{AA}}(S_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha,\mu_\beta+\eta_\beta),\mu_\vartheta+\eta_\vartheta)$
$-S_\gamma^{\mathbf{AA}}(S_\gamma^{\mathbf{AA}}(\mu_\alpha,\mu_\beta),\mu_\vartheta), T_\gamma^{\mathbf{AA}}(T_\gamma^{\mathbf{AA}}(\nu_\alpha,\nu_\beta),\nu_\vartheta)\rangle$
$= \langle S_\gamma^{\mathbf{AA}}(\mu_\alpha, S_\gamma^{\mathbf{AA}}(\mu_\beta,\mu_\vartheta)), S_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha, S_\gamma^{\mathbf{AA}}(\mu_\beta+\eta_\beta,\mu_\vartheta+\eta_\vartheta))$
$-S_\gamma^{\mathbf{AA}}(\mu_\alpha, S_\gamma^{\mathbf{AA}}(\mu_\beta,\mu_\vartheta)), T_\gamma^{\mathbf{AA}}(\nu_\alpha, T_\gamma^{\mathbf{AA}}(\nu_\beta,\nu_\vartheta))\rangle$
$= \alpha \oplus \langle S_\gamma^{\mathbf{AA}}(\mu_\beta,\mu_\vartheta), S_\gamma^{\mathbf{AA}}(\mu_\beta+\eta_\beta,\mu_\vartheta+\eta_\vartheta) - S_\gamma^{\mathbf{AA}}(\mu_\beta,\mu_\vartheta), T(\nu_\beta,\nu_\vartheta)\rangle$
$= \alpha \oplus (\beta \oplus \vartheta)$;

(v) $(\xi\alpha) \oplus (\lambda\alpha) = \langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),$
$(\tau_\gamma^{\mathbf{AA}})^{-1}(\xi \cdot \tau_\gamma^{\mathbf{AA}}(\mu_\alpha))\rangle \oplus \langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha))$
$-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)), (\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \tau_\gamma^{\mathbf{AA}}(\mu_\alpha))\rangle(\xi+\lambda)\alpha$
$= \langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha) + \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha) + \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha))$

$-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),(\tau_\gamma^{\mathbf{AA}})^{-1}(\xi\cdot\tau_\gamma^{\mathbf{AA}}(\mu_\alpha)+\lambda\cdot\tau_\gamma^{\mathbf{AA}}(\mu_\alpha))\rangle$
$=\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}((\xi+\lambda)\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),(\zeta_\gamma^{\mathbf{AA}})^{-1}((\xi+\lambda)\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha))$
$-(\zeta_\gamma^{\mathbf{AA}})^{-1}((\xi+\lambda)\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),(\tau_\gamma^{\mathbf{AA}})^{-1}((\xi+\lambda)\cdot\tau_\gamma^{\mathbf{AA}}(\mu_\alpha))\rangle$
$=(\xi+\lambda)\alpha.$

(vii) $(\lambda\alpha)\oplus(\lambda\beta)=\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha))-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),$
$(\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\tau_\gamma^{\mathbf{AA}}(\mu_\alpha))\rangle\oplus\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu)),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu+\eta))$
$-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu)),(\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\tau_\gamma^{\mathbf{AA}}(\mu))\rangle$
$=\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)+\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha)+\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\beta+\eta_\beta))$
$-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)+\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),(\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\tau_\gamma^{\mathbf{AA}}(\nu_\alpha)+\lambda\cdot\tau_\gamma^{\mathbf{AA}}(\nu_\beta))\rangle$
$=\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda(\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)+\zeta_\gamma^{\mathbf{AA}}(\mu_\beta))),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda(\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha)+\zeta_\gamma^{\mathbf{AA}}(\mu_\beta+\eta_\beta)))$
$-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda(\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)+\zeta_\gamma^{\mathbf{AA}}(\mu_\beta))),(\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda(\tau_\gamma^{\mathbf{AA}}(\nu_\alpha)+\tau_\gamma^{\mathbf{AA}}(\nu_\beta)))\rangle$
$=\lambda\cdot\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)+\zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),(\zeta_\gamma^{\mathbf{AA}})^{-1}$
$(\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha)+\zeta_\gamma^{\mathbf{AA}}(\mu_\beta+\eta_\beta))-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)+\zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),(\tau_\gamma^{\mathbf{AA}})^{-1}(\tau_\gamma^{\mathbf{AA}}(\nu_\alpha)+\tau_\gamma^{\mathbf{AA}}(\nu_\beta))\rangle$
$=\lambda(\alpha\oplus\beta).$

(ix) $\xi(\lambda\alpha)=\xi\cdot\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha))$
$-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),\ (\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda\cdot\tau_\gamma^{\mathbf{AA}}(\mu_\alpha))\rangle$
$=\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi\cdot\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi\cdot\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha+\eta_\alpha))$
$-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\xi\cdot\lambda\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_\alpha)),(\tau_\gamma^{\mathbf{AA}})^{-1}(\xi\cdot\lambda\cdot\tau_\gamma^{\mathbf{AA}}(\mu_\alpha))\rangle$
$=(\xi\cdot\lambda)\alpha.$

By using similar arguments to the proofs of (iii), (v), (vii), and (ix), it can be verified that the statements (iv), (vi), (viii), and (x) hold. □

Theorem 3. *Let $\alpha_j=\langle\mu_j,\eta_j,\nu_j\rangle\in\mathscr{P}$ $(j=1,\ldots,n)$ and T be a strict t-norm. Then,*

$$\begin{aligned}&\alpha_1\oplus\alpha_2\oplus\cdots\oplus\alpha_n\\=&\Big\langle(S_\gamma^{\mathbf{AA}})^{(n)}(\mu_1,\ldots,\mu_n),\\&\quad(S_\gamma^{\mathbf{AA}})^{(n)}(\mu_1+\eta_1,\ldots,\mu_n+\eta_n)-(S_\gamma^{\mathbf{AA}})^{(n)}(\mu_1,\ldots,\mu_n),\\&\quad(T_\gamma^{\mathbf{AA}})^{(n)}(\nu_1,\ldots,\nu_n)\Big\rangle,\end{aligned}\tag{4}$$

and

$$\begin{aligned}&\alpha_1\otimes\alpha_2\otimes\cdots\otimes\alpha_n\\=&\Big\langle(T_\gamma^{\mathbf{AA}})^{(n)}(\mu_1,\ldots,\mu_n),\\&\quad(S_\gamma^{\mathbf{AA}})^{(n)}(\eta_1+\nu_1,\ldots,\eta_n+\nu_n)-(S_\gamma^{\mathbf{AA}})^{(n)}(\nu_1,\ldots,\nu_n),\\&\quad(S_\gamma^{\mathbf{AA}})^{(n)}(\nu_1,\ldots,\nu_n)\Big\rangle.\end{aligned}\tag{5}$$

Proof. We only prove the formula (4) in view of the mathematical induction on n. The formula (5) can be proved analogously.

(1) When $n=2$, it is clear that $\alpha_1\oplus\alpha_2=\langle S_\gamma^{\mathbf{AA}}(\mu_1,\mu_2),S_\gamma^{\mathbf{AA}}(\mu_1+\eta_1,\mu_2+\eta_2)-S_\gamma^{\mathbf{AA}}(\mu_1,\mu_2),T_\gamma^{\mathbf{AA}}(\nu_1,\nu_2)\rangle$ by Definition 9.

(2) Suppose that the formula (4) holds when $n=m$; i.e.,

$$\begin{aligned}&\alpha_1\oplus\alpha_2\oplus\cdots\oplus\alpha_m\\=&\Big\langle(S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1,\ldots,\mu_m),\end{aligned}$$

$$(S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1+\eta_1,\ldots,\mu_m+\eta_m)-(S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1,\ldots,\mu_m),$$
$$(T_\gamma^{\mathbf{AA}})^{(m)}(\nu_1,\ldots,\nu_m)\Big\rangle.$$

When $n=m+1$, by Definition 9 and Theorem 2 (iii), we get

$$\begin{aligned}
&\alpha_1\oplus\alpha_2\oplus\cdots\oplus\alpha_{m+1}=(\alpha_1\oplus\alpha_2\oplus\cdots\oplus\alpha_m)\oplus\alpha_{m+1}\\
=&\Big\langle(S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1,\ldots,\mu_m),\\
&(S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1+\eta_1,\ldots,\mu_m+\eta_m)-(S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1,\ldots,\mu_m),\\
&(T_\gamma^{\mathbf{AA}})^{(m)}(\nu_1,\ldots,\nu_m)\Big\rangle\oplus\langle\mu_{m+1},\eta_{m+1},\nu_{m+1}\rangle\\
=&\Big\langle S_\gamma^{\mathbf{AA}}((S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1,\ldots,\mu_m),\mu_{m+1}),\\
&S_\gamma^{\mathbf{AA}}((S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1+\eta_1,\ldots,\mu_m+\eta_m),\mu_{m+1}+\eta_{m+1})\\
&-S_\gamma^{\mathbf{AA}}((S_\gamma^{\mathbf{AA}})^{(m)}(\mu_1,\ldots,\mu_m),\mu_{m+1}),\\
&T_\gamma^{\mathbf{AA}}((T_\gamma^{\mathbf{AA}})^{(m)}(\nu_1,\ldots,\nu_m),\nu_{m+1})\Big\rangle,\\
=&\Big\langle(S_\gamma^{\mathbf{AA}})^{(m+1)}(\mu_1,\ldots,\mu_m,\mu_{m+1}),\\
&(S_\gamma^{\mathbf{AA}})^{(m+1)}(\mu_1+\eta_1,\ldots,\mu_m+\eta_m,\mu_{m+1}+\eta_{m+1})\\
&-(S_\gamma^{\mathbf{AA}})^{(m+1)}(\mu_1,\ldots,\mu_m,\mu_{m+1}),\\
&(T_\gamma^{\mathbf{AA}})^{(m+1)}(\nu_1,\ldots,\nu_m,\nu_{m+1})\Big\rangle,
\end{aligned}$$

and thus formula (4) holds when $n=m+1$.

Therefore, by (1) and (2), we obtain that formula (4) holds for all n. □

4. PF Aczél-Alsina interactional aggregation operators

Based on the Aczél-Alsina interactional operational laws defined in Section 3, this section is devoted to aggregating the PF information.

4.1. PF Aczél-Alsina interactional weighted average (geometric) operator.

Definition 10. Let $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^\top$ be the weight vector such that $\omega_j\in(0,1]$ and $\sum_{j=1}^{n}\omega_j=1$. Define the PF Aczél-Alsina interactional weighted average operator PFAAIWA and PF Aczél-Alsina interactional weighted geometric operator PFAAIWG by:

$$\begin{aligned}
&\mathrm{PFAAIWA}:\mathscr{P}^n\longrightarrow\mathscr{P}\\
&(\alpha_1,\alpha_2,\ldots,\alpha_n)\longmapsto\omega_1\alpha_1\oplus\omega_2\alpha_2\oplus\cdots\oplus\omega_n\alpha_n,
\end{aligned}\tag{6}$$

and

$$\begin{aligned}
&\mathrm{PFAAIWG}:\mathscr{P}^n\longrightarrow\mathscr{P}\\
&(\alpha_1,\alpha_2,\ldots,\alpha_n)\longmapsto\alpha_1^{\omega_1}\otimes\alpha_2^{\omega_2}\otimes\cdots\otimes\alpha_n^{\omega_n},
\end{aligned}\tag{7}$$

respectively.

Theorem 4. *Let $\alpha_j=(\mu_j,\eta_j,\nu_j)$ $(j=1,2,\ldots,n)$ be a collection of PFNs and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^\top$ be the weight vector of α_j $(j=1,2,\ldots,n)$ such that $\omega_j\in(0,1]$ and $\sum_{j=1}^{n}\omega_j=1$. Then,*

$$
\begin{aligned}
&\mathrm{PFAAIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left\langle (\zeta_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\zeta_\gamma^{\boldsymbol{AA}}(\mu_j)),(\zeta_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\zeta_\gamma^{\boldsymbol{AA}}(\mu_j+\eta_j))\right.\\
&\left.-(\zeta_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\zeta_\gamma^{\boldsymbol{AA}}(\mu_j)),\,(\tau_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\tau_\gamma^{\boldsymbol{AA}}(\nu_j))\right\rangle,\\
&=\left\langle 1-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mu_j))^\gamma}},e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mu_j))^\gamma}}-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mu_j-\eta_j))^\gamma}},\right.\\
&\qquad\left.e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln\nu_j)^\gamma}}\right\rangle,
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
&\mathrm{PFAAIWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left\langle (\tau_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\tau_\gamma^{\boldsymbol{AA}}(\mu_j)),(\zeta_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\zeta_\gamma^{\boldsymbol{AA}}(\eta_j+\nu_j))\right.\\
&\qquad\left.-(\zeta_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\zeta_\gamma^{\boldsymbol{AA}}(\nu_j)),\,(\zeta_\gamma^{\boldsymbol{AA}})^{-1}(\sum_{j=1}^{n}\omega_j\cdot\zeta_\gamma^{\boldsymbol{AA}}(\nu_j))\right\rangle,\\
&=\left\langle e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln\mu_j)^\gamma}},e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\nu_j))^\gamma}}-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\eta_j-\nu_j))^\gamma}},\right.\\
&\qquad\left.1-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\nu_j))^\gamma}}\right\rangle.
\end{aligned}
\tag{9}
$$

Proof. Noting that $(\tau_\gamma^{\mathbf{AA}})^{-1}(y)=e^{-\sqrt[\gamma]{y}}$ and $(\zeta_\gamma^{\mathbf{AA}})^{-1}(y)=1-e^{-\sqrt[\gamma]{y}}$, by formulas (2), (3), (6) and (7), Theorem 3, and Definition 9, we have

$$
\begin{aligned}
&\mathrm{PFAAIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\big\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_1)),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_1+\eta_1))-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_1)),\\
&\quad(\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\tau_\gamma^{\mathbf{AA}}(\nu_1))\big\rangle\\
&\qquad\oplus\cdots\oplus\\
&\big\langle(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_n)),(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_n+\eta_n))-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad(\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\tau_\gamma^{\mathbf{AA}}(\nu_n))\big\rangle\\
&=\Big\langle(S_\gamma^{\mathbf{AA}})^{(n)}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_1)),\ldots,(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_n))),\\
&\quad(S_\gamma^{\mathbf{AA}})^{(n)}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_1+\eta_1)),\ldots,(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_n+\eta_n)))\\
&\qquad-(S_\gamma^{\mathbf{AA}})^{(n)}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_1)),\ldots,(\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\zeta_\gamma^{\mathbf{AA}}(\mu_n))),\\
&\quad(T_\gamma^{\mathbf{AA}})^{(n)}((\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1\cdot\tau_\gamma^{\mathbf{AA}}(\nu_1)),\ldots,(\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_n\cdot\tau_\gamma^{\mathbf{AA}}(\nu_n)))\Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \big\langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n + \eta_n))\\
&\quad - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\nu_n))\big\rangle,\\
&= \left\langle 1 - e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln(1-\mu_j))^\gamma}}, e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln(1-\mu_j))^\gamma}} - e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln(1-\mu_j-\eta_j))^\gamma}},\right.\\
&\quad \left. e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln \nu_j)^\gamma}} \right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{PFAAIWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \big\langle (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\mu_1)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\eta_1 + \nu_1)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_1)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_1))\big\rangle\\
&\qquad \oplus \cdots \oplus\\
&\quad \big\langle (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\mu_n)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\eta_n + \nu_n)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_n)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_n))\big\rangle\\
&= \Big\langle (T_\gamma^{\mathbf{AA}})^{(n)}((\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\mu_1)), \ldots, (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\mu_n))),\\
&\quad (S_\gamma^{\mathbf{AA}})^{(n)}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\eta_1 + \nu_1)), \ldots, (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\eta_n + \nu_n)))\\
&\qquad - (S_\gamma^{\mathbf{AA}})^{(n)}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_1)), \ldots, (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_n))),\\
&\quad (S_\gamma^{\mathbf{AA}})^{(n)}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_1)), \ldots, (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_n)))\Big\rangle\\
&= \big\langle (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\eta_1 + \nu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\eta_n + \nu_n))\\
&\quad - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_n)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\nu_n))\big\rangle,\\
&= \left\langle e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln \mu_j)^\gamma}}, e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln(1-\nu_j))^\gamma}} - e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln(1-\eta_j-\nu_j))^\gamma}},\right.\\
&\quad \left. 1 - e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j(-\ln(1-\nu_j))^\gamma}} \right\rangle.
\end{aligned}
$$

□

The operators PFAAIWA and PFAAIWG have the following desirable properties.

Theorem 5 (Monotonicity)**.** *Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle$ $(j = 1, 2, \ldots, n)$ and $\beta_j = \langle \mu_{\beta_j}, \eta_{\beta_j}, \nu_{\beta_j} \rangle$ $(j = 1, 2, \ldots, n)$ be two collections of PFNs such that $\mu_{\alpha_j} \le \mu_{\beta_j}$, $\eta_{\alpha_j} \le \eta_{\beta_j}$, and $\nu_{\alpha_j} \ge \nu_{\beta_j}$, i.e., $\alpha_j \subseteq \beta_j$. Then, one has*

$$\mathrm{PFAAIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \preceq_{\mathrm{W}} \mathrm{PFAAIWA}(\beta_1, \beta_2, \ldots, \beta_n).$$

Proof. By Theorem 4, we have

$$
\begin{aligned}
&\mathrm{PFIAAWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left\langle 1-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}},e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}}-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j}-\eta_{\alpha_j})\right)^{\gamma}}},\right.\\
&\qquad\left.e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln\nu_{\alpha_j}\right)^{\gamma}}}\right\rangle,\triangleq\langle\tilde{\mu},\tilde{\eta},\tilde{\nu}\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{PFIAAWA}(\beta_1,\beta_2,\ldots,\beta_n)\\
&=\left\langle 1-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j})\right)^{\gamma}}},e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j})\right)^{\gamma}}}-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j}-\eta_{\beta_j})\right)^{\gamma}}},\right.\\
&\qquad\left.e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln\nu_{\beta_j}\right)^{\gamma}}}\right\rangle,\triangleq\langle\hat{\mu},\hat{\eta},\hat{\nu}\rangle.
\end{aligned}
$$

By $\mu_{\alpha_j}\le\mu_{\beta_j}$ and $\nu_{\alpha_j}\ge\nu_{\beta_j}$, we get that

$$
\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}\le\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j})\right)^{\gamma}}
$$

and

$$
\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln\nu_{\alpha_j}\right)^{\gamma}}\le\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln\nu_{\beta_j}\right)^{\gamma}},
$$

implying that

$$
\tilde{\mu}=1-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}}\le 1-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j})\right)^{\gamma}}}=\hat{\mu}
$$

and

$$
\tilde{\nu}=e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln\nu_{\alpha_j}\right)^{\gamma}}}\ge e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln\nu_{\beta_j}\right)^{\gamma}}}=\hat{\nu}.
$$

To prove that $\langle\tilde{\mu},\tilde{\eta},\tilde{\nu}\rangle\preceq_{\mathrm{W}}\langle\hat{\mu},\hat{\eta},\hat{\nu}\rangle$, we consider the following cases:

(1) If there exists $1\le j_0\le n$ such that $\mu_{\alpha_{j_0}}=1$, then $\mu_{\beta_{j_0}}=1$ by $\mu_{\alpha_{j_0}}\le\mu_{\beta_{j_0}}$, implying that $\tilde{\mu}=\hat{\mu}=1$. Thus, $\langle\tilde{\mu},\tilde{\eta},\tilde{\nu}\rangle=\langle 1,0,0\rangle=\langle\hat{\mu},\hat{\eta},\hat{\nu}\rangle$.

(2) If $\mu_{\alpha_j}<1$ holds for all $1\le j\le n$, we consider the following subcases:

2-1) If $\mu_{\alpha_j}=\mu_{\beta_j}$ holds for all $1\le j\le n$, then $\tilde{\mu}=\hat{\mu}$ and $\mu_{\alpha_j}+\eta_{\alpha_j}\le\mu_{\alpha_j}+\eta_{\beta_j}$ by $\eta_{\alpha_j}\le\eta_{\beta_j}$, implying that

$$
\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j}-\eta_{\alpha_j})\right)^{\gamma}}\le\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j}-\eta_{\beta_j})\right)^{\gamma}},
$$

and thus

$$
e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j}-\eta_{\alpha_j})\right)^{\gamma}}}\ge e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j}-\eta_{\beta_j})\right)^{\gamma}}}.
$$

This, together with $\mu_{\alpha_j}=\mu_{\beta_j}$ $(1\le j\le n)$, implies that

$$
\begin{aligned}
\tilde{\eta}&=e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}}-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j}-\eta_{\alpha_j})\right)^{\gamma}}}\\
&\le e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j})\right)^{\gamma}}}-e^{-\sqrt[\gamma]{\sum\limits_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j}-\eta_{\beta_j})\right)^{\gamma}}}=\hat{\eta}.
\end{aligned}
$$

Then, $\langle\tilde{\mu},\tilde{\eta},\tilde{\nu}\rangle \subseteq \langle\hat{\mu},\hat{\eta},\hat{\nu}\rangle$ by $\tilde{\mu}=\hat{\mu}$ and $\tilde{\nu}\geq\hat{\nu}$. Therefore, $\langle\tilde{\mu},\tilde{\eta},\tilde{\nu}\rangle \preceq_{\mathrm{W}} \langle\hat{\mu},\hat{\eta},\hat{\nu}\rangle$.

2-2) If there exists $1\leq j_0\leq n$ such that $\mu_{\alpha_{j_0}}<\mu_{\beta_{j_0}}$, then $\mu_{\alpha_{j_0}}<\frac{1}{2}(\mu_{\alpha_{j_0}}+\mu_{\beta_{j_0}})<1$. This, together with $\mu_{\alpha_j}<1$ and $\mu_{\alpha_j}\leq\mu_{\beta_j}$ $(1\leq j\leq n)$, implies that

$$\begin{aligned}0&\leq\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}\\&<\sqrt[\gamma]{\omega_{j_0}\left[-\ln\left(1-\frac{1}{2}(\mu_{\alpha_{j_0}}+\mu_{\beta_{j_0}})\right)\right]^{\gamma}+\sum_{j=1,j\neq j_0}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}\\&<+\infty,\end{aligned}$$

and

$$\begin{aligned}&\sqrt[\gamma]{\omega_{j_0}\left[-\ln\left(1-\frac{1}{2}(\mu_{\alpha_{j_0}}+\mu_{\beta_{j_0}})\right)\right]^{\gamma}+\sum_{j=1,j\neq j_0}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}\\&\leq\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j})\right)^{\gamma}}.\end{aligned}$$

Then,

$$\begin{aligned}1-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}}&<1-e^{-\sqrt[\gamma]{\omega_{j_0}\left[-\ln\left(1-\frac{1}{2}(\mu_{\alpha_{j_0}}+\mu_{\beta_{j_0}})\right)\right]^{\gamma}+\sum_{j=1,j\neq j_0}^{n}\omega_j\left(-\ln(1-\mu_{\alpha_j})\right)^{\gamma}}}\\&\leq 1-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j\left(-\ln(1-\mu_{\beta_j})\right)^{\gamma}}},\end{aligned}$$

i.e., $\tilde{\mu}<\hat{\mu}$. By $\tilde{\nu}\geq\hat{\nu}$, we have $S(\langle\tilde{\mu},\tilde{\eta},\tilde{\nu}\rangle)=\tilde{\mu}-\tilde{\nu}<\hat{\mu}-\hat{\nu}=S(\langle\hat{\mu},\hat{\eta},\hat{\nu}\rangle)$. Therefore, $\langle\tilde{\mu},\tilde{\eta},\tilde{\nu}\rangle \prec_{\mathrm{W}} \langle\hat{\mu},\hat{\eta},\hat{\nu}\rangle$ by Definition 4. □

Theorem 6 (Idempotency)**.** *If $\alpha_j=\langle\mu,\eta,\nu\rangle\in\mathscr{P}$ for all $j=1,2,\ldots,n$, then*

$$\mathrm{PFAAIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\mu,\eta,\nu\rangle.$$

Proof. By Theorem 4, we have

$$\begin{aligned}&\mathrm{PFIAAWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\=&\left\langle 1-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mu))^{\gamma}}},e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mu))^{\gamma}}}-e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln(1-\mu-\eta))^{\gamma}}},\right.\\&\left.e^{-\sqrt[\gamma]{\sum_{j=1}^{n}\omega_j(-\ln\nu)^{\gamma}}}\right\rangle\\=&\langle\mu,\eta,\nu\rangle.\end{aligned}$$

□

Theorem 7 (Boundedness)**.** *Let $\alpha_j=\langle\mu_j,\eta_j,\nu_j\rangle\in\mathscr{P}$ $(j=1,2,\ldots,n)$, then*

$$\begin{aligned}&\langle\min_{1\leq j\leq n}\{\mu_j\},\min_{1\leq j\leq n}\{\eta_j\},\max_{1\leq j\leq n}\{\nu_j\}\rangle\\&\preceq_{\mathrm{W}}\mathrm{PFAAIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&\preceq_{\mathrm{W}}\langle\max_{1\leq j\leq n}\{\mu_j\},1-(\max_{1\leq j\leq n}\{\mu_j\}+\min_{1\leq j\leq n}\{\nu_j\}),\min_{1\leq j\leq n}\{\nu_j\}\rangle.\end{aligned}$$

Proof. For convenience, denote $\langle \min_{1\le j\le n}\{\mu_j\}, \min_{1\le j\le n}\{\eta_j\}, \max_{1\le j\le n}\{\nu_j\}\rangle = \langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1\rangle$, PFAAIWA $(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \mu, \eta, \nu\rangle$, and $\langle \max_{1\le j\le n}\{\mu_j\}, 1-(\max_{1\le j\le n}\{\mu_j\} + \min_{1\le j\le n}\{\nu_j\}), \min_{1\le j\le n}\{\nu_j\}\rangle = \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle$. It is easy to see that $\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1\rangle$, $\langle \mu, \eta, \nu\rangle$, and $\langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle$ are PFNs. Noting that $\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1\rangle \subseteq \alpha_j$ holds for all $1 \le j \le n$, by Theorems 5 and 6, we have

$$\begin{aligned}\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1\rangle &= \text{PFAAIWA}(\langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1\rangle, \ldots, \langle \tilde{\mu}_1, \tilde{\eta}_1, \tilde{\nu}_1\rangle) \quad \text{(by Theorems 6)}\\ &\preceq_{\text{W}} \text{PFAAIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \text{(by Theorems 5).}\end{aligned}$$

Clearly, $\mu_j \le \tilde{\mu}_2$ and $\nu_j \ge \tilde{\nu}_2$ holds for all $1 \le j \le n$. This, together with Theorem 4, implies that

$$\mu = 1 - e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j (-\ln(1-\mu_j))^{\gamma}}} \le 1 - e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j (-\ln(1-\tilde{\mu}_2))^{\gamma}}} = \tilde{\mu}_2,$$

and

$$\nu = e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j (-\ln \nu_j)^{\gamma}}} \ge e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j (-\ln \tilde{\nu}_2)^{\gamma}}} = \tilde{\nu}_2.$$

To prove $\langle \mu, \eta, \nu\rangle \preceq_{\text{W}} \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle$, we consider the following cases:

(1) If $\tilde{\mu}_2 = 1$, then $\tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle = \langle 1, 0, 0\rangle$. It is clear that $\langle \mu, \eta, \nu\rangle \preceq_{\text{W}} \langle 1, 0, 0\rangle = \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle$ since $\langle 1, 0, 0\rangle$ is the top element of $\mathscr{P}$.

(2) If $\tilde{\mu}_2 < 1$, we consider the following subcases:

2-1) If $\mu_1 = \mu_2 = \cdots = \mu_n = \tilde{\mu}_2$, then, for any $1 \le k \le n$, it holds that $\tilde{\eta}_2 = 1 - (\max_{1\le j\le n}\{\mu_j\} + \min_{1\le j\le n}\{\nu_j\}) = 1 - \tilde{\mu}_2 - \min_{1\le j\le n}\{\nu_j\} \ge 1 - \mu_k - \nu_k \ge \eta_k$ and $\tilde{\nu}_2 = \min_{1\le j\le n}\{\nu_j\} \le \nu_k$, implying that $\alpha_k \subseteq \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle$. This, together with Theorems 5 and 6, implies that

$$\begin{aligned}\langle \mu, \eta, \nu\rangle = \text{PFAAIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \preceq_{\text{W}} \text{PFAAIWA}(\langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle, \ldots,\\ \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle) = \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle.\end{aligned}$$

2-2) If there exists $1 \le j_0 \le n$ such that $\mu_{j_0} \ne \tilde{\mu}_2$, by $\mu_{j_0} \le \tilde{\mu}_2$, then $\mu_{j_0} < \tilde{\mu}_2 < 1$. This, together with $\mu_j \le \tilde{\mu}_2$ $(1 \le j \le n)$, implies that

$$0 \le \sqrt[\gamma]{\sum_{j=1}^{n} \omega_j \left(-\ln(1-\mu_j)\right)^{\gamma}} < \sqrt[\gamma]{\sum_{j=1}^{n} \omega_j \left(-\ln(1-\tilde{\mu}_2)\right)^{\gamma}} = -\ln(1-\tilde{\mu}_2) < +\infty,$$

and thus

$$\mu = 1 - e^{-\sqrt[\gamma]{\sum_{j=1}^{n} \omega_j (-\ln(1-\mu_j))^{\gamma}}} < 1 - e^{-\ln(1-\tilde{\mu}_2)} = \tilde{\mu}_2.$$

Therefore, $S(\langle \mu, \eta, \nu\rangle) = \mu - \nu < \tilde{\mu}_2 - \tilde{\nu}_2 = S(\langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle)$ by $\nu \ge \tilde{\nu}_2$. This means that $\langle \mu, \eta, \nu\rangle \preceq_{\text{W}} \langle \tilde{\mu}_2, \tilde{\eta}_2, \tilde{\nu}_2\rangle$ by Definition 4. □

Theorem 8 (Shift Invariance)**.** *Let $\alpha_j = \langle \mu_j, \eta_j, \nu_j\rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$ and $\beta = \langle \mu_\beta, \eta_\beta, \nu_\beta\rangle \in \mathscr{P}$. Then,*

$$\text{PFAAIWA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \ldots, \alpha_n \oplus \beta) = \text{PFAAIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \beta.$$

Proof. By Definition 9, we have

$$\begin{aligned}\alpha_j \oplus \beta = \big\langle &(\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_j) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),\\ &(\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_j + \eta_j) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta + \eta_\beta)) - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}(\mu_j) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),\\ &(\tau_\gamma^{\mathbf{AA}})^{-1}(\tau_\gamma^{\mathbf{AA}}(\nu_j) + \tau_\gamma^{\mathbf{AA}}(\nu_\beta))\big\rangle.\end{aligned}$$

This, together with Theorem 4, implies that

$$
\begin{aligned}
&\mathrm{PFAAIWA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \ldots, \alpha_n \oplus \beta)\\
=&\big\langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n + \eta_n) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta + \eta_\beta))\\
&\qquad - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),\\
&\quad (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\nu_n) + \tau_\gamma^{\mathbf{AA}}(\nu_\beta))\big\rangle\\
=&\big\langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n))) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n + \eta_n))) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta + \eta_\beta))\\
&\qquad - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\zeta_\gamma^{\mathbf{AA}}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n))) + \zeta_\gamma^{\mathbf{AA}}(\mu_\beta)),\\
&\quad (\tau_\gamma^{\mathbf{AA}})^{-1}(\tau_\gamma^{\mathbf{AA}}((\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\nu_n))) + \tau_\gamma^{\mathbf{AA}}(\nu_\beta))\big\rangle\\
=&\big\langle S_\gamma^{\mathbf{AA}}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)), \mu_\beta),\\
&\quad S_\gamma^{\mathbf{AA}}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n + \eta_n)), \mu_\beta + \eta_\beta)\\
&\quad - S_\gamma^{\mathbf{AA}}((\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)), \mu_\beta),\\
&\quad T_\gamma^{\mathbf{AA}}((\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\nu_n)), \nu_\beta)\big\rangle\\
=&\mathrm{PFAAIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \beta \quad \text{(by formula (8))}.
\end{aligned}
$$

□

Theorem 9 (Homogeneity). *Let $\alpha_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \mathscr{P}$ $(j = 1, 2, \ldots, n)$ and $\lambda > 0$. Then,*

$$\mathrm{PFAAIWA}(\lambda \cdot \alpha_1, \lambda \cdot \alpha_2, \ldots, \lambda \cdot \alpha_n) = \lambda \cdot \mathrm{PFAAIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Proof. Noting that

$$
\begin{aligned}
&\lambda \cdot \alpha_j = \langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_j)), (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_j + \eta_j))\\
&-(\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_j)), (\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda \cdot \tau_\gamma^{\mathbf{AA}}(\mu_j))\rangle \ (j = 1, 2, \ldots, n),
\end{aligned}
$$

by Definition 9 and Theorem 4, we have

$$
\begin{aligned}
&\mathrm{PFAAIWA}(\lambda \cdot \alpha_1, \lambda \cdot \alpha_2, \ldots, \lambda \cdot \alpha_n)\\
=&\big\langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \cdots + \omega_n \cdot \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n + \eta_n))\\
&\qquad - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \lambda \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \lambda \cdot \tau_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \lambda \cdot \tau_\gamma^{\mathbf{AA}}(\nu_n))\big\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda \cdot \mathrm{PFAAIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
=&\lambda \cdot \big\langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n + \eta_n))\\
&\qquad - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n)),\\
&\quad (\tau_\gamma^{\mathbf{AA}})^{-1}(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\nu_n))\big\rangle,\\
=&\big\langle (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n))),\\
&\quad (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1 + \eta_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n + \eta_n)))\\
&\qquad - (\zeta_\gamma^{\mathbf{AA}})^{-1}(\lambda(\omega_1 \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_1) + \cdots + \omega_n \cdot \zeta_\gamma^{\mathbf{AA}}(\mu_n))),\\
&\quad (\tau_\gamma^{\mathbf{AA}})^{-1}(\lambda(\omega_1 \cdot \tau_\gamma^{\mathbf{AA}}(\nu_1) + \cdots + \omega_n \cdot \tau_\gamma^{\mathbf{AA}}(\nu_n)))\big\rangle,
\end{aligned}
$$

implying that $\mathrm{PFAAIWA}(\lambda\cdot\alpha_1,\lambda\cdot\alpha_2,\ldots,\lambda\cdot\alpha_n)=\lambda\cdot\mathrm{PFAAIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)$. □

Similarly, we have the following results.

Theorem 10 (Monotonicity). *Let $\alpha_j=\langle\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j}\rangle$ $(j=1,2,\ldots,n)$ and $\beta_j=\langle\mu_{\beta_j},\eta_{\beta_j},\nu_{\beta_j}\rangle$ $(j=1,2,\ldots,n)$ be two collections of PFNs such that $\mu_{\alpha_j}\le\mu_{\beta_j}$, $\eta_{\alpha_j}\le\eta_{\beta_j}$, and $\nu_{\alpha_j}\ge\nu_{\beta_j}$, i.e., $\alpha_j\subseteq\beta_j$. Then*

$$\mathrm{PFAAIWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)\preceq_{\mathrm{W}}\mathrm{PFAAIWG}(\beta_1,\beta_2,\ldots,\beta_n).$$

Theorem 11 (Idempotency). *If $\alpha_j=\langle\mu,\eta,\nu\rangle\in\mathscr{P}$ for all $j=1,2,\ldots,n$, then*

$$\mathrm{PFAAIWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\mu,\eta,\nu\rangle.$$

Theorem 12 (Boundedness). *Let $\alpha_j=\langle\mu_j,\eta_j,\nu_j\rangle\in\mathscr{P}$ $(j=1,2,\ldots,n)$, then*

$$\begin{aligned}&\langle\min_{1\le j\le n}\{\mu_j\},\min_{1\le j\le n}\{\eta_j\},\max_{1\le j\le n}\{\nu_j\}\rangle\\&\preceq_{\mathrm{W}}\mathrm{PFAAIWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&\preceq_{\mathrm{W}}\langle\max_{1\le j\le n}\{\mu_j\},1-(\max_{1\le j\le n}\{\mu_j\}+\min_{1\le j\le n}\{\nu_j\}),\min_{1\le j\le n}\{\nu_j\}\rangle.\end{aligned}$$

Theorem 13 (Shift Invariance). *Let $\alpha_j=\langle\mu_j,\eta_j,\nu_j\rangle\in\mathscr{P}$ $(j=1,2,\ldots,n)$ and $\beta=\langle\mu_\beta,\eta_\beta,\nu_\beta\rangle\in\mathscr{P}$. Then,*

$$\mathrm{PFAAIWG}(\alpha_1\otimes\beta,\alpha_2\otimes\beta,\ldots,\alpha_n\otimes\beta)=\mathrm{PFAAIWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)\otimes\beta.$$

Theorem 14 (Homogeneity). *Let $\alpha_j=\langle\mu_j,\eta_j,\nu_j\rangle\in\mathscr{P}$ $(j=1,2,\ldots,n)$ and $\lambda>0$. Then,*

$$\mathrm{PFAAIWG}((\alpha_1)^\lambda,(\alpha_2)^\lambda,\ldots,(\alpha_n)^\lambda)=(\mathrm{PFAAIWG}(\alpha_1,\alpha_2,\ldots,\alpha_n))^\lambda.$$

5. A PF MADM method and a numerical example

5.1. **A novel PF MCDM method.** Consider an MADM problem under the PF environment, let $A=\{A_1,A_2,\ldots,A_m\}$ represent a set of alternatives, and let $G=\{G_1,G_2,\ldots,G_n\}$ represent a set of evaluation attributes. The weight vector for the attributes is denoted as $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^\top$, where $\omega_j\in(0,1]$ and $\sum_{j=1}^n\omega_j=1$. Assume that the rating of an alternative A_i $(i=1,\ldots,m)$ based on the attribute G_j $(j=1,\ldots,n)$ is assessed by the decision-maker, forming PFNs $\alpha_{ij}=\langle\mu_{ij},\eta_{ij},\nu_{ij}\rangle$, where μ_{ij} and η_{ij} are the degrees of the positive and the neutral memberships, respectively, and ν_{ij} is the degree that the alternative A_i does not satisfy the attribute G_j. To rank alternatives, the following steps are taken:

Step 1: (Construct the decision matrix) The decision-maker provides their preferences in the form of PFNs $\alpha_{ij}=\langle\mu_{ij},\eta_{ij},\nu_{ij}\rangle$ for each alternative A_i under each attribute G_j. This forms the PF decision matrix $D=(\alpha_{ij})_{m\times n}$, structured as follows:

Table 1. The PF decision matrix D.

	G_1	G_2	$\cdots$	G_n
A_1	$\langle\mu_{11},\eta_{11},\nu_{11}\rangle$	$\langle\mu_{12},\eta_{12},\nu_{12}\rangle$	$\cdots$	$\langle\mu_{1n},\eta_{1n},\nu_{1n}\rangle$
A_2	$\langle\mu_{21},\eta_{21},\nu_{21}\rangle$	$\langle\mu_{22},\eta_{22},\nu_{22}\rangle$	$\cdots$	$\langle\mu_{2n},\eta_{2n},\nu_{2n}\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_m	$\langle\mu_{m1},\eta_{m1},\nu_{m1}\rangle$	$\langle\mu_{m2},\eta_{m2},\nu_{m2}\rangle$	$\cdots$	$\langle\mu_{mn},\eta_{mn},\nu_{mn}\rangle$

Step 2: (Normalize the decision matrix) Transform the matrix $D=(\alpha_{ij})_{m\times n}$ into a normalized matrix $R=(r_{ij})_{m\times n}$ as follows:

$$r_{ij}=\begin{cases}\alpha_{ij}, & \text{for benefit attribute } G_j,\\ \alpha_{ij}^{\complement}, & \text{for cost attribute } G_j,\end{cases}$$

where $\alpha_{ij}^{\complement}$ is the complement of α_{ij}.

Step 3: (Compute the aggregated value) Using the normalized decision matrix R from Step 2, calculate the aggregated value for each alternative A_i $(i = 1, \ldots, m)$ under the different attributes G_1, G_2, …, G_n using the PFAAIWA or PFAAIWG operators. The collective aggregated value r_i is obtained for each alternative A_i.

Step 4: (Rank the alternatives) Rank the alternatives $A_1, A_2, \ldots, A_m$ using the total order defined in Definition 4. Select the most desirable alternative based on the ranking.

5.2. **An illustrative example.** A financing organization is planning to choose the best organization to put away the cash from the following five potential choices (adapted from [18, 40]): A_1 is an automobile organization; A_2 is a nourishment organization; A_3 is a laptop organization; A_4 is a weapon organization; A_5 is a television organization. The financing organization consider the following four attributes to judge the five potential choices: (1) the "hazard investigation" denoted by G_1; (2) the "growth investigation" denoted by G_2; (3) the "social-political effect investigation" denoted by G_3; (4) the "environmental effect investigation" denoted by G_4, where the weight vector is $\omega = (0.2, 0.1, 0.3, 0.4)^{\top}$, which was given by the decision-maker.

Step 1: Using normalization, the normalized decision matrix $R = (r_{ij})_{6\times 4}$, transformed from the expert's decision matrix, is shown in Table 2.

TABLE 2. The normalized PF decision matrix R.

	G_1	G_2	G_3	G_4
A_1	$\langle 0.72, 0.18, 0.10\rangle$	$\langle 0.08, 0.82, 0.07\rangle$	$\langle 0.40, 0.37, 0.19\rangle$	$\langle 0.52, 0.32, 0.11\rangle$
A_2	$\langle 0.15, 0.15, 0.65\rangle$	$\langle 0.67, 0.21, 0.09\rangle$	$\langle 0.12, 0.72, 0.15\rangle$	$\langle 0.84, 0.09, 0.06\rangle$
A_3	$\langle 0.44, 0.46, 0.10\rangle$	$\langle 0.35, 0.35, 0.20\rangle$	$\langle 0.25, 0.40, 0.35\rangle$	$\langle 0.30, 0.20, 0.50\rangle$
A_4	$\langle 0.70, 0.10, 0.20\rangle$	$\langle 0.74, 0.16, 0.10\rangle$	$\langle 0.67, 0.25, 0.07\rangle$	$\langle 0.12, 0.75, 0.12\rangle$
A_5	$\langle 0.43, 0.34, 0.23\rangle$	$\langle 0.78, 0.13, 0.08\rangle$	$\langle 0.09, 0.85, 0.05\rangle$	$\langle 0.91, 0.02, 0.07\rangle$

Step 2: Using the PFAAIWA operator and formula (8), the overall aggregated values for each alternative A_i are calculated as follows (with $\gamma = 1$): $r_1 = \langle 0.5082, 0.3368, 0.1215\rangle$, $r_2 = \langle 0.5994, 0.2506, 0.1325\rangle$, $r_3 = \langle 0.3216, 0.3690, 0.2971\rangle$, $r_4 = \langle 0.5320, 0.3487, 0.1110\rangle$, and $r_5 = \langle 0.7150, 0.1980, 0.0814\rangle$.

Step 3: By direct calculation, the score values of the aggregated ratings r_1, r_2, …, r_5 are as follows: $S(r_1) = 0.3867$, $S(r_2) = 0.4669$, $S(r_3) = 0.0245$, $S(r_4) = 0.4210$, and $S(r_5) = 0.6337$. Thus, the alternatives are ranked as follows: $r_5 \succ_{\mathrm{W}} r_2 \succ_{\mathrm{W}} r_4 \succ_{\mathrm{W}} r_1 \succ_{\mathrm{W}} r_3$ according to Definition 4. Therefore, the best choice is the television organization, i.e., A_5.

To study the changing tendency of the scores and the rankings for A_1, A_2, A_3, A_4, and A_5 with the variation of the parameter γ, we use the following Fig.2 for illustrating this subject.

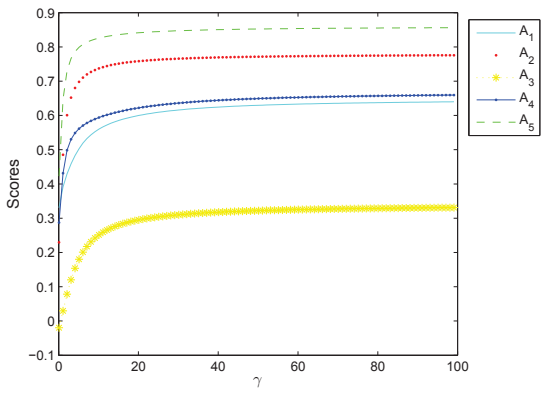


FIGURE 2. Scores for alternatives in different values of γ obtained by PFAAIWA.

Fig.2 draws the scores for the alternatives derived by the operator PFAAIWA, as the parameter γ changes from 0 to 100. Then, we observe that (1) the score functions of all alternatives increase monotonously; (2) A_5 is the best choice.

In summary, the parameter γ can be interpreted as a reflection of the decision-makers' preferences. As γ changes within a certain range, the scores of the alternatives vary accordingly. However, the overall ranking order remains consistent as $A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$. This ranking is consistent with the results obtained in [24, 40, 50], as shown in Table 3.

Table 3. Comparative analysis with some exist methods.

Methods	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Ranking
Senapati [40]	0.7762	0.8756	0.6735	0.8181	0.9325	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Wei [50]	0.6865	0.7548	0.5723	0.7040	0.8343	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Khan et al. [24]	0.7762	0.8756	0.6735	0.8181	0.9325	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 1$)	0.3867	0.4669	0.0245	0.4210	0.6337	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 2$)	0.4259	0.5931	0.0736	0.4939	0.7247	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 3$)	0.4559	0.6482	0.1163	0.5279	0.7636	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 4$)	0.4807	0.6778	0.1508	0.5473	0.7847	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 5$)	0.5010	0.6965	0.1778	0.5602	0.7978	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 6$)	0.5177	0.7094	0.1991	0.5698	0.8068	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 7$)	0.5312	0.7189	0.2161	0.5773	0.8134	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 8$)	0.5424	0.7262	0.2298	0.5835	0.8184	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 9$)	0.5516	0.7320	0.2411	0.5889	0.8224	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 10$)	0.5594	0.7367	0.2504	0.5935	0.8256	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 20$)	0.5997	0.7584	0.2949	0.6219	0.8412	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 40$)	0.6244	0.7693	0.3175	0.6441	0.8504	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 60$)	0.6330	0.7729	0.3251	0.6527	0.8536	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 80$)	0.6373	0.7747	0.3288	0.6570	0.8552	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$
Our method ($\gamma = 100$)	0.6399	0.7757	0.3310	0.6596	0.8562	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$

Therefore, we have sufficient reasons to conclude that this financing organization's best choice is A_5–"**television organization**".

6. Conclusion

Being an important family of t-norms, Aczél-Alsina t-norms have been widely applied for MADM under various fuzzy environments [19, 40–43]. However, the PF Aczél-Alsina aggregation operators introduced in [30, 35, 40] are not closed on the set of PFNs. This motivates us to investigate closed operational laws in PFNs and their applications. Based on Aczel-Alsina t-norms and t-conorms, we first derive new closed PF interactional operations, including addition, product, scalar multiplication, and power. Based on these four operations, we propose the PFAAIWA and PFAAIWG operators, which are shown to satisfy monotonicity, idempotent, bounded, shift-invariant, and homogeneous. In particular, these operators serve as aggregation operators for PFNs. For applications, we develop a novel MADM method to solve MADM problems involving PFNs and present an investment selection example for a financing organization to illustrate our method. The proposed approach can be broadly applied to other MADM problems within PFSs and related three-dimensional fuzzy environments, such as IvPFSs, SFSs, and complex SFSs, in the future.

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