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TWO-LEVEL ARROW-HURWICZ ALGORITHMS FOR THE THERMALLY COUPLED INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

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Abstract. In this paper, for the stationary thermally coupled incompressible magnetohydrodynamics system, we design several two-level Arrow-Hurwicz algorithms, which combine the Arrow-Hurwicz iteration and a two-level strategy. We first apply the Arrow-Hurwicz iteration to obtain a solution on a coarse mesh, and then based on the obtained solution, we use three different iterative schemes (Oseen-type, Stokes-type, and Newton-type) to obtain the desired numerical solution on fine mesh. With the refinement of the finite element mesh, the computational time can be reduced by using these two-level Arrow-Hurwicz algorithms, compared with the classical Arrow-Hurwicz iterative algorithm. Finally, numerical experiments demonstrate the effectiveness and efficiency of the proposed algorithms.

Keywords: Two-Level Strategy, Arrow-Hurwicz Iteration, Thermally Coupled Magnetohydrodynamics, Error Analysis, Iterative Scheme, Stokes-Type Equations.

AMS Subject Classification: 65N12, 65N30.

1. Introduction

This paper is concerned with the following steady-state thermally coupled incompressible magnetohydrodynamics equations [10,17,22,23,26,28], i.e., the stationary incompressible Navier-Stokes equations and Maxwell's equations coupled to the heat equation by the Boussinesq approximation. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a convex polygonal/polyhedral domain. Following [22,23], we seek the velocity $\mathbf{u}$, the magnetic field $\mathbf{H}$, the fluid pressure p and the temperature T such that

$$\begin{aligned} -Re^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + s\mathbf{H}\times \mathrm{curl}\mathbf{H} &= \mathbf{f} + \beta T\mathbf{j}, \quad \text{in } \Omega, \\ sRm^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{H}) - s\mathrm{curl}(\mathbf{u}\times\mathbf{H}) &= \mathbf{g}, \quad \text{in } \Omega, \\ -\kappa\Delta T + \mathbf{u}\cdot\nabla T &= \gamma, \quad \text{in } \Omega, \\ \mathrm{div}\mathbf{u} = 0,\ \mathrm{div}\mathbf{H} &= 0, \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where Re, Rm, β, κ and s denote the hydrodynamic Reynolds number, the magnetic Reynolds number, the thermal expansion coefficient, the thermal conductivity and the coupling number, respectively. In addition, $\mathbf{g}$ represents the known applied current with $\mathrm{div}\mathbf{g} = 0$, $\mathbf{f}$ is an external force term, γ is a given heat source, and $\mathbf{j}$ denotes a unit vector in the direction opposite to the gravity. Furthermore, as in [5, 7, 22], in this paper the system (1) is considered in conjunction with the following boundary conditions:

$$\begin{aligned} &\mathbf{u}|_{\partial\Omega} = 0, \quad \text{(no-slip condition)}, \\ &\mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ and } \mathbf{n}\times\mathrm{curl}\mathbf{H}|_{\partial\Omega} = 0, \quad \text{(perfectly conducting wall)}, \\ &T|_{\Gamma_D} = 0 \text{ and } \nabla T\cdot\mathbf{n}|_{\Gamma_N} = 0, \quad \text{(insulated wall)}, \end{aligned} \tag{2}$$

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where $\mathbf{n}$ the outer unit normal vector to $\partial\Omega$. Let us decompose the boundary $\partial\Omega$ into two disjoint parts, such that Γ_D and Γ_N, and suppose that the temperature is given on Γ_D while the heat flux is prescribed on Γ_N. Assume that $\Gamma_D = \partial\Omega \setminus \Gamma_N$ is a regular open subset of $\partial\Omega$.

The thermally coupled incompressible magnetohydrodynamics model can describe the dynamic behavior of an electrically conducting fluid under the influence of a magnetic field, where the buoyancy effect cannot be neglected in the momentum equation due to temperature variations in the conducting fluid. It is used to study the changing relationship between the physical quantities of the moving conductive fluid and the magnetic field. The problem of steady-state thermally coupled incompressible magnetohydrodynamics model has attracted considerable attention in recent years. It is mainly used in magnetic propulsion devices, magnetic fluid engines, liquid metal magnetic fluid power generation, nuclear reactor technology, electromagnetic pump design, continuous metal casting, crystal and semiconductor manufacturing, etc. With the rapid development of computer technology, the numerical simulation of the steady-state thermally coupled incompressible magnetohydrodynamics model has advanced significantly developed, theoretical analysis and numerical simulation complement each other. In fact, the numerical method for the steady-state thermally coupled incompressible magnetohydrodynamics model has long been an active research topic in the field of computational mathematics and applied mathematics. As an important subject in current science and engineering applications, it continues to attract increasing research interest.

In order to numerically solve and simulate the steady-state thermally coupled incompressible magnetohydrodynamics model, many computational techniques have been studied. In [22, 23], the author has proved the existence and uniqueness to solutions of the steady-state thermally coupled incompressible magnetohydrodynamics model, and investigated its finite element approximation. Further, the authors of the paper [24] have proposed a general approach to the steady-state, electromagnetically and thermally driven liquid-metal flows. In [4], the authors have established existence results of weak solutions for two steady-state magnetohydrodynamics systems of equations including Joule heating. In [3], the authors have extended block preconditioning techniques to the thermally coupled incompressible inductionless magnetohydrodynamics problem. In addition, a stabilized finite element method has been designed to solve thermally coupled magnetohydrodynamics flows in [6]. In order to avoid breakdown of solver and increase of computing time with increasing the stabilization parameter, a grad-div stabilization with the Jacobi iteration has been constructed in [21]. Recently, in [29], the authors have proposed three iterative methods with a finite element discretization for the steady-state thermally coupled incompressible magnetohydrodynamics equations. In addition, the choice of the different iterative methods is given for solving the considered equations with different physical parameters.

However, many existing numerical methods for solving the steady-state thermally coupled incompressible magnetohydrodynamics equations need to solve saddle point systems at each iteration due to the incompressibility constraints. Although the previous works obtain many efficient numerical results, relatively little attention is thrown to the development of efficient numerical methods to deal with the considered problem concerning saddle-point structure. It is well known that the Arrow-Hurwicz algorithm is a very efficient method for the saddle-point system. Therefore, the Arrow-Hurwicz iterative finite element method is proposed for the steady-state thermally coupled incompressible magnetohydrodynamics problem [18]. Although the convergence rate of the Arrow-Hurwicz algorithm is independent of the mesh size, with the refinement of the finite element mesh, the computational time increases rapidly if the parameter is not chosen appropriately. Therefore, it is necessary to update the proposed algorithm in [18].

As is known, the two-level strategy is used to improve the efficiency of numerical method. The basic idea of this strategy involves solving one small nonlinear problem on coarse mesh and one linear system on fine mesh. In [20], the authors have considered a two-level method with backtracking for resolving the nonlinearity in finite element approximation of the equilibrium Navier-Stokes equations. In [14], the authors have reported a simplified two-level finite element

method for the steady Navier-Stokes problem. Two-level defect-correction Oseen iterative stabilized finite element methods for the steady-state Navier-Stokes equations was given in [16]. Based on the two-level method, the authors have analyzed some iterative finite element methods for steady Navier-Stokes equations with different viscosities in [27]. Additionally, the authors have proposed two-grid Arrow-Hurwicz methods for the steady incompressible Navier-Stokes equations in [9]. Some one- and two-level Arrow-Hurwicz-type iterative algorithms have been considered for the stationary Smagorinsky model [19].

In this paper, we derive some two-level Arrow-Hurwicz algorithms for the steady-state thermally coupled incompressible magnetohydrodynamics flow. We first use the Arrow-Hurwicz algorithm to obtain a coarse-mesh solution, and then, based on this solution, employ three iterative schemes (Oseen-type, Stokes-type and Newton-type) to obtain the desired numerical solution on fine mesh. The remainder of this paper is organized as follows. In Section 2, some notation, function spaces and significant results of the thermally coupled incompressible magnetohydrodynamics problem are introduced. In Section 3, we show the finite element method and the Arrow-Hurwicz algorithm based on the finite element discretization for the considered problem. In Section 4, we design two-level Arrow-Hurwicz algorithms, and establish stability and error estimates for the presented algorithms. In Section 5, some numerical experiments are performed to validate the accuracy and the efficiency of the numerical algorithms.

2. Preliminaries

For $1 \leq q \leq \infty$ and $m \in \mathbb{N}^+$, we define the usual Sobolev space $W^{m,q}(\Omega)$ norm and Lebegue space $L^q(\Omega)$ norm by $\|\cdot\|_{W^{m,q}}$ and $\|\cdot\|_{L^q}$, respectively. Particularly, $L^2(\Omega)$ norm and its inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$. In addition, we write $H^m(\Omega)$ for $W^{m,2}(\Omega)$.

Next, for mathematical setting of the system (1)-(2), we introduce the following necessary function spaces:

$$\begin{aligned}
&\boldsymbol{X} = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\partial\Omega} = \mathbf{0}\}, \quad \boldsymbol{W} = \{\mathbf{B} \in H^1(\Omega)^d : \mathbf{B}\cdot\mathbf{n}|_{\partial\Omega} = 0\},\\
&Q = \{S \in H^1(\Omega) : S|_{\Gamma_D} = 0\}, \quad M = \{q \in L^2(\Omega) : (q,1) = 0\},\\
&\boldsymbol{V} = \{\mathbf{v} \in \boldsymbol{X}, \mathrm{div}\mathbf{v} = 0\}.
\end{aligned}$$

In addition, we introduce the product space $\boldsymbol{D} = \boldsymbol{X} \times \boldsymbol{W}$, which is equipped with the norms for all $(\mathbf{w}, \boldsymbol{\Phi}) \in \boldsymbol{D}$, $\|(\mathbf{w}, \boldsymbol{\Phi})\|_{\boldsymbol{D}}^2 = \|\mathbf{w}\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\Phi}\|_{\boldsymbol{W}}^2$ with $\|\mathbf{w}\|_{\boldsymbol{X}}^2 = \|\nabla\mathbf{w}\|_0^2$ and $\|\boldsymbol{\Phi}\|_{\boldsymbol{W}}^2 = \|\mathrm{curl}\boldsymbol{\Phi}\|_0^2 + \|\mathrm{div}\boldsymbol{\Phi}\|_0^2$, and $\|(\mathbf{w}, \boldsymbol{\Phi})\|_0^2 = \|\mathbf{w}\|_0^2 + \|\boldsymbol{\Phi}\|_0^2$.

Moreover, we define three continuous bilinear forms $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ on $Q \times Q$, $\boldsymbol{X} \times \boldsymbol{X}$ and $\boldsymbol{W} \times \boldsymbol{W}$, respectively, by

$$\begin{aligned}
a_0(T, S) =& \kappa(\nabla T, \nabla S), \qquad a_1(\mathbf{u}, \mathbf{v}) = Re^{-1}(\nabla\mathbf{u}, \nabla\mathbf{v}),\\
a_2(\mathbf{H}, \mathbf{B}) =& sRm^{-1}\big((\mathrm{curl}\mathbf{H}, \mathrm{curl}\mathbf{B}) + (\mathrm{div}\mathbf{H}, \mathrm{div}\mathbf{B})\big),
\end{aligned}$$

and three trilinear forms $b_0(\cdot,\cdot,\cdot)$, $b_1(\cdot,\cdot,\cdot)$ and $b_2(\cdot,\cdot,\cdot)$ on $\boldsymbol{X}\times Q\times Q$, $\boldsymbol{X}\times\boldsymbol{X}\times\boldsymbol{X}$ and $\boldsymbol{W}\times\boldsymbol{W}\times\boldsymbol{X}$, by

$$\begin{aligned}
b_0(\mathbf{u}, T, S) =& (\mathbf{u}\cdot\nabla T, S) + 0.5((\mathrm{div}\mathbf{u})T, S) = 0.5(\mathbf{u}\cdot\nabla T, S) - 0.5(\mathbf{u}\cdot\nabla S, T),\\
b_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) =& ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}) + 0.5((\mathrm{div}\mathbf{u})\mathbf{w}, \mathbf{v}) = 0.5((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}) - 0.5((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}),\\
b_2(\mathbf{H}, \mathbf{B}, \mathbf{v}) =& s(\mathbf{H}\times\mathrm{curl}\mathbf{B}, \mathbf{v}).
\end{aligned}$$

Next, we recall the following classical inequalities [1, 13, 30]:

$$\begin{aligned}
&\|v\|_{L^\infty} \leq c_0\|\Delta v\|_0, \quad \forall v \in H^2(\Omega), \quad \|\mathbf{v}\|_{L^4} \leq c_1\|\nabla\mathbf{v}\|_0, \quad \forall\mathbf{v} \in \boldsymbol{X},\\
&\|\mathbf{v}\|_{L^3} \leq \hat{c}_0\|\nabla\mathbf{v}\|_0, \quad \forall\mathbf{v} \in \boldsymbol{X}, \quad \|\mathrm{curl}\mathbf{v}\|_0 \leq \sqrt{2}\|\nabla\mathbf{v}\|_0, \quad \forall\mathbf{v} \in \boldsymbol{W},
\end{aligned} \tag{3}$$

where $c_0, \hat{c}_0, c_1 > 0$ are constants depending on Ω. In addition, we need the Poincaré inequality [1]

$$\|v\|_0 \leq C_F \|\nabla v\|_0, \ \forall v \in H_0^1(\Omega),$$

where C_F is the Poincaré constant depending on Ω.

The trilinear form satisfies the following property [15, 31]:

$$|b_0(\mathbf{u}, T, S)| \leq N_0 \|\nabla \mathbf{u}\|_0 \|\nabla T\|_0 \|\nabla S\|_0, \quad \forall \mathbf{u} \in \boldsymbol{X}, \quad \forall T, S \in Q, \tag{4}$$

where $N_0 > 0$ is a constant depending on Ω.

Then, the variational formulation for the considered problem (1)-(2) for all $((\mathbf{v}, \mathbf{B}), S, q) \in \boldsymbol{D} \times Q \times M$, $((\mathbf{u}, \mathbf{H}), T, p) \in \boldsymbol{D} \times Q \times M$ can be written such that:

$$\begin{cases} a_0(T, S) + b_0(\mathbf{u}, T, S) = (\gamma, S), \\ A_0((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{u}, \mathbf{H}), (\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), p) + d((\mathbf{u}, \mathbf{H}), q) \\ \quad = (\mathbf{F}, (\mathbf{v}, \mathbf{B})) + G(T, (\mathbf{v}, \mathbf{B})), \end{cases} \tag{5}$$

where $A_1((\mathbf{u}, \mathbf{H}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{v}, \mathbf{B})) = b_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b_2(\mathbf{H}, \boldsymbol{\Phi}, \mathbf{v}) - b_2(\mathbf{H}, \mathbf{B}, \mathbf{w})$, $A_0((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})) = a_1(\mathbf{u}, \mathbf{v}) + a_2(\mathbf{H}, \mathbf{B})$, $(\mathbf{F}, (\mathbf{v}, \mathbf{B})) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{B})$, $G(T, (\mathbf{v}, \mathbf{B})) = \beta(T\mathbf{j}, \mathbf{v})$ and $d((\mathbf{v}, \mathbf{B}), p) = (\mathrm{div}\mathbf{v}, p)$.

Note that in this paper we consider the case that the magnetic field belongs to H^1. The problem for the magnetic field includes well regularity, i.e., solutions that belong to H^1, rather than only to $H(\mathrm{curl})$.

Now, we recall the coercive and continuous properties of $A_0((\cdot,\cdot),(\cdot,\cdot))$ and the continuous property of $A_1((\cdot,\cdot),(\cdot,\cdot),(\cdot,\cdot))$.

Lemma 1. *[13, 18] For all* $(\mathbf{u}, \mathbf{H}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{v}, \mathbf{B}) \in \boldsymbol{D}$, *there hold*

$$\begin{aligned} A_0((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})) &\leq c_A \|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \|(\mathbf{v}, \mathbf{B})\|_{\boldsymbol{D}}, \\ A_0((\mathbf{u}, \mathbf{H}), (\mathbf{u}, \mathbf{H})) &\geq \nu_A \|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}}^2, \\ A_1((\mathbf{u}, \mathbf{H}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{v}, \mathbf{B})) &\leq N \|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \|(\mathbf{w}, \boldsymbol{\Phi})\|_{\boldsymbol{D}} \|(\mathbf{v}, \mathbf{B})\|_{\boldsymbol{D}}, \end{aligned}$$

where $c_A = \max\{Re^{-1}, sRm^{-1}\}$, $\nu_A = \min\{Re^{-1}, sRm^{-1}\}$ *and* $N = c_1^2 \max\{1, s\}$.

In addition, we show the following existence and uniqueness results for the problem (1)-(2) [18, 22, 29].

Theorem 1. *Let* $\gamma \in Q'$, $\mathbf{F} \in \mathbf{D}'$ *and* ν_A *satisfy the following uniqueness condition:*

$$0 < \delta = \delta_1 + \delta_2 := \nu_A^{-2} N(\|\mathbf{F}\|_{-1} + C_F^2 \kappa^{-1} \beta \|\gamma\|_{-1}) + \nu_A^{-1} C_F^2 \kappa^{-2} \beta N_0 \|\gamma\|_{-1} < 1,$$

where $\|\mathbf{F}\|_{-1} = \sup\limits_{(\mathbf{u},\mathbf{H}) \in \mathbf{D}, (\mathbf{u},\mathbf{H}) \neq (\mathbf{0},\mathbf{0})} \frac{(\mathbf{F}, (\mathbf{u}, \mathbf{H}))}{\|\nabla (\mathbf{u}, \mathbf{H})\|_0}$, $\|\gamma\|_{-1} = \sup\limits_{T \in Q, T \neq 0} \frac{(\gamma, T)}{\|\nabla T\|_0}$. *Then, the problem* (1)-(2) *admits a unique solution* $((\mathbf{u}, \mathbf{H}), p, T) \in \mathbf{D} \times M \times Q$ *such that*

$$\nu_A \|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \leq \|\mathbf{F}\|_{-1} + C_F^2 \beta \kappa^{-1} \|\gamma\|_{-1}, \qquad \kappa \|\nabla T\|_0 \leq \|\gamma\|_{-1}.$$

3. Arrow-Hurwicz algorithm based on finite element discretization

From now on, let μ be a real positive parameter and $\mu = h$ or H. Here, $1 > H \gg h$. The conforming finite element subspaces $(\boldsymbol{X}_\mu, \boldsymbol{W}_\mu, Q_\mu, M_\mu)$ of $(\boldsymbol{X}, \boldsymbol{W}, Q, M)$ are characterized by $K_\mu = K_\mu(\Omega)$, a partitioning of $\overline{\Omega}$ into triangles or tetrahedra K, assumed to be uniformly regular as $\mu \to 0$. Next, we define the product space $\boldsymbol{D}_\mu = \boldsymbol{X}_\mu \times \boldsymbol{W}_\mu$.

In this paper, we use the pair of the MINI element [2] to approximate the velocity and pressure, piecewise linear element for the magnetic field and temperature. From [2], the couple

$\boldsymbol{D}_\mu \times M_\mu$ admits the following discrete inf-sup condition [23]: for each $q_\mu \in M_\mu$, there exists $(\mathbf{v}_\mu, \mathbf{B}_\mu) \in \boldsymbol{D}_\mu$, such that

$$\sup_{(\mathbf{v}_\mu,\mathbf{B}_\mu)\in\boldsymbol{D}_\mu,(\mathbf{v}_\mu,\mathbf{B}_\mu)\neq(\mathbf{0},\mathbf{0})} \frac{|d((\mathbf{v}_\mu,\mathbf{B}_\mu),q_\mu)|}{\|(\mathbf{v}_\mu,\mathbf{B}_\mu)\|_{\boldsymbol{D}}} \geq \widetilde{\beta}\|q_\mu\|_0,$$

where $\widetilde{\beta} > 0$ is a constant depending on Ω.

Next, the above finite element spaces satisfy the following standard properties [8,11–13,25,31].

Lemma 2. *There exist a mapping $r_\mu : H^2(\Omega)^d \cap \boldsymbol{X} \to \boldsymbol{X}_\mu$ satisfying*

$$(\nabla\cdot(\mathbf{v}-r_\mu\mathbf{v}),q)=0,\quad \|\nabla(\mathbf{v}-r_\mu\mathbf{v})\|_0\leq c_2\mu\|\Delta\mathbf{v}\|_0,\quad \forall\mathbf{v}\in H^2(\Omega)^d\cap\boldsymbol{X},q\in M_\mu,$$

and a mapping $R_\mu : H^2(\Omega)^d \cap \boldsymbol{W} \to \boldsymbol{W}_\mu$ satisfying

$$\begin{aligned}&(\nabla\times R_\mu\boldsymbol{\Phi},\nabla\times\mathbf{B})+(\nabla\cdot R_\mu\boldsymbol{\Phi},\nabla\cdot\mathbf{B})=(\nabla\times\boldsymbol{\Phi},\nabla\times\mathbf{B})+(\nabla\cdot\boldsymbol{\Phi},\nabla\cdot\mathbf{B})\\&\quad=(\nabla\times\boldsymbol{\Phi},\nabla\times\mathbf{B}),\quad\forall\mathbf{B}\in\boldsymbol{W}_\mu,\\&\|\boldsymbol{\Phi}-R_\mu\boldsymbol{\Phi}\|_0+\mu\|\nabla(\boldsymbol{\Phi}-R_\mu\boldsymbol{\Phi})\|_0\leq c_2\mu^2\|\Delta\boldsymbol{\Phi}\|_0,\quad\forall\boldsymbol{\Phi}\in H^2(\Omega)^d\cap\boldsymbol{W},\end{aligned}$$

and a mapping $\hbar_\mu : H^2(\Omega) \cap Q \to Q_\mu$ satisfying

$$(\nabla(T-\hbar_\mu T),\nabla s)=0,\quad \|\nabla(T-\hbar_\mu T)\|_0\leq c_2\mu\|\Delta T\|_0,\quad\forall T\in H^2(\Omega)\cap Q,s\in Q_\mu,$$

and an L^2-orthogonal projection operator $\aleph_\mu : M \to M_\mu$ satisfying

$$\|q-\aleph_\mu q\|_0\leq c_2\mu\|\nabla q\|_0,\quad\forall q\in H^1(\Omega)\cap M.$$

In fact, from the approximation properties of the finite element spaces $\boldsymbol{X}_\mu$, Q_μ and the regularity of $\mathbf{v} \in H^2(\Omega)^d$, $T \in H^2(\Omega)$, one has error estimates concerning mappings r_μ, $\hbar_\mu$. Moreover, by standard interpolation estimate for L^2-projection, we get the error estimate on $\aleph_\mu$. Finally, by applying the approximation property of the finite element space $\boldsymbol{W}_\mu$ and the regularity of $\boldsymbol{\Phi}$, we can deduce the desired estimate.

Now, according to the above definition of the finite element subspaces, the finite element approximation for (5) is to seek $((\mathbf{u}_\mu, \mathbf{H}_\mu), T_\mu, p_\mu) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ such that for all $((\mathbf{v}, \mathbf{B}), S, q) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$

$$\begin{cases} a_0(T_\mu,S)+b_0(\mathbf{u}_\mu,T_\mu,S)=(\gamma,S),\\ A_0((\mathbf{u}_\mu,\mathbf{H}_\mu),(\mathbf{v},\mathbf{B}))+A_1((\mathbf{u}_\mu,\mathbf{H}_\mu),(\mathbf{u}_\mu,\mathbf{H}_\mu),(\mathbf{v},\mathbf{B}))-d((\mathbf{v},\mathbf{B}),p_\mu)+d((\mathbf{u}_\mu,\mathbf{H}_\mu),q)\\ \quad=(\mathbf{F},(\mathbf{v},\mathbf{B}))+G(T_\mu,(\mathbf{v},\mathbf{B})).\end{cases}\tag{6}$$

The following results can be found in [23, 29], which describe the stability of the numerical solutions obtained by (6).

Next, we introduce the discrete analogue of the space $\boldsymbol{V}$ as follows:

$$\boldsymbol{V}_\mu=\{\mathbf{v}\in\boldsymbol{X}_\mu,\quad(\mathrm{div}\mathbf{v},q)=0,\quad\forall q\in M_\mu\}.$$

Denote $P_\mu : L^2(\Omega) \to \boldsymbol{V}_\mu$ and $R_\mu : L^2(\Omega) \to \boldsymbol{W}_\mu$ by L^2-orthogonal projectors.

Here, we introduce discrete Stokes operator $A_{1\mu} = -P_\mu\Delta_\mu$ (see [8,31])

$$-(\Delta_\mu\phi_\mu,\psi_\mu)=(\nabla\phi_\mu,\nabla\psi_\mu),\quad\forall\varphi_\mu,\psi_\mu\in\boldsymbol{X}_\mu\quad\text{or}\quad Q_\mu.$$

Meanwhile, we introduce another operator $A_{2\mu}\mathbf{H}_\mu = R_\mu(\nabla_\mu\times\nabla\times\mathbf{H}_\mu+\nabla_\mu\nabla\cdot\mathbf{H}_\mu) \in \boldsymbol{W}_\mu$ (see [8]) such that

$$(A_{2\mu}\mathbf{H}_\mu,\mathbf{B})=(A_{2\mu}^{\frac{1}{2}}\mathbf{H}_\mu,A_{2\mu}^{\frac{1}{2}}\mathbf{B})=(\nabla\times\mathbf{H}_\mu,\nabla\times\mathbf{B})+(\nabla\cdot\mathbf{H}_\mu,\nabla\cdot\mathbf{B}),\quad\mathbf{H}_\mu,\mathbf{B}\in\boldsymbol{W}_\mu.$$

Now, we turn to discrete estimates

$$\|\nabla\mathbf{v}_\mu\|_{L^6}\leq\hat{c}_1\|A_{1\mu}\mathbf{v}_\mu\|_0,\quad\|\nabla\mathbf{v}_\mu\|_{L^3}+\|\mathbf{v}_\mu\|_{L^\infty}\leq\hat{c}_2\|\nabla\mathbf{v}_\mu\|_0^{\frac{1}{2}}\|A_{1\mu}\mathbf{v}_\mu\|_0^{\frac{1}{2}},\quad\forall\mathbf{v}_\mu\in\boldsymbol{X}_\mu,$$

where $\hat{c}_1, \hat{c}_2 > 0$ are constants depending on Ω (see [8]).

Lemma 3. *[8, 31] For all* $(\mathbf{u}_\mu, \mathbf{H}_\mu), (\mathbf{v}, \mathbf{B}) \in \boldsymbol{D}_\mu, T_\mu, S \in Q_\mu$, *the trilinear form* $b_0(\cdot,\cdot,\cdot)$ *and* $A_1(\cdot,\cdot,\cdot)$ *satisfy the following estimations:*

$$\begin{aligned}
|b_0(\mathbf{u}_\mu, T_\mu, S)| &\le \hat{c}_0\hat{c}_1\|\mathbf{u}_\mu\|_0\|A_{1\mu}T_\mu\|_0\|\nabla S\|_0,\\
|b_0(\mathbf{u}_\mu, T_\mu, S)| &\le \hat{c}_2\|A_{1\mu}\mathbf{u}_\mu\|_0\|T_\mu\|_0\|\nabla S\|_0,\\
A_1((\mathbf{u}_\mu, \mathbf{H}_\mu), (\mathbf{w}_\mu, \boldsymbol{\Phi}_\mu), (\mathbf{v}, \mathbf{B})) &\le \hat{N}_0\|(\mathbf{u}_\mu, \mathbf{H}_\mu)\|_0\|(A_{1\mu}\mathbf{w}_\mu, A_{2\mu}\boldsymbol{\Phi}_\mu)\|_0\|\nabla(\mathbf{v}, \mathbf{B})\|_0,\\
A_1((\mathbf{u}_\mu, \mathbf{H}_\mu), (\mathbf{w}_\mu, \boldsymbol{\Phi}_\mu), (\mathbf{v}, \mathbf{B})) &\le \bar{N}_0\|(A_{1\mu}\mathbf{u}_\mu, A_{2\mu}\mathbf{H}_\mu)\|_0\|(\mathbf{w}_\mu, \boldsymbol{\Phi}_\mu)\|_0\|\nabla(\mathbf{v}, \mathbf{B})\|_0,
\end{aligned}$$

where $\hat{N}_0 = \hat{c}_0\hat{c}_1\max\{1, \sqrt{2}s\}, \bar{N}_0 = \hat{c}_2\max\{1, \sqrt{2}s\}$.

Theorem 2. *Let* $((\mathbf{u}_\mu, \mathbf{H}_\mu), T_\mu, p_\mu) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ *be a solution of the finite element discretization (6). Then, under the assumptions of Theorem 1, there hold*

$$\nu_A\|(\mathbf{u}_\mu, \mathbf{H}_\mu)\|_{\boldsymbol{D}} \le \|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}, \qquad \kappa\|\nabla T_\mu\|_0 \le \|\gamma\|_{-1}.$$

Lemma 4. *[8, 31] Let* $((\mathbf{u}_\mu, \mathbf{H}_\mu), T_\mu, p_\mu) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ *be a solution of the finite element discretization (6). If we assume*

$$\hat{N}_1 = \kappa - \hat{c}_0\hat{c}_1\nu_A N^{-1}\delta_1 > 0, \quad \hat{N}_2 = \nu_A - \hat{N}_0\nu_A N^{-1}\delta_1 > 0,$$

then, under assumptions of Theorem 2, the following estimations hold:

$$\|(A_{1\mu}\mathbf{u}_\mu, A_{2\mu}\mathbf{H}_\mu)\|_0 \le \hat{N}_1^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}), \qquad \|A_{1\mu}T_\mu\|_0 \le \hat{N}_2^{-1}\|\gamma\|_{-1}.$$

Furthermore, we derive the following bound of the errors.

Theorem 3. *Under assumptions of Theorem 2, we have following error estimations:*

$$\|\nabla(T - T_\mu)\|_0 + \|(\mathbf{u} - \mathbf{u}_\mu, \mathbf{H} - \mathbf{H}_\mu)\|_{\boldsymbol{D}} + \|p - p_\mu\|_0 \le C\mu,$$

where $C >$ *is a constant independent on* μ.

Proof. By subtracting the first equation of (5) from the first equation of (6), we have

$$a_0(T - T_\mu, S) + b_0(\mathbf{u}_\mu, T - T_\mu, S) + b_0(\mathbf{u} - \mathbf{u}_\mu, T, S) = 0. \tag{7}$$

By choosing $S = \hbar_\mu T - T_\mu$ in (7), we obtain

$$a_0(T - T_\mu, \hbar_\mu T - T_\mu) + b_0(\mathbf{u} - \mathbf{u}, T, \hbar_\mu T - T_\mu) + b_0(\mathbf{u}_\mu, T - T_\mu, \hbar_\mu T - T_\mu) = 0.$$

Split the above equation and apply the skew-symmetric property of the trilinear form to show

$$\begin{aligned}
&a_0(\hbar_\mu T - T_\mu, \hbar_\mu T - T_\mu) + b_0(r_\mu\mathbf{u} - \mathbf{u}_\mu, T, \hbar_\mu T - T_\mu)\\
&= a_0(\hbar_\mu T - T, \hbar_\mu T - T_\mu) + b_0(r_\mu\mathbf{u} - \mathbf{u}, T, \hbar_\mu T - T_\mu) + b_0(\mathbf{u}_\mu, \hbar_\mu T - T, \hbar_\mu T - T_\mu),
\end{aligned}$$

which combines with (4) and Theorem 1-2 to give

$$\begin{aligned}
&\kappa\|\nabla(\hbar_\mu T - T_\mu)\|_0\\
&\le \big(\kappa + N_0\nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\big)\|\nabla(\hbar_\mu T - T)\|_0\\
&\quad + N_0\kappa^{-1}\|\gamma\|_{-1}\|(r_\mu\mathbf{u} - \mathbf{u}, R_\mu\mathbf{H} - \mathbf{H})\|_{\boldsymbol{D}} + N_0\kappa^{-1}\|\gamma\|_{-1}\|(r_\mu\mathbf{u} - \mathbf{u}_\mu, R_\mu\mathbf{H} - \mathbf{H}_\mu)\|_{\boldsymbol{D}},
\end{aligned} \tag{8}$$

where we have applied $\|(\mathbf{w}, \boldsymbol{\Phi})\|_{\boldsymbol{D}}^2 \ge \|\mathbf{w}\|_{\boldsymbol{X}}^2$. Hence, applying Lemma 2 and the triangle inequality to (8), we have

$$\kappa\|\nabla(T - T_\mu)\|_0 \le C\mu + N_0\kappa^{-1}\|\gamma\|_{-1}\|(r_\mu\mathbf{u} - \mathbf{u}_\mu, R_\mu\mathbf{H} - \mathbf{H}_\mu)\|_{\boldsymbol{D}}. \tag{9}$$

Next, subtract the second equation of (5) from the second equation of (6) to get

$$\begin{aligned}
&A_0\big((\mathbf{u} - \mathbf{u}_\mu, \mathbf{H} - \mathbf{H}_\mu), (\mathbf{v}, \mathbf{B})\big) + A_1\big((\mathbf{u} - \mathbf{u}_\mu, \mathbf{H} - \mathbf{H}_\mu), (\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{B})\big)\\
&\quad + A_1\big((\mathbf{u}_\mu, \mathbf{H}_\mu), (\mathbf{u} - \mathbf{u}_\mu, \mathbf{H} - \mathbf{H}_\mu), (\mathbf{v}, \mathbf{B})\big) - d((\mathbf{v}, \mathbf{B}), p - p_\mu)\\
&\quad + d((\mathbf{u} - \mathbf{u}_\mu, \mathbf{H} - \mathbf{H}_\mu), q) = G(T - T_\mu, (\mathbf{v}, \mathbf{B})).
\end{aligned} \tag{10}$$

Choosing $(\mathbf{v},\mathbf{B})=(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu)$ and $q=\aleph_\mu p-p_\mu$ in (10), we obtain

$$\begin{aligned}
&A_0\big((r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu),(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu)\big)\\
&=G(\hbar_\mu T-T_\mu,(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu))-G(\hbar_\mu T-T,(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu))\\
&\quad+A_0\big((r_\mu\mathbf{u}-\mathbf{u},R_\mu\mathbf{H}-\mathbf{H}),(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu)\big)\\
&\quad-A_1\big((r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u},\mathbf{H}),(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu)\big)\\
&\quad+A_1\big((\mathbf{u}_\mu,\mathbf{H}_\mu),(r_\mu\mathbf{u}-\mathbf{u},R_\mu\mathbf{H}-\mathbf{H}),(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu)\big)\\
&\quad+A_1\big((r_\mu\mathbf{u}-\mathbf{u},R_\mu\mathbf{H}-\mathbf{H}),(\mathbf{u},\mathbf{H}),(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu)\big)\\
&\quad-d((r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu),p-p_\mu)+d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\aleph_\mu p-p_\mu).
\end{aligned}\tag{11}$$

In view of Lemma 2, we get

$$\begin{aligned}
&d((r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu),p-p_\mu)-d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\aleph_\mu p-p_\mu)\\
&=d((r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu),p-\aleph_\mu p).
\end{aligned}$$

Hence, according to (8), (11), Lemma 1 and Theorem 1, we arrive at

$$\begin{aligned}
&\nu_A(1-\delta)\|(r_\mu\mathbf{u}-\mathbf{u}_\mu,R_\mu\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}\\
&\le\big(2\beta C_F^2+\beta C_F^2N_0\kappa^{-1}\nu_A^{-1}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\big)\|\nabla(\hbar_\mu T-T)\|_0+\|p-\aleph_\mu p\|_0\\
&\quad+(c_A+2\nu_A\delta)\|(r_\mu\mathbf{u}-\mathbf{u},R_\mu\mathbf{H}-\mathbf{H})\|_{\boldsymbol{D}}.
\end{aligned}\tag{12}$$

Then, using the triangle inequality and Lemma 2, we rewrite (12) as

$$\nu_A(1-\delta)\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}\le C\mu.\tag{13}$$

Then, according to (12), we rewrite (9) as

$$\kappa\|\nabla(T-T_\mu)\|_0\le C\mu.\tag{14}$$

Moreover, by choosing $q=0$ in (10), one finds

$$\begin{aligned}
&d((\mathbf{v},\mathbf{B}),p-p_\mu)\\
&=A_0\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{v},\mathbf{B})\big)+A_1\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{B})\big)\\
&\quad+A_1\big((\mathbf{u}_\mu,\mathbf{H}_\mu),(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{v},\mathbf{B})\big)-G(T-T_\mu,(\mathbf{v},\mathbf{B})).
\end{aligned}$$

Hence, according to the discrete inf-sup condition, Lemma 1 and Theorem 1-2, and the Cauchy-Schwartz inequality and Poincaré inequality, we obtain

$$\begin{aligned}
\tilde{\beta}\|p-p_\mu\|_0\le&\big(c_A+2N\nu_A^{-1}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\big)\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}\\
&+\beta C_F^2\|\nabla(T-T_\mu)\|_0,
\end{aligned}$$

which applies (13) and (14) to finish the proof of this theorem. □

Further, in order to get L^2-norm error estimate, we need to introduce a dual problem [8]: seek $((\mathbf{w},\mathbf{\Phi}),\theta,\eta)\in\boldsymbol{D}\times Q\times M$ such that

$$\begin{cases}
a_0(S,\theta)+b_0(\mathbf{u},S,\theta)+b_0(S,\mathbf{u},\theta)=(T-T_\mu,S),\\
A_0((\mathbf{v},\mathbf{B}),(\mathbf{w},\mathbf{\Phi}))+A_1((\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{B}),(\mathbf{w},\mathbf{\Phi}))\\
\quad+A_1((\mathbf{v},\mathbf{B}),(\mathbf{u},\mathbf{H}),(\mathbf{w},\mathbf{\Phi}))-d((\mathbf{v},\mathbf{B}),\eta)+d((\mathbf{w},\mathbf{\Phi}),q)\\
=((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{v},\mathbf{B}))+G(T-T_\mu,(\mathbf{v},\mathbf{B})),
\end{cases}\tag{15}$$

for all $((\mathbf{v},\mathbf{B}),S,q)\in\boldsymbol{D}\times Q\times M$.

Theorem 4. *If $B_1,B_2>0$ and the solution of (15) satisfies $(\mathbf{w},\mathbf{\Phi})\in\boldsymbol{D}\cap H^2(\Omega)^d,\theta\in Q\cap H^2(\Omega),\eta\in M\cap H^1(\Omega)$, then there hold the following results:*

$$\begin{aligned}
B_1\|\Delta(\mathbf{w},\mathbf{\Phi})\|_0+\|\nabla\eta\|_0&\le\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_0+\beta\|T-T_\mu\|_0,\\
B_2\|\Delta\theta\|_0&\le\|T-T_\mu\|_0,
\end{aligned}\tag{16}$$

where $B_1 := \nu_A - 2\nu_A\delta_1 c_0$ *and* $B_2 := \kappa - 2N^{-1}\nu_A\delta_1 c_0$.

Proof. Firstly, according to the first equation of (15), Theorem 1 and (3), we get

$$\begin{aligned}\kappa\|\Delta\theta\|_0 &\le 2\|\nabla\mathbf{u}\|_0\|\theta\|_{L^\infty} + \|T - T_\mu\|_0\\ &\le 2\nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})c_0\|\Delta\theta\|_0 + \|T - T_\mu\|_0,\end{aligned}$$

which implies

$$\big(\kappa - 2N^{-1}\nu_A\delta_1 c_0\big)\|\Delta\theta\|_0 \le \|T - T_\mu\|_0. \tag{17}$$

Secondly, according to the second equation of (15), Lemma 1, Theorem 1 and (3), we have

$$\begin{aligned}&\nu_A\|\Delta(\mathbf{w},\mathbf{\Phi})\|_0 + \|\nabla\eta\|_0\\ &\le \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0 + 2\nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})c_0\|\Delta(\mathbf{w},\mathbf{\Phi})\|_0 + \beta\|T - T_\mu\|_0.\end{aligned}$$

Thus, we find that

$$\big(\nu_A - 2\nu_A\delta_1 c_0\big)\|\Delta(\mathbf{w},\mathbf{\Phi})\|_0 + \|\nabla\eta\|_0 \le \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0 + \beta\|T - T_\mu\|_0. \tag{18}$$

Combining (17) with (18), we obtain the desired result (16). The proof is completed. □

Theorem 5. *Let* $((\mathbf{u}_\mu, \mathbf{H}_\mu), T_\mu, p_\mu) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ *be a solution of the finite element discretization (6), and* $0 < B_3 < 1$. *Then, under the assumptions of Theorem 4 the following estimation holds:*

$$\|T - T_\mu\|_0 + \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0 \le C\mu^2,$$

where $B_3 := c_0 B_2^{-1}\nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}) + B_1^{-1}\beta^2 C_F^2 + B_4$ *and* $B_4 := \frac{c_0}{2}B_2^{-1}\kappa^{-1}\|\gamma\|_{-1} + \frac{1}{2}B_1^{-1}\beta C_F^2 + \frac{1}{2}\beta$.

Proof. Combining (7) and (10) with $q = 0$, we have

$$\begin{cases} a_0(T - T_\mu, S) + b_0(\mathbf{u}, T - T_\mu, S) - b_0(\mathbf{u}-\mathbf{u}_\mu, T - T_\mu, S) + b_0(\mathbf{u}-\mathbf{u}_\mu, T, S) = 0,\\ A_0\big((\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{v},\mathbf{B})\big) - A_1\big((\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{v},\mathbf{B})\big)\\ \quad + A_1\big((\mathbf{u},\mathbf{H}), (\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{v},\mathbf{B})\big) + A_1\big((\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{u},\mathbf{H}), (\mathbf{v},\mathbf{B})\big)\\ \quad - d((\mathbf{v},\mathbf{B}), p - p_\mu) = G(T - T_\mu, (\mathbf{v},\mathbf{B})).\end{cases} \tag{19}$$

Taking $(\mathbf{v},\mathbf{B}) = (\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), S = T - T_\mu, q = 0$ in (15), we get

$$\begin{cases} a_0(T - T_\mu, \theta) + b_0(\mathbf{u}, T - T_\mu, \theta) + b_0(T - T_\mu, \mathbf{u}, \theta) = (T - T_\mu, T - T_\mu),\\ A_0((\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{w},\mathbf{\Phi})) + A_1((\mathbf{u},\mathbf{H}), (\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{w},\mathbf{\Phi}))\\ \quad + A_1((\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{u},\mathbf{H}), (\mathbf{w},\mathbf{\Phi})) - d((\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), \eta)\\ = ((\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu), (\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)) + G(T - T_\mu, (\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)).\end{cases} \tag{20}$$

Next, by subtracting the first equation of (20) from the first equation of (19), we have

$$\begin{aligned}&a_0(T - T_\mu, \theta - S) + b_0(\mathbf{u}, T - T_\mu, \theta - S) + b_0(T - T_\mu, \mathbf{u}, \theta)\\ &= (T - T_\mu, T - T_\mu) + b_0(\mathbf{u}-\mathbf{u}_\mu, T_\mu, S),\end{aligned}$$

which implies

$$\begin{aligned}\|T - T_\mu\|_0^2 &= a_0(T - T_\mu, \theta - S) + b_0(\mathbf{u}, T - T_\mu, \theta - S) + b_0(T - T_\mu, \mathbf{u}, \theta)\\ &\quad + b_0(\mathbf{u}-\mathbf{u}_\mu, T_\mu, \theta - S) + b_0(\mathbf{u}-\mathbf{u}_\mu, T_\mu, \theta).\end{aligned}$$

Then, choosing $S = \hbar_\mu\theta$ and using (3), (4) and Lemma 2, we get

$$\begin{aligned}\|T - T_\mu\|_0^2 &\le c_2\mu\kappa\|\nabla(T - T_\mu)\|_0\|\Delta\theta\|_0 + c_2\mu N_0\|\nabla\mathbf{u}\|_0\|\nabla(T - T_\mu)\|_0\|\Delta\theta\|_0\\ &\quad + c_0\|\nabla\mathbf{u}\|_0\|T - T_\mu\|_0\|\Delta\theta\|_0 + c_2\mu N_0\|\nabla(\mathbf{u}-\mathbf{u}_\mu)\|_0\|\nabla T_\mu\|_0\|\Delta\theta\|_0\\ &\quad + c_0\|\mathbf{u}-\mathbf{u}_\mu\|_0\|\nabla T_\mu\|_0\|\Delta\theta\|_0,\end{aligned}$$

which combines with the fact that $B_2\|\Delta\theta\|_0 \le \|T-T_\mu\|_0$ in Theorem 4 to get

$$\begin{aligned}\|T-T_\mu\|_0^2 &\le c_2\mu B_2^{-1}(\kappa\|\nabla(T-T_\mu)\|_0 + N_0\|\nabla\mathbf{u}\|_0\|\nabla(T-T_\mu)\|_0)\|T-T_\mu\|_0\\ &\quad + c_0B_2^{-1}\|\nabla\mathbf{u}\|_0\|T-T_\mu\|_0^2 + c_2\mu N_0B_2^{-1}\|\nabla(\mathbf{u}-\mathbf{u}_\mu)\|_0\|\nabla T_\mu\|_0\|T-T_\mu\|_0\\ &\quad + c_0B_2^{-1}\|\mathbf{u}-\mathbf{u}_\mu\|_0\|\nabla T_\mu\|_0\|T-T_\mu\|_0\\ &\le c_2\mu B_2^{-1}(\kappa\|\nabla(T-T_\mu)\|_0 + N_0\|\nabla\mathbf{u}\|_0\|\nabla(T-T_\mu)\|_0\\ &\quad + N_0\|\nabla(\mathbf{u}-\mathbf{u}_\mu)\|_0\|\nabla T_\mu\|_0)\|T-T_\mu\|_0\\ &\quad + c_0B_2^{-1}\nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\|T-T_\mu\|_0^2\\ &\quad + \frac{c_0}{2}B_2^{-1}\kappa^{-1}\|\gamma\|_{-1}(\|\mathbf{u}-\mathbf{u}_\mu\|_0^2 + \|T-T_\mu\|_0^2).\end{aligned} \tag{21}$$

Rewrite (21) to obtain

$$\begin{aligned}&\big(1 - c_0B_2^{-1}\nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}) - \frac{c_0}{2}B_2^{-1}\kappa^{-1}\|\gamma\|_{-1}\big)\|T-T_\mu\|_0^2\\ &\le c\mu B_2^{-1}(\kappa\|\nabla(T-T_\mu)\|_0 + N_0\|\nabla\mathbf{u}\|_0\|\nabla(T-T_\mu)\|_0\\ &\quad + N_0\|\nabla(\mathbf{u}-\mathbf{u}_\mu)\|_0\|\nabla T_\mu\|_0)\|T-T_\mu\|_0 + \frac{c_0}{2}B_2^{-1}\kappa^{-1}\|\gamma\|_{-1}\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_0^2.\end{aligned} \tag{22}$$

Moreover, by subtracting the second equation of (20) from the second equation of (19), we have

$$\begin{aligned}\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_0^2 &= A_0\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad - A_1\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad + A_1\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w},\boldsymbol{\Phi})\big)\\ &\quad + A_1\big((\mathbf{u},\mathbf{H}),(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad + A_1\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u},\mathbf{H}),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad + d((\mathbf{v},\mathbf{B}),p-p_\mu) - d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\eta)\\ &\quad + G(T-T_\mu,(\mathbf{v},\mathbf{B})) - G(T-T_\mu,(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)).\end{aligned} \tag{23}$$

In addition, due to Lemma 2, there holds

$$\begin{aligned}&d((\mathbf{v},\mathbf{B}),p-p_\mu) - d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\eta)\\ &= -d((\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B}),p-p_\mu) - d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\eta) + d((\mathbf{w},\boldsymbol{\Phi}),p-p_\mu)\\ &= -d((\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B}),p-p_\mu) - d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\aleph_\mu\eta)\\ &\quad - d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\eta-\aleph_\mu\eta) + d((\mathbf{w},\boldsymbol{\Phi}),p-p_\mu),\end{aligned}$$

which combines with (23) to get

$$\begin{aligned}\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_0^2 &= A_0\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad - A_1\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad + A_1\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w},\boldsymbol{\Phi})\big)\\ &\quad + A_1\big((\mathbf{u},\mathbf{H}),(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad + A_1\big((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),(\mathbf{u},\mathbf{H}),(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})\big)\\ &\quad - d((\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B}),p-p_\mu) - d((\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu),\eta-\aleph_\mu\eta)\\ &\quad + G(T-T_\mu,(\mathbf{w},\boldsymbol{\Phi})) - G(T-T_\mu,(\mathbf{w}-\mathbf{v},\boldsymbol{\Phi}-\mathbf{B})) - G(T-T_\mu,(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)).\end{aligned} \tag{24}$$

Now, taking $(\mathbf{v}, \mathbf{B}) = (r_\mu \mathbf{w}, R_\mu \mathbf{\Phi})$ in (24) and using Lemma 1-2, Theorem 4 and the Cauchy-Schwartz inequality and Poincaré inequality, we get

$$
\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0^2 \\
&\le c_A \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} \|(\mathbf{w}-r_\mu \mathbf{w}, \mathbf{\Phi}-R_\mu \mathbf{\Phi})\|_{\boldsymbol{D}} \\
&\quad + N\|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 (\|(\mathbf{w}, \mathbf{\Phi})\|_{\boldsymbol{D}} + \|(\mathbf{w}-r_\mu \mathbf{w}, \mathbf{\Phi}-R_\mu \mathbf{\Phi})\|_{\boldsymbol{D}}) \\
&\quad + 2N\|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} \|(\mathbf{w}-r_\mu \mathbf{w}, \mathbf{\Phi}-R_\mu \mathbf{\Phi})\|_{\boldsymbol{D}} \\
&\quad + \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} \|\eta - \aleph_\mu \eta\|_0 + \|(\mathbf{w}-r_\mu \mathbf{w}, \mathbf{\Phi}-R_\mu \mathbf{\Phi})\|_{\boldsymbol{D}} \|p-p_\mu\|_0 \\
&\quad + \beta C_F^2 \|(\mathbf{w}-r_\mu \mathbf{w}, \mathbf{\Phi}-R_\mu \mathbf{\Phi})\|_{\boldsymbol{D}} \|\nabla(T-T_\mu)\|_0 + \beta C_F \|(\mathbf{w}, \mathbf{\Phi})\|_{\boldsymbol{D}} \|T-T_\mu\|_0 \\
&\quad + \beta \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0 \|T-T_\mu\|_0 \\
&\le c\mu \big(c_A \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + N\|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \\
&\quad + 2N\|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + \|p-p_\mu\|_0 + \beta C_F^2 \|\nabla(T-T_\mu)\|_0 \big) \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0 \\
&\quad + C_F N \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0 + c\mu \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} \|\nabla \eta\|_0 \\
&\quad + \beta C_F^2 \|T-T_\mu\|_0 \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0 + \frac{1}{2}\beta \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0^2 + \frac{1}{2}\beta \|T-T_\mu\|_0^2 \\
&\le c\mu \big(c_A \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + N\|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \\
&\quad + 2N\|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + \|p-p_\mu\|_0 + \beta C_F^2 \|\nabla(T-T_\mu)\|_0 \big) \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0 \\
&\quad + NC_F \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0 + c\mu \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} \|\nabla \eta\|_0 \\
&\quad + (\frac{1}{2} B_1^{-1} \beta C_F^2 + \frac{1}{2}\beta) \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0^2 + (\frac{1}{2} B_1^{-1} \beta C_F^2 + B_1^{-1} \beta^2 C_F^2 + \frac{1}{2}\beta) \|T-T_\mu\|_0^2,
\end{aligned}
$$

which combines with (22) to provide

$$
\begin{aligned}
&\big(1-B_3\big)\|T-T_\mu\|_0^2 + \big(1-B_4\big)\|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0^2 \\
&\le c\mu B_2^{-1} \beta^{-1} (\kappa \|\nabla(T-T_\mu)\|_0 + N_0 \|\nabla \mathbf{u}\|_0 \|\nabla(T-T_\mu)\|_0 \\
&\quad + N_0 \|\nabla(\mathbf{u}-\mathbf{u}_\mu)\|_0 \|\nabla T_\mu\|_0)(\beta \|T-T_\mu\|_0 + \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0) \\
&\quad + c\mu \big(c_A \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + N\|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \\
&\quad + 2N\|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + \|p-p_\mu\|_0 + \beta C_F^2 \|\nabla(T-T_\mu)\|_0 \big) \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0 \\
&\quad + c\mu \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} \|\nabla \eta\|_0 + NC_F \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0,
\end{aligned}
\tag{25}
$$

where $B_3 := c_0 B_2^{-1} \nu_A^{-1} (\|\mathbf{F}\|_{-1} + C_F^2 \beta \kappa^{-1} \|\gamma\|_{-1}) + B_1^{-1} \beta^2 C_F^2 + B_4$ and $B_4 := \frac{c_0}{2} B_2^{-1} \kappa^{-1} \|\gamma\|_{-1} + \frac{1}{2} B_1^{-1} \beta C_F^2 + \frac{1}{2}\beta$. Note that $B_3 > B_4$.

Now, combining (25) with $(a+b)^2 \le 2(a^2+b^2)$, we find that

$$
\begin{aligned}
&\frac{1}{2} B_5 (\beta \|T-T_\mu\|_0 + \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0)^2 \\
&\le c\mu B_2^{-1} \beta^{-1} (\kappa \|\nabla(T-T_\mu)\|_0 + N_0 \|\nabla \mathbf{u}\|_0 \|\nabla(T-T_\mu)\|_0 \\
&\quad + N_0 \|\nabla(\mathbf{u}-\mathbf{u}_\mu)\|_0 \|\nabla T_\mu\|_0)(\beta \|T-T_\mu\|_0 + \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0) \\
&\quad + c\mu \big(c_A \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + N\|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \\
&\quad + 2N\|(\mathbf{u}, \mathbf{H})\|_{\boldsymbol{D}} \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + \|p-p_\mu\|_0 + \beta C_F^2 \|\nabla(T-T_\mu)\|_0 \big) \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0 \\
&\quad + c\mu \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} \|\nabla \eta\|_0 + NC_F \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 \|\Delta(\mathbf{w}, \mathbf{\Phi})\|_0,
\end{aligned}
\tag{26}
$$

where $B_5 := \min\{1-B_4, (1-B_3)\beta^{-2}\}$.

Hence, combining (26) with Theorem 4, we have

$$
\frac{1}{2} B_5 (\beta \|T-T_\mu\|_0 + \|(\mathbf{u}-\mathbf{u}_\mu, \mathbf{H}-\mathbf{H}_\mu)\|_0)
$$

$$
\begin{aligned}
&\leq c\mu B_2^{-1}\beta^{-1}(\kappa\|\nabla(T-T_\mu)\|_0 + N_0\|\nabla\mathbf{u}\|_0\|\nabla(T-T_\mu)\|_0 + N_0\|\nabla(\mathbf{u}-\mathbf{u}_\mu)\|_0\|\nabla T_\mu\|_0)\\
&\quad + c\mu\big(c_A\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + N\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2\\
&\quad + 2N\|(\mathbf{u},\mathbf{H})\|_{\boldsymbol{D}}\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + \|p-p_\mu\|_0 + \beta C_F^2\|\nabla(T-T_\mu)\|_0\big)B_1^{-1}\\
&\quad + c\mu\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}} + NC_F\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2 B_1^{-1},
\end{aligned}
$$

which combines with Theorem 3 to complete the proof. □

Next, for the thermally coupled incompressible magnetohydrodynamics, we recall the Arrow-Hurwicz method and its convergence analysis results which are showed in [18]. For convenience, we set some iterative errors $\mathbf{e}_\mu^n = \mathbf{u}_\mu - \mathbf{u}_\mu^n$, $\boldsymbol{\xi}_\mu^n = \mathbf{H}_\mu - \mathbf{H}_\mu^n$, $\eta_\mu^n = p_\mu - p_\mu^n$ and $\theta_\mu^n = T_\mu - T_\mu^n$, $n \geq 0$.

Arrow-Hurwicz Algorithm [18]. Let ρ and α be two positive parameters. Then, we find $((\mathbf{u}_\mu^{n+1},\mathbf{H}_\mu^{n+1}),T_\mu^{n+1},p_\mu^{n+1}) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ by the following two steps.

Step 1. Choose an initial function pair $((\mathbf{u}_\mu^0,\mathbf{H}_\mu^0),T_\mu^0,p_\mu^0) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ defined by solving the following equations:

$$
\begin{cases}
a_0(T_\mu^0,S) = (\gamma,S), \quad \forall S \in Q_\mu,\\
A_0((\mathbf{u}_\mu^0,\mathbf{H}_\mu^0),(\mathbf{v},\mathbf{B})) - d((\mathbf{v},\mathbf{B}),p_\mu^0) = (\mathbf{F},(\mathbf{v},\mathbf{B})) + G(T_\mu^0,(\mathbf{v},\mathbf{B})), \quad \forall(\mathbf{v},\mathbf{B}) \in \boldsymbol{D}_\mu,\\
d((\mathbf{u}_\mu^0,\mathbf{H}_\mu^0),q) = 0, \quad \forall q \in M_\mu.
\end{cases} \tag{27}
$$

Step 2. For $n = 1,2,\ldots,m-1$, given $((\mathbf{u}_\mu^n,\mathbf{H}_\mu^n),T_\mu^n,p_\mu^n) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$, find $((\mathbf{u}_\mu^{n+1},\mathbf{H}_\mu^{n+1}),$ $T_\mu^{n+1},p_\mu^{n+1}) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ as solution of

$$
\begin{cases}
\rho^{-1}Re\kappa(\nabla T_\mu^{n+1} - \nabla T_\mu^n,\nabla S) + a_0(T_\mu^n,S) + b_0(\mathbf{u}_\mu^n,T_\mu^{n+1},S) = (\gamma,S),\\
\rho^{-1}(\nabla\mathbf{u}_\mu^{n+1} - \nabla\mathbf{u}_\mu^n,\nabla\mathbf{v}) + \rho^{-1}Rea_2(\mathbf{H}_\mu^{n+1} - \mathbf{H}_\mu^n,\mathbf{B}) + A_0((\mathbf{u}_\mu^n,\mathbf{H}_\mu^n),(\mathbf{v},\mathbf{B}))\\
\quad - d((\mathbf{v},\mathbf{B}),p_\mu^n) + A_1((\mathbf{u}_\mu^n,\mathbf{H}_\mu^n),(\mathbf{u}_\mu^{n+1},\mathbf{H}_\mu^{n+1}),(\mathbf{v},\mathbf{B})) = (\mathbf{F},(\mathbf{v},\mathbf{B})) + G(T_\mu^{n+1},(\mathbf{v},\mathbf{B})),\\
\alpha(p_\mu^{n+1} - p_\mu^n,q) + \rho d((\mathbf{u}_\mu^{n+1},\mathbf{H}_\mu^{n+1}),q) = 0,
\end{cases}
$$

for all $(\mathbf{v},\mathbf{B}) \in \boldsymbol{D}_\mu$, $S \in Q_\mu$ and $q \in M_\mu$.

The following lemma shows the iterative error of the initial guess.

Lemma 5. *[18] Let $((\mathbf{u}_h^0,\mathbf{H}_\mu^0),p_\mu^0,T_\mu^0) \in \boldsymbol{D}_\mu \times Q_\mu \times M_\mu$ be the solution of (27). Then, under the assumptions of Theorem 2, we have the following results:*

$$
\begin{aligned}
\|\nabla\theta_\mu^0\|_0 &\leq N_0\kappa^{-2}\nu_A^{-1}\|\gamma\|_{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),\\
\|(\mathbf{e}_\mu^0,\boldsymbol{\xi}_\mu^0)\|_{\boldsymbol{D}} &\leq \nu_A^{-1}\delta(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),\\
\|\eta_\mu^0\|_0 &\leq \widetilde{\beta}^{-1}\delta(c_A\nu_A^{-1} + 1)(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}).
\end{aligned}
$$

Theorem 6. *[18] Assume that $\{(\mathbf{u}_\mu^n,\mathbf{H}_\mu^n),p_\mu^n,T_\mu^n\}$ is the function sequence of the Arrow-Hurwicz algorithm. If we choose the parameters ρ and α satisfy the conditions*

$$
2B_6 < \zeta, \quad \rho + 2\kappa|Re-\rho| < Re\kappa,
$$

where $\zeta := 2Re\nu_A - 2\rho\beta^2C_F^2 - \rho^2\alpha^{-1} - 2\rho N_0^2\kappa^{-2}\|\gamma\|_{-1}^2$ and $B_6 := 2(c_A|Re-\rho| + \rho\nu_A\delta_1)$, then under the assumptions of Theorem 2, we have

$$
D_1\|(\mathbf{e}_\mu^{n+1},\boldsymbol{\xi}_\mu^{n+1})\|_{\boldsymbol{D}}^2 + \alpha\|\eta_\mu^{n+1}\|_0^2 + D_2\|\nabla\theta_\mu^{n+1}\|_0^2 \leq D_1\|(\mathbf{e}_\mu^n,\boldsymbol{\xi}_\mu^n)\|_{\boldsymbol{D}}^2 + \alpha\|\eta_\mu^n\|_0^2 + D_2\|\nabla\theta_\mu^n\|_0^2,
$$

where $D_1 := (2Re\nu_A - 2\rho\beta^2C_F^2 - \rho^2\alpha^{-1} + 2\rho N_0^2\kappa^{-2}\|\gamma\|_{-1}^2 + \sqrt{\Delta_1})/2$ and $D_2 := Re\kappa - 2\rho + \sqrt{\Delta_2}$, $\Delta_1 := \zeta^2 - 4B_6^2$, $\Delta_2 := (Re\kappa - 2\rho)^2 - 4\kappa^2|Re-\rho|^2$.

Now, we recall the convergence analysis for the Arrow-Hurwicz algorithm given in [18].

Theorem 7. *[18] Under the assumptions of Theorem 6, the following estimates hold:*

$$\varrho_1\|(\mathbf{e}_\mu^{n+1},\boldsymbol{\xi}_\mu^{n+1})\|_{\boldsymbol{D}}^2+\alpha\|\eta_\mu^{n+1}\|_0^2+\varrho_2\|\nabla\theta_\mu^{n+1}\|_0^2\le\varpi(\varrho_1\|(\mathbf{e}_\mu^{n},\boldsymbol{\xi}_\mu^{n})\|_{\boldsymbol{D}}^2+\alpha\|\eta_\mu^{n}\|_0^2+\varrho_2\|\nabla\theta_\mu^{n}\|_0^2),$$

where $0<\varrho_1<Re\nu_A+\rho N_0^2\kappa^{-2}\|\gamma\|_{-1}^2$, $0<\varrho_2<\frac{1}{2}Re\kappa$ *and* $0<\varpi<1$ *are three generic constants independent of* n *and* h.

Next, based on the convergence result in Theorem 7, we have the stability concerning the iterative solutions.

Theorem 8. *Let* $((\mathbf{u}_\mu^m,\mathbf{H}_\mu^m),T_\mu^m,p_\mu^m)$ *be generated by the Arrow-Hurwicz algorithm. Then, under the assumptions of Theorem 5 and 7, we have the following stability:*

$$\|(\mathbf{u}_\mu^m,\mathbf{H}_\mu^m)\|_{\boldsymbol{D}}\le A_3(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),\quad \|\nabla T_\mu^m\|_0\le A_4(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),$$

where $B_8:=\varrho_1\nu_A^{-2}\delta^2+\alpha\widetilde{\beta}^{-2}\delta^2(c_A\nu_A^{-1}+1)^2+\varrho_2N_0^2\kappa^{-4}\nu_A^{-2}\|\gamma\|_{-1}^2$, $A_3:=\sqrt{2(\nu_A^{-2}+\varrho_1^{-2}\varpi^mB_8)}$, $A_4:=\sqrt{2(\kappa^{-2}\|\gamma\|_{-1}^{-2}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})^{-2}+\alpha^{-2}\varpi^mB_8)}$.

Proof. According to Theorem 7, we obtain

$$\varrho_1\|(\mathbf{e}_\mu^{m},\boldsymbol{\xi}_\mu^{m})\|_{\boldsymbol{D}}^2+\alpha\|\eta_\mu^{m}\|_0^2+\varrho_2\|\nabla\theta_\mu^{m}\|_0^2\le\varpi^m(\varrho_1\|(\mathbf{e}_\mu^{0},\boldsymbol{\xi}_\mu^{0})\|_{\boldsymbol{D}}^2+\alpha\|\eta_\mu^{0}\|_0^2+\varrho_2\|\nabla\theta_\mu^{0}\|_0^2),$$

which combines with Lemma 5, we get

$$\begin{aligned}\|(\mathbf{e}_\mu^m,\boldsymbol{\xi}_\mu^m)\|_{\boldsymbol{D}}&\le\varrho_1^{-1}\sqrt{\varpi^mB_8}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),\\ \|\nabla\theta_\mu^m\|_0&\le\alpha^{-1}\sqrt{\varpi^mB_8}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),\\ \|\eta_\mu^m\|_0&\le\varrho_2^{-1}\sqrt{\varpi^mB_8}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),\end{aligned}\tag{28}$$

where $B_8:=\varrho_1\nu_A^{-2}\delta^2+\alpha\widetilde{\beta}^{-2}\delta^2(c_A\nu_A^{-1}+1)^2+\varrho_2N_0^2\kappa^{-4}\nu_A^{-2}\|\gamma\|_{-1}^2$. Finally, applying the triangle inequality and Theorem 2, we get

$$\begin{aligned}\|(\mathbf{u}_\mu^m,\mathbf{H}_\mu^m)\|_{\boldsymbol{D}}^2&\le2(\|(\mathbf{u}_\mu,\mathbf{H}_\mu)\|_{\boldsymbol{D}}^2+\|(\mathbf{e}_\mu^m,\boldsymbol{\xi}_\mu^m)\|_{\boldsymbol{D}}^2)\\ &\le2(\nu_A^{-2}+\varrho_1^{-2}\varpi^mB_8)(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})^2,\\ \|\nabla T_\mu^m\|_0^2&\le2(\|\nabla T_\mu\|_0^2+\|\nabla\theta_\mu^m\|_0^2)\\ &\le2(\kappa^{-2}\|\gamma\|_{-1}^{-2}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})^{-2}+\alpha^{-2}\varpi^mB_8)(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})^2.\end{aligned}$$

□

At the end of this section, combining the finite element errors and the iterative errors, we have the following estimation.

Theorem 9. *Let* $((\mathbf{u},\mathbf{H}),T,p)$ *and* $((\mathbf{u}_\mu^m,\mathbf{H}_\mu^m),T_\mu^m,p_\mu^m)$ *be the solutions of problem (5) and Arrow-Hurwicz algorithm, respectively. Then, under the assumptions of Theorem 8 the following estimation holds:*

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_\mu^m,\mathbf{H}-\mathbf{H}_\mu^m)\|_{\boldsymbol{D}}+\|\nabla(T-T_\mu^m)\|_0+\|p-p_\mu^m\|_0\\ &\le C\mu+r\sqrt{\varpi^mB_8}(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\end{aligned}$$

where $r:=\max\{\sqrt{2}\varrho_1^{-1},\alpha^{-1},\varrho_2^{-1}\}$.

Proof. According to the triangle inequality, we get

$$\begin{aligned}\|(\mathbf{u}-\mathbf{u}_\mu^m,\mathbf{H}-\mathbf{H}_\mu^m)\|_{\boldsymbol{D}}&\le\sqrt{2}(\|(\mathbf{u}-\mathbf{u}_\mu,\mathbf{H}-\mathbf{H}_\mu)\|_{\boldsymbol{D}}+\|(\mathbf{e}_\mu^m,\boldsymbol{\xi}_\mu^m)\|_{\boldsymbol{D}}),\\ \|\nabla(T-T_\mu^m)\|_0&\le\|\nabla(T-T_\mu)\|_0+\|\nabla(T_\mu-T_\mu^m)\|_0,\\ \|p-p_\mu^m\|_0&\le\|p-p_\mu\|_0+\|p_\mu-p_\mu^m\|_0,\end{aligned}$$

which combines with (28) and Theorem 3 to have

$$\|(\mathbf{u}-\mathbf{u}_\mu^m,\mathbf{H}-\mathbf{H}_\mu^m)\|_{\boldsymbol{D}}+\|\nabla(T-T_\mu^m)\|_0+\|p-p_\mu^m\|_0$$

$$\leq C\mu + r\sqrt{\varpi^m B_8}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),$$

where $r := \max\{\sqrt{2}\varrho_1^{-1}, \alpha^{-1}, \varrho_2^{-1}\}$. □

4. Two-level Arrow-Hurwicz algorithms

Now, we present two-level Arrow-Hurwicz iterative algorithms combined with a mixed finite element method for the steady-state thermally coupled incompressible magnetohydrodynamics problem and give the stabilities and error estimates of these algorithms.

Algotirhm 1. *(Arrow-Hurwicz and One-Step-Oseen iteration)*
Step 1. *Choose an initial function pair* $((\mathbf{u}_H^0, \mathbf{H}_H^0), T_H^0, p_H^0) \in \boldsymbol{D}_H \times Q_H \times M_H$ *defined by solving the following equations:*

$$\begin{cases} a_0(T_H^0, S) = (\gamma, S), \quad \forall S \in Q_H, \\ A_0((\mathbf{u}_H^0, \mathbf{H}_H^0), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), p_H^0) = (\mathbf{F}, (\mathbf{v}, \mathbf{B})) + G(T_H^0, (\mathbf{v}, \mathbf{B})), \quad \forall (\mathbf{v}, \mathbf{B}) \in \boldsymbol{D}_H, \\ d((\mathbf{u}_H^0, \mathbf{H}_H^0), q) = 0, \quad \forall q \in M_H. \end{cases} \tag{29}$$

Step 2. *For* $n = 1, 2, \ldots, m-1$, *given* $((\mathbf{u}_H^n, \mathbf{H}_H^n), T_H^n, p_H^n) \in \boldsymbol{D}_H \times Q_H \times M_H$, *find* $((\mathbf{u}_H^{n+1}, \mathbf{H}_H^{n+1}), T_H^{n+1}, p_H^{n+1}) \in \boldsymbol{D}_H \times Q_H \times M_H$ *as the solution of*

$$\begin{cases} \rho^{-1}Re\kappa(\nabla T_H^{n+1} - \nabla T_H^n, \nabla S) + a_0(T_H^n, S) + b_0(\mathbf{u}_H^n, T_H^{n+1}, S) = (\gamma, S), \\ \rho^{-1}(\nabla \mathbf{u}_H^{n+1} - \nabla \mathbf{u}_H^n, \nabla \mathbf{v}) + \rho^{-1}Rea_2(\mathbf{H}_H^{n+1} - \mathbf{H}_H^n, \mathbf{B}) + A_0((\mathbf{u}_H^n, \mathbf{H}_H^n), (\mathbf{v}, \mathbf{B})) \\ \quad - d((\mathbf{v}, \mathbf{B}), p_H^n) + A_1((\mathbf{u}_H^n, \mathbf{H}_H^n), (\mathbf{u}_H^{n+1}, \mathbf{H}_H^{n+1}), (\mathbf{v}, \mathbf{B})) \\ = (\mathbf{F}, (\mathbf{v}, \mathbf{B})) + G(T_H^{n+1}, (\mathbf{v}, \mathbf{B})), \\ \alpha(p_H^{n+1} - p_H^n, q) + \rho d((\mathbf{u}_H^{n+1}, \mathbf{H}_H^{n+1}), q) = 0, \end{cases} \tag{30}$$

for all $(\mathbf{v}, \mathbf{B}) \in \boldsymbol{D}_H$, $S \in Q_H$ *and* $q \in M_H$.
Step 3. *Solve an Oseen-type linear problem on fine grid, i.e., find* $((\mathbf{u}_h^m, \mathbf{H}_h^m), T_h^m, p_h^m) \in \boldsymbol{D}_h \times Q_h \times M_h$ *as solution of*

$$\begin{cases} a_0(T_h^m, S) + b_0(\mathbf{u}_H^m, T_h^m, S) = (\gamma, S), \\ A_0((\mathbf{u}_h^m, \mathbf{H}_h^m), (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{u}_H^m, \mathbf{H}_H^m), (\mathbf{u}_h^m, \mathbf{H}_h^m), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), p_h^m) \\ \quad + d((\mathbf{u}_h^m, \mathbf{H}_h^m), q) = (\mathbf{F}, (\mathbf{v}, \mathbf{B})) + G(T_h^m, (\mathbf{v}, \mathbf{B})), \end{cases} \tag{31}$$

for all $(\mathbf{v}, \mathbf{B}) \in \boldsymbol{D}_h$, $S \in Q_h$ *and* $q \in M_h$.

Now, we show the stability and convergence of Algorithm 1 in the following theorems.

Theorem 10. *Let* $\big((\mathbf{u}_h^m, \mathbf{H}_h^m), T_h^m\big)$ *be the solution of Algorithm 1. Then, we have*

$$\|(\mathbf{u}_h^m, \mathbf{H}_h^m)\|_{\boldsymbol{D}} \leq \nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}), \qquad \|\nabla T_h^m\|_0 \leq \kappa^{-1}\|\gamma\|_{-1}.$$

Proof. Taking $S = T_h^m$ in the first equation of (31), we get

$$\|\nabla T_h^m\|_0 \leq \kappa^{-1}\|\gamma\|_{-1}. \tag{32}$$

Note that $b_0(\mathbf{u}_H^m, T_h^m, T_h^m) = 0$.

Then, choosing $(\mathbf{v}, \mathbf{B}) = (\mathbf{u}_h^m, \mathbf{H}_h^m)$, $q = p_h^m$ in the second equation of (31) and using (32) and the Cauchy-Schwartz inequality and Poincaré inequality, we obtain

$$\|(\mathbf{u}_h^m, \mathbf{H}_h^m)\|_{\boldsymbol{D}} \leq \nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\|\nabla T_h^m\|_0) \leq \nu_A^{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}).$$

Note that $A_1((\mathbf{u}_H^m, \mathbf{H}_H^m), (\mathbf{u}_h^m, \mathbf{H}_h^m), (\mathbf{u}_h^m, \mathbf{H}_h^m)) = 0$. The proof is estimation completed. □

Theorem 11. *Let* $\big((\mathbf{u},\mathbf{H}),T,p\big)$ *and* $\big((\mathbf{u}_h^m,\mathbf{H}_h^m),T_h^m,p_h^m\big)$ *be the solutions of the problem (5) and Algorithm 1, respectively. Then, under assumptions of Theorem 9 and Lemma 4, the following estimate holds:*

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_h^m,\mathbf{H}-\mathbf{H}_h^m)\|_{\boldsymbol{D}}+\|\nabla\big(T-T_h^m\big)\|_0+\|p-p_h^m\|_0\\&\le Ch+(CH^2+\varrho_1^{-1}\sqrt{\varpi^m B_8}C_F\nu_A^2N^{-1}\delta_1)\widehat{A}_1,\end{aligned}$$

where $\widehat{A}_1:=\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\|\gamma\|_{-1}+(\hat{N}_0\hat{N}_1^{-1}N^{-1}\nu_A\delta_1+\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\kappa N_0^{-1}\delta_2)(1+\widetilde{\beta}^{-1}(1+c_A+\delta_1\nu_A^2A_3))$.

Proof. Subtracting the first equation of (31) from the first equation of (6) with $\mu=h$, we have

$$a_0(\theta_h^m,S)+b_0(\mathbf{u}_h-\mathbf{u}_H^m,T_h,S)+b_0(\mathbf{u}_H^m,\theta_h^m,S)=0. \tag{33}$$

Choosing $S=\theta_h^m$ in (33) and notice that $b_0(\mathbf{u}_H^m,\theta_h^m,\theta_h^m)=0$. Then, using Lemma 3-4, Theorem 5, (28) and the Poincaré inequality, we get

$$\begin{aligned}\kappa\|\nabla\theta_h^m\|_0&\le\hat{c}_0\hat{c}_1\|A_{1h}T_h\|_0\|\mathbf{u}_h-\mathbf{u}_H^m\|_0\\&\le\hat{c}_0\hat{c}_1\|A_{1h}T_h\|_0\big(\|\mathbf{u}-\mathbf{u}_h\|_0+\|\mathbf{u}-\mathbf{u}_H\|_0+\|\mathbf{u}_H-\mathbf{u}_H^m\|_0\big)\\&\le\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\|\gamma\|_{-1}(CH^2+\varrho_1^{-1}\sqrt{\varpi^m B_8}C_F(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})),\end{aligned} \tag{34}$$

which and Theorem 3 imply

$$\begin{aligned}&\|\nabla\big(T-T_h^m\big)\|_0\le\|\nabla\big(T-T_h\big)\|_0+\|\nabla\theta_h^m\|_0\\&\le Ch+\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\kappa^{-1}\|\gamma\|_{-1}(CH^2+\varrho_1^{-1}\sqrt{\varpi^m B_8}C_F(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})).\end{aligned} \tag{35}$$

Next, subtracting the second equation of (31) from the second equation of (6) with $\mu=h$, we get

$$\begin{aligned}&A_0((\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),(\mathbf{v},\mathbf{B}))+A_1((\mathbf{u}_h-\mathbf{u}_H^m,\mathbf{H}_h-\mathbf{H}_H^m),(\mathbf{u}_h,\mathbf{H}_h),(\mathbf{v},\mathbf{B}))\\&\quad+A_1((\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),(\mathbf{v},\mathbf{B}))-d((\mathbf{v},\mathbf{B}),\eta_h^m)\\&\quad+d((\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),q)=G(T_h-T_h^m,(\mathbf{v},\mathbf{B})).\end{aligned} \tag{36}$$

Taking $(\mathbf{v},\mathbf{B})=(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)$, $q=\eta_h^m$ in (36), and combining Lemma 1, 3, 4, the Cauchy-Schwartz inequality and Poincaré inequality, Theorem 5, (28) and (34), we arrive at

$$\begin{aligned}&\nu_A\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}\\&\le\hat{N}_0\|(A_{1h}\mathbf{u}_h,A_{2h}\mathbf{H}_h)\|_0\|(\mathbf{u}_h-\mathbf{u}_H^m,\mathbf{H}_h-\mathbf{H}_H^m)\|_0+\beta C_F^2\|\nabla\theta_h^m\|_0\\&\le\hat{N}_0\|(A_{1h}\mathbf{u}_h,A_{21h}\mathbf{H}_h)\|_0\big(\|(\mathbf{u}-\mathbf{u}_h,\mathbf{H}-\mathbf{H}_h)\|_0+\|(\mathbf{u}-\mathbf{u}_H,\mathbf{H}-\mathbf{H}_H)\|_0\\&\quad+\|(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m)\|_0\big)+\beta C_F^2\|\nabla\theta_h^m\|_0\\&\le(CH^2+\varrho_1^{-1}\sqrt{\varpi^m B_8}C_F\nu_A^2N^{-1}\delta_1)(\hat{N}_0\hat{N}_1^{-1}N^{-1}\nu_A\delta_1+\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\kappa N_0^{-1}\delta_2)\nu_A,\end{aligned} \tag{37}$$

where we have used the uniqueness condition in Theorem 1 for the last inequality. Next, using (37) and Theorem 3, we obtain

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_h^m,\mathbf{H}-\mathbf{H}_h^m)\|_{\boldsymbol{D}}\le\|(\mathbf{u}-\mathbf{u}_h,\mathbf{H}-\mathbf{H}_h)\|_{\boldsymbol{D}}+\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}\\&\le Ch+(CH^2+\varrho_1^{-1}\sqrt{\varpi^m B_8}C_F\nu_A^2N^{-1}\delta_1)(\hat{N}_0\hat{N}_1^{-1}N^{-1}\nu_A\delta_1+\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\kappa N_0^{-1}\delta_2).\end{aligned} \tag{38}$$

Finally, applying the discrete inf-sup condition to (36) with $q=0$, Lemma 1, 3, and the Cauchy-Schwartz inequality and Poincaré inequality, we obtain

$$\begin{aligned}\widetilde{\beta}\|\eta_h^m\|_0&\le c_A\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}+\hat{N}_0\|(A_{1h}\mathbf{u}_h,A_{2h}\mathbf{H}_h)\|_0\|(\mathbf{u}_h-\mathbf{u}_H^m,\mathbf{H}_h-\mathbf{H}_H^m)\|_0\\&\quad+N\|(\mathbf{u}_H^m,\mathbf{H}_H^m)\|_{\boldsymbol{D}}\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}+\beta C_F^2\|\nabla\theta_h^m\|_0\\&\le(CH^2+\varrho_1^{-1}\sqrt{\varpi^m B_8}C_F\nu_A^2N^{-1}\delta_1)(\hat{N}_0\hat{N}_1^{-1}N^{-1}\nu_A\delta_1+\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\kappa N_0^{-1}\delta_2)\\&\quad\times(1+c_A+\delta_1\nu_A^2A_3),\end{aligned} \tag{39}$$

where we have applied Lemma 3, Theorem 5, 8, and (28), (34), (37) for the last inequality. Then, combine (39) and Theorem 3 to get

$$\begin{aligned}\|p-p_h^m\|_0 &\le \|p-p_h\|_0+\|\eta_h^m\|_0\\ &\le Ch+\widetilde{\beta}^{-1}(CH^2+\varrho_1^{-1}\sqrt{\varpi^m B_8}C_F\nu_A^2N^{-1}\delta_1)(\hat{N}_0\hat{N}_1^{-1}N^{-1}\nu_A\delta_1\\ &\quad+\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\kappa N_0^{-1}\delta_2)(1+c_A+\delta_1\nu_A^2A_3).\end{aligned}\tag{40}$$

According to (35), (38) and (40), we finish the proof of this theorem. □

Algotirhm 2. *(Arrow-Hurwicz and One-Step-Stokes iteration)*
Step 1. *Solve (29) to get the initial function* $((\mathbf{u}_h^0,\mathbf{H}_H^0),T_H^0,p_H^0)\in \boldsymbol{D}_H\times Q_H\times M_H$.
Step 2. *For* $n=1,2,\ldots,m-1$, *given* $((\mathbf{u}_H^n,\mathbf{H}_H^n),T_H^n,p_H^n)\in \boldsymbol{D}_H\times Q_H\times M_H$, *find* $((\mathbf{u}_H^{n+1},\mathbf{H}_H^{n+1}),T_H^{n+1},p_H^{n+1})\in \boldsymbol{D}_H\times Q_H\times M_H$ *as solution of (30).*
Step 3. *Solve a Stokes-type linear problem on fine grid, i.e., find* $((\mathbf{u}_h^m,\mathbf{H}_h^m),T_h^m,p_h^m)\in \boldsymbol{D}_h\times Q_h\times M_h$ *as solution of*

$$\begin{cases}a_0(T_h^m,S)+b_0(\mathbf{u}_H^m,T_H^m,S)=(\gamma,S),\\ A_0((\mathbf{u}_h^m,\mathbf{H}_h^m),(\mathbf{v},\mathbf{B}))+A_1((\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{v},\mathbf{B}))-d((\mathbf{v},\mathbf{B}),p_h^m)\\ \quad+d((\mathbf{u}_h^m,\mathbf{H}_h^m),q)=(\mathbf{F},(\mathbf{v},\mathbf{B}))+G(T_h^m,(\mathbf{v},\mathbf{B})),\end{cases}\tag{41}$$

for all $(\mathbf{v},\mathbf{B})\in \boldsymbol{D}_h$, $S\in Q_h$ *and* $q\in M_h$.

We now prove the stability and convergence of Algorithm 2.

Theorem 12. *Let* $\big((\mathbf{u}_h^m,\mathbf{H}_h^m),T_h^m\big)$ *be the solution of Algorithm 2. Then, under the assumptions of Theorem 9, we have*

$$\begin{aligned}\|(\mathbf{u}_h^m,\mathbf{H}_h^m)\|_{\boldsymbol{D}}&\le \nu_A^3\delta_1^2N^{-1}A_3^2+\nu_AN^{-1}\delta_1+\beta C_F^2\kappa^{-1}N_0A_3A_4N^{-2}\nu_A^3\delta_1^2,\\ \|\nabla T_h^m\|_0&\le \kappa^{-1}N_0A_3A_4N^{-2}\nu_A^4\delta_1^2+\kappa^{-1}\|\gamma\|_{-1}.\end{aligned}$$

Proof. Taking $S=T_h^m$ in the first equation of (41) and using (4), Theorem 8, we write

$$\begin{aligned}\|\nabla T_h^m\|_0&\le \kappa^{-1}N_0\|\nabla\mathbf{u}_H^m\|_0\|\nabla T_H^m\|_0+\kappa^{-1}\|\gamma\|_{-1}\\ &\le \kappa^{-1}N_0A_3A_4(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})^2+\kappa^{-1}\|\gamma\|_{-1}\\ &\le \kappa^{-1}N_0A_3A_4N^{-2}\nu_A^4\delta_1^2+\kappa^{-1}\|\gamma\|_{-1},\end{aligned}\tag{42}$$

where we have used the uniqueness condition in Theorem 1. Next, take $(\mathbf{v},\mathbf{B})=(\mathbf{u}_h^m,\mathbf{H}_h^m)$ and $q=p_h^m$ in the second equation of (41). Then, apply Lemma 1, Theorem 8 and (42).

$$\begin{aligned}\|(\mathbf{u}_h^m,\mathbf{H}_h^m)\|_{\boldsymbol{D}}&\le \nu_A^{-1}(N\|(\mathbf{u}_H^m,\mathbf{H}_H^m)\|_{\boldsymbol{D}}^2+\|\mathbf{F}\|_{-1}+C_F^2\beta\|\nabla T_h^m\|_0)\\ &\le \nu_A^3\delta_1^2N^{-1}A_3^2+\nu_AN^{-1}\delta_1+C_F^2\beta\kappa^{-1}N_0A_3A_4N^{-2}\nu_A^3\delta_1^2.\end{aligned}$$

□

Theorem 13. *Let* $\big((\mathbf{u},\mathbf{H}),T,p\big)$ *and* $\big((\mathbf{u}_h^m,\mathbf{H}_h^m),T_h^m,p_h^m\big)$ *be the solutions of the problem (5) and Algorithm 2, respectively. Then, under assumptions of Theorem 9 and Lemma 4, the following estimate holds:*

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_h^m,\mathbf{H}-\mathbf{H}_h^m)\|_{\boldsymbol{D}}+\|\nabla\big(T-T_h^m\big)\|_0+\|p-p_h^m\|_0\\ &\le Ch+\big(CH^2\big(\hat{N}_1^{-1}N^{-1}\nu_A\delta_1(\hat{N}_0^{-1}+\bar{N}_0^{-1})+\nu_A^{-1}\kappa^{-1}\hat{c}_0\hat{c}_1\hat{N}_2{}^{-1}\|\gamma\|_{-1}+\kappa^{-1}\hat{c}_2\hat{N}_1{}^{-1}\nu_A^2N^{-1}\delta_1\big)\\ &\quad+\sqrt{\varpi^mB_8}\nu_AN^{-1}\delta_1A_6\big)\big(\widetilde{\beta}^{-1}(1+c_A)+1+\nu_A\beta^{-1}C_F^{-2}\big),\end{aligned}$$

where $A_6:=\beta C_F^2N_0\kappa^{-1}(\kappa^{-1}\|\gamma\|_{-1}\varrho_1^{-1}+A_3\alpha^{-1}\nu_AN^{-1}\delta_1)+\varrho_1^{-1}\nu_A\delta_1(\nu_AA_3+1)$.

Proof. Subtracting the first equation of (41) from the first equation of (6) with $\mu = h$, we obtain

$$\begin{aligned} &a_0(\theta_h^m, S) + b_0(\mathbf{u}_h - \mathbf{u}_H, T_h, S) + b_0(\mathbf{u}_H, T_h - T_H, S) \\ &\quad + b_0(\mathbf{e}_H^m, T_H, S) + b_0(\mathbf{u}_H^m, \theta_H^m, S) = 0. \end{aligned} \tag{43}$$

Taking $S = \theta_h^m$ in (43), and using Lemma 3, (4) and the triangle inequality to write

$$\begin{aligned} \kappa\|\nabla\theta_h^m\|_0 \le\ & \hat{c}_0\hat{c}_1\|A_{1h}T_h\|_0(\|\mathbf{u} - \mathbf{u}_H\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0) \\ &+ \hat{c}_2\|A_{1h}\mathbf{u}_H\|_0(\|T - T_H\|_0 + \|T - T_h\|_0) \\ &+ N_0\|\nabla T_H\|_0\|\nabla\mathbf{e}_H^m\|_0 + N_0\|\nabla\mathbf{u}_H^m\|_0\|\nabla\theta_H^m\|_0, \end{aligned}$$

which combines Lemma 3, Theorem 2, 5 and 8 with (28) to provide

$$\begin{aligned} \kappa\|\nabla\theta_h^m\|_0 &\le CH^2\big(\hat{c}_0\hat{c}_1\hat{N}_2^{-1}|\gamma\|_{-1} + \hat{c}_2\hat{N}_1^{-1}\nu_A^2N^{-1}\delta_1\big) \\ &\quad + N_0\kappa^{-1}\|\gamma\|_{-1}\varrho_1^{-1}\sqrt{\varpi^m B_8}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}) \\ &\quad + N_0A_3\alpha^{-1}\sqrt{\varpi^m B_8}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})^2 \\ &\le CH^2\big(\hat{c}_0\hat{c}_1\hat{N}_2^{-1}|\gamma\|_{-1} + \hat{c}_2\hat{N}_1^{-1}\nu_A^2N^{-1}\delta_1\big) \\ &\quad + N_0\sqrt{\varpi^m B_8}\nu_A^2N^{-1}\delta_1(\kappa^{-1}\|\gamma\|_{-1}\varrho_1^{-1} + A_3\alpha^{-1}\nu_A^2N^{-1}\delta_1). \end{aligned} \tag{44}$$

Note that the uniqueness condition in Theorem 1 is implemented.

Next, subtracting the second equation of (41) from the second equation of (6) with $\mu = h$, we obtain

$$\begin{aligned} &A_0((\mathbf{e}_h^m, \boldsymbol{\xi}_h^m), (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{u}_h - \mathbf{u}_H, \mathbf{H}_h - \mathbf{H}_H), (\mathbf{u}_h, \mathbf{H}_h), (\mathbf{v}, \mathbf{B})) \\ &\quad + A_1((\mathbf{u}_H, \mathbf{H}_H), (\mathbf{u}_h - \mathbf{u}_H, \mathbf{H}_h - \mathbf{H}_H), (\mathbf{v}, \mathbf{B})) \\ &\quad + A_1((\mathbf{u}_H^m, \mathbf{H}_H^m), (\mathbf{e}_H^m, \boldsymbol{\xi}_H^m), (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{e}_H^m, \boldsymbol{\xi}_H^m), (\mathbf{u}_H, \mathbf{H}_H), (\mathbf{v}, \mathbf{B})) \\ &\quad - d((\mathbf{v}, \mathbf{B}), \eta_h^m) + d((\mathbf{e}_h^m, \boldsymbol{\xi}_h^m), q) = G(\theta_h^m, (\mathbf{v}, \mathbf{B})). \end{aligned} \tag{45}$$

Choosing $(\mathbf{v}, \mathbf{B}) = (\mathbf{e}_h^m, \boldsymbol{\xi}_h^m)$, $q = \eta_h^m$ in (45), and using Lemma 1, 3, we obtain

$$\begin{aligned} \nu_A\|(\mathbf{e}_h^m, \boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}} \le\ & \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{H}_h - \mathbf{H}_H)\|_0(\hat{N}_0\|(A_{1h}\mathbf{u}_h, A_{2h}\mathbf{H}_h)\|_0 + \bar{N}_0\|(A_{1H}\mathbf{u}_H, A_{2H}\mathbf{H}_H)\|_0) \\ &+ \beta C_F^2\|\nabla\theta_h^m\|_0 + N\|(\mathbf{e}_H^m, \boldsymbol{\xi}_H^m)\|_{\boldsymbol{D}}(\|(\mathbf{u}_H^m, \mathbf{H}_H^m)\|_{\boldsymbol{D}} + \|(\mathbf{u}_H, \mathbf{H}_H)\|_{\boldsymbol{D}}), \end{aligned}$$

which combines with Lemma 4, Theorem 2, 5, 8, (44) and (28) to get

$$\begin{aligned} \nu_A\|(\mathbf{e}_h^m, \boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}} \le\ & CH^2\hat{N}_1^{-1}N^{-1}\nu_A^2\delta_1(\hat{N}_0^{-1} + \bar{N}_0^{-1}) + CH^2\kappa^{-1}\big(\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\|\gamma\|_{-1} + \hat{c}_2\hat{N}_1^{-1}\nu_A^2N^{-1}\delta_1\big) \\ &+ \beta C_F^2N_0\sqrt{\varpi^m B_8}\nu_A^2N^{-1}\delta_1\kappa^{-1}(\kappa^{-1}\|\gamma\|_{-1}\varrho_1^{-1} + A_3\alpha^{-1}\nu_A^2N^{-1}\delta_1) \\ &+ \varrho_1^{-1}\sqrt{\varpi^m B_8}\nu_A^3N^{-1}\delta_1^2(\nu_AA_3 + 1). \end{aligned}$$

Then, rewriting the above estimation, we find

$$\begin{aligned} \|(\mathbf{e}_h^m, \boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}} \le\ & CH^2\nu_A^{-1}\big(\hat{N}_1^{-1}N^{-1}\nu_A^2\delta_1(\hat{N}_0^{-1} + \bar{N}_0^{-1}) + \kappa^{-1}\hat{c}_0\hat{c}_1\hat{N}_2^{-1}\|\gamma\|_{-1} \\ &+ \kappa^{-1}\hat{c}_2\hat{N}_1^{-1}\nu_A^2N^{-1}\delta_1\big) + \sqrt{\varpi^m B_8}\nu_AN^{-1}\delta_1A_6, \end{aligned} \tag{46}$$

where $A_6 := \beta C_F^2N_0\kappa^{-1}(\kappa^{-1}\|\gamma\|_{-1}\varrho_1^{-1} + A_3\alpha^{-1}\nu_A^2N^{-1}\delta_1) + \varrho_1^{-1}\nu_A\delta_1(\nu_AA_3 + 1)$.

Furthermore, applying the discrete inf-sup condition to (45) with $q = 0$ and using Lemma 1, Lemma 3, 4, the Cauchy-Schwartz inequality and Poincaré inequality, Theorem 2, 5, 8, and by

considering (29), (45), (47), we have

$$\begin{aligned}
&\widetilde{\beta}\|\eta_h^m\|_0\\
&\le c_A\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}+\|(\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H)\|_0(\hat{N}_0\|(A_{1h}\mathbf{u}_h,A_{2h}\mathbf{H}_h)\|_0\\
&\quad+\bar{N}_0\|(A_{1H}\mathbf{u}_H,A_{2H}\mathbf{H}_H)\|_0)+N\|(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m)\|_{\boldsymbol{D}}(\|(\mathbf{u}_H^m,\mathbf{H}_H^m)\|_{\boldsymbol{D}}\\
&\quad+\|(\mathbf{u}_H,\mathbf{H}_H)\|_{\boldsymbol{D}})+\beta C_F^2\|\nabla\theta_h^m\|_0\\
&\le\big(CH^2\nu_A^{-1}\big(\hat{N}_1^{-1}N^{-1}\nu_A^2\delta_1(\hat{N}_0^{-1}+\bar{N}_0^{-1})+\kappa^{-1}\hat{c}_0\hat{c}_1{\hat{N}_2}^{-1}\|\gamma\|_{-1}+\kappa^{-1}\hat{c}_2{\hat{N}_1}^{-1}\nu_A^2N^{-1}\delta_1\big)\\
&\quad+\sqrt{\varpi^mB_8}\nu_AN^{-1}\delta_1A_6\big)(1+c_A).
\end{aligned}\tag{47}$$

Finally, owing to the triangle inequality, we get

$$\begin{aligned}
\|(\mathbf{u}-\mathbf{u}_h^m,\mathbf{H}-\mathbf{H}_h^m)\|_{\boldsymbol{D}}&\le\|(\mathbf{u}-\mathbf{u}_h,\mathbf{H}-\mathbf{H}_h)\|_{\boldsymbol{D}}+\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}},\\
\|\nabla\big(T-T_h^m\big)\|_0&\le\|\nabla\big(T-T_h\big)\|_0+\|\nabla\theta_h^m\|_0,\\
\|p-p_h^m\|_0&\le\|p-p_h\|_0+\|\eta_h^m\|_0,
\end{aligned}$$

which combines with (44), (46), (47) and Theorem 3 to complete the proof. □

Algotirhm 3. *(Arrow-Hurwicz and One-Step-Newton iteration)*
Step 1. *Solve (29) to obtain the initial function* $((\mathbf{u}_H^0,\mathbf{H}_H^0),T_H^0,p_H^0)\in\boldsymbol{D}_H\times Q_H\times M_H$.
Step 2. *For* $n=1,2,\ldots,m-1$, *given* $((\mathbf{u}_H^n,\mathbf{H}_H^n),T_H^n,p_H^n)\in\boldsymbol{D}_H\times Q_H\times M_H$, *find* $((\mathbf{u}_H^{n+1},\mathbf{H}_H^{n+1}),T_H^{n+1},p_H^{n+1})\in\boldsymbol{D}_H\times Q_H\times M_H$ *as solution of (30).*
Step 3. *Solve a Newton-type linear problem on fine grid, i.e., find* $((\mathbf{u}_h^m,\mathbf{H}_h^m),T_h^m,p_h^m)\in\boldsymbol{D}_h\times Q_h\times M_h$ *as solution of*

$$\begin{cases}
a_0(T_h^m,S)+b_0(\mathbf{u}_H^m,T_h^m,S)+b_0(\mathbf{u}_h^m,T_H^m,S)=(\gamma,S)+b_0(\mathbf{u}_H^m,T_H^m,S),\\
A_0((\mathbf{u}_h^m,\mathbf{H}_h^m),(\mathbf{v},\mathbf{B}))+A_1((\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{u}_h^m,\mathbf{H}_h^m),(\mathbf{v},\mathbf{B}))\\
\quad+A_1((\mathbf{u}_h^m,\mathbf{H}_h^m),(\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{v},\mathbf{B}))-d((\mathbf{v},\mathbf{B}),p_h^m)+d((\mathbf{u}_h^m,\mathbf{H}_h^m),q)\\
=(\mathbf{F},(\mathbf{v},\mathbf{B}))+G(T_h^m,(\mathbf{v},\mathbf{B}))+A_1((\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{v},\mathbf{B})),
\end{cases}\tag{48}$$

for all $(\mathbf{v},\mathbf{B})\in\boldsymbol{D}_h$, $S\in Q_h$ *and* $q\in M_h$.

We further prove the stability and convergence of Algorithm 3.

Theorem 14. *Let* $\big((\mathbf{u}_h^m,\mathbf{H}_h^m),T_h^m\big)$ *be the solution of Algorithm 3 and*

$$A_7:=1-\nu_AA_3\delta_1-\beta C_F^2\kappa^{-1}N_0A_4\nu_AN^{-1}\delta_1>0.$$

Then, under assumptions of Theorem 9, we have

$$\begin{aligned}
\|(\mathbf{u}_h^m,\mathbf{H}_h^m)\|_{\boldsymbol{D}}&\le A_7^{-1}(A_3^2\nu_A^3N^{-1}\delta_1^2+\beta C_F^2\kappa^{-1}N_0A_3A_4\nu_A^3N^{-2}\delta_1^2+N^{-1}\delta_1\nu_A),\\
\|\nabla T_h^m\|_0&\le\kappa^{-1}N_0A_3A_4\nu_A^4N^{-2}\delta_1^2+\kappa^{-1}N_0A_4\nu_A^2N^{-1}\delta_1A_8+\kappa^{-1}\|\gamma\|_{-1}.
\end{aligned}$$

Proof. Taking $S=T_h^m$ in the first equation of (48) and using (4), Theorem 8, we get

$$\begin{aligned}
\|\nabla T_h^m\|_0&\le\kappa^{-1}N_0\|\nabla\mathbf{u}_H^m\|_0\|\nabla T_H^m\|_0+\kappa^{-1}N_0\|\nabla\mathbf{u}_h^m\|_0\|\nabla T_H^m\|_0+\kappa^{-1}\|\gamma\|_{-1}\\
&\le\kappa^{-1}N_0A_3A_4(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})^2+\kappa^{-1}\|\gamma\|_{-1}\\
&\quad+\kappa^{-1}N_0A_4(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\|\nabla\mathbf{u}_h^m\|_0\\
&\le\kappa^{-1}N_0A_3A_4\nu_A^4N^{-2}\delta_1^2+\kappa^{-1}N_0A_4\nu_A^2N^{-1}\delta_1\|\nabla\mathbf{u}_h^m\|_0+\kappa^{-1}\|\gamma\|_{-1}.
\end{aligned}\tag{49}$$

Note that $b_0(\mathbf{u}_H^m,T_h^m,T_h^m)=0$.

Then, choosing $(\mathbf{v},\mathbf{B})=(\mathbf{u}_h^m,\mathbf{H}_h^m)$, $q=p_h^m$ in the second equation of (48) and using Lemma 1, Theorem 8 and (49), we have

$$\begin{aligned}
\nu_A\|(\mathbf{u}_h^m,\mathbf{H}_h^m)\|_{\boldsymbol{D}}&\le N\|\nabla\mathbf{u}_H^m\|_0^2+N\|\nabla\mathbf{u}_h^m\|_0\|\nabla\mathbf{u}_H^m\|_0+\|\mathbf{F}\|_{-1}+\beta C_F^2\|\nabla T_h^m\|_0\\
&\le A_3^2\nu_A^4N^{-1}\delta_1^2+A_3\nu_A^2\delta_1\|(\mathbf{u}_h^m,\mathbf{H}_h^m)\|_{\boldsymbol{D}}+\|\mathbf{F}\|_{-1}+\beta C_F^2\kappa^{-1}N_0A_3A_4\nu_A^4N^{-2}\delta_1^2
\end{aligned}$$

$$+\beta C_F^2\kappa^{-1}N_0A_4\nu_A^2N^{-1}\delta_1\|(\mathbf{u}_h^m,\mathbf{H}_h^m)\|_{\boldsymbol{D}}+\beta C_F^2\kappa^{-1}\|\gamma\|_{-1},$$

where we notice that $A_1((\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{u}_h^m,\mathbf{H}_h^m),(\mathbf{u}_h^m,\mathbf{H}_h^m))=0$. Thus, we find the following estimate by rewriting the above inequality

$$\begin{aligned}&(1-\nu_AA_3\delta_1-\beta C_F^2\kappa^{-1}N_0A_4\nu_AN^{-1}\delta_1)\|(\mathbf{u}_h^m,\mathbf{H}_h^m)\|_{\boldsymbol{D}}\\&\le A_3^2\nu_A^3N^{-1}\delta_1^2+\beta C_F^2\kappa^{-1}N_0A_3A_4\nu_A^3N^{-2}\delta_1^2+N^{-1}\delta_1\nu_A.\end{aligned}$$

The proof is completed. □

Theorem 15. *Let* $\big((\mathbf{u},\mathbf{H}),T,p\big)$ *and* $\big((\mathbf{u}_h^m,\mathbf{H}_h^m),T_h^m,p_h^m\big)$ *be the solutions of the problem (5) and Algorithm 3, respectively. If we assume*

$$A_8:=\nu_A-\beta C_F^2\kappa^{-1}N_0A_4\nu_A^2\delta_1N^{-1}-A_3\nu_A^2\delta_1>0,$$

then, under assumptions of Theorem 9, the following estimation holds:

$$\begin{aligned}&\|(\mathbf{u}-\mathbf{u}_h^m,\mathbf{H}-\mathbf{H}_h^m)\|_{\boldsymbol{D}}+\|\nabla\big(T-T_h^m\big)\|_0+\|p-p_h^m\|_0\\&\le Ch+A_8^{-1}\big(CH^2N+2C\nu_A^2\delta_1\varrho_1^{-1}\sqrt{\varpi^mB_8}+N^{-1}\nu_A^4\delta_1^2\varrho_1^{-2}\varpi^mB_8\\&\quad+N_0CH^2\beta C_F^2\kappa^{-1}+N_0C\nu_A^2N^{-1}\delta_1\sqrt{\varpi^mB_8}(\varrho_1^{-1}+\alpha^{-1})\beta C_F^2\kappa^{-1}\\&\quad+N_0\varrho_1^{-1}\alpha^{-1}\varpi^mB_8\nu_A^2N^{-2}\delta_1^2\beta C_F^2\kappa^{-1}\big)\big(\widetilde{\beta}^{-1}(1+c_A)+1+\kappa^{-1}N_0A_4\nu_A^2\delta_1N^{-1}+\kappa^{-1}\beta^{-1}C_F^{-2}\big).\end{aligned}$$

Proof. Subtracting the first equation of (48) from the first equation of (6) with $\mu=h$, we have

$$a_0(\theta_h^m,S)+b_0(\mathbf{u}_h-\mathbf{u}_h^m,T_H^m,S)+b_0(\mathbf{u}_H^m,\theta_h^m,S)+b_0(\mathbf{u}_h-\mathbf{u}_H^m,T_h-T_H^m,S)=0.\tag{50}$$

Taking $S=\theta_h^m$ in (50) and considering $b_0(\mathbf{u}_H^m,\theta_h^m,\theta_h^m)=0$, we get

$$\kappa\|\nabla\theta_h^m\|_0+b_0(\mathbf{e}_h^m,T_H^m,\theta_h^m)=-b_0(\mathbf{u}_h-\mathbf{u}_H^m,T_h-T_H,\theta_h^m)-b_0(\mathbf{u}_h-\mathbf{u}_H^m,\theta_H^m,\theta_h^m),$$

which combines with (4), (28), Theorem 3, 8 and the triangle inequality to give

$$\begin{aligned}\kappa\|\nabla\theta_h^m\|_0&\le N_0\|\nabla\mathbf{e}_h^m\|_0\|\nabla T_H^m\|_0+N_0\|\nabla(\mathbf{u}_h-\mathbf{u}_H)\|_0\|\nabla(T_h-T_H)\|_0\\&\quad+N_0\|\nabla\mathbf{e}_H^m\|_0\|\nabla(T_h-T_H)\|_0+N_0\|\nabla(\mathbf{u}_h-\mathbf{u}_H)\|_0\|\nabla\theta_H^m\|_0+N_0\|\nabla\mathbf{e}_H^m\|_0\|\nabla\theta_H^m\|_0\\&\le N_0A_4(\|\mathbf{F}\|_{-1}+C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\|\nabla\mathbf{e}_h^m\|_0+N_0CH^2\\&\quad+N_0CH\nu_A^2N^{-1}\delta_1\sqrt{\varpi^mB_8}(\varrho_1^{-1}+\alpha^{-1})+N_0\varrho_1^{-1}\alpha^{-1}\varpi^mB_8\nu_A^2N^{-2}\delta_1^2.\end{aligned}\tag{51}$$

Next, subtracting the second equation of (48) from the second equation of (6) with $\mu=h$, we obtain

$$\begin{aligned}&A_0((\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),(\mathbf{v},\mathbf{B}))+A_1((\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),(\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{v},\mathbf{B}))+A_1((\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),(\mathbf{v},\mathbf{B}))\\&\quad+A_1((\mathbf{u}_h-\mathbf{u}_H^m,\mathbf{H}_h-\mathbf{H}_H^m),(\mathbf{u}_h-\mathbf{u}_H^m,\mathbf{H}_h-\mathbf{H}_H^m),(\mathbf{v},\mathbf{B}))-d((\mathbf{v},\mathbf{B}),\eta_h^m)\\&\quad+d((\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),q)=G(\theta_h^m,(\mathbf{v},\mathbf{B})).\end{aligned}\tag{52}$$

Taking $(\mathbf{v},\mathbf{B})=(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),q=\eta_h^m$ in (52), we get

$$\begin{aligned}&A_0((\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m))+A_1((\mathbf{e}_h^m,\boldsymbol{\xi}_h^m),(\mathbf{u}_H^m,\mathbf{H}_H^m),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m))\\&=-A_1((\mathbf{u}_h-\mathbf{u}_H^m,\mathbf{H}_h-\mathbf{H}_H^m),(\mathbf{u}_h-\mathbf{u}_H^m,\mathbf{H}_h-\mathbf{H}_H^m),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m))+G(\theta_h^m,(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)).\\&=-A_1((\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H),(\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m))\\&\quad-A_1((\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H),(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m))\\&\quad-A_1((\mathbf{e}_H^m,\boldsymbol{\xi}_H^m),(\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m))\\&\quad-A_1((\mathbf{e}_H^m,\boldsymbol{\xi}_H^m),(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m),(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m))+G(\theta_h^m,(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)),\end{aligned}$$

which combines with Lemma 1, Theorem 3, 8, (28) and (51) to show

$$\begin{aligned}\nu_A\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}&\le N\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}\|(\mathbf{u}_H^m,\mathbf{H}_H^m)\|_{\boldsymbol{D}}+N\|(\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H)\|_{\boldsymbol{D}}^2\\&\quad+2N\|(\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H)\|_{\boldsymbol{D}}\|(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m)\|_{\boldsymbol{D}}+N\|(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m)\|_{\boldsymbol{D}}^2+\beta C_F^2\|\nabla\theta_h^m\|_0\end{aligned}$$

$$
\begin{aligned}
&\leq A_3\nu_A^2\delta_1\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}} + NCH^2 + 2CH\nu_A^2\delta_1\varrho_1^{-1}\sqrt{\varpi^m B_8}\\
&\quad + N^{-1}\nu_A^4\delta_1^2\varrho_1^{-2}\varpi^m B_8 + \beta C_F^2\kappa^{-1}N_0A_4(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1})\|\nabla\mathbf{e}_h^m\|_0\\
&\quad + N_0CH^2\beta C_F^2\kappa^{-1} + N_0CH\nu_A^2N^{-1}\delta_1\sqrt{\varpi^m B_8}(\varrho_1^{-1}+\alpha^{-1})\beta C_F^2\kappa^{-1}\\
&\quad + N_0\varrho_1^{-1}\alpha^{-1}\varpi^m B_8\nu_A^2N^{-2}\delta_1^2\beta C_F^2\kappa^{-1}.
\end{aligned}
$$

Thus, one can see

$$
\begin{aligned}
&\big(\nu_A - \beta C_F^2\kappa^{-1}N_0A_4\nu_A^2\delta_1N^{-1} - A_3\nu_A^2\delta_1\big)\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}\\
&\leq NCH^2 + 2CH\nu_A^2\delta_1\varrho_1^{-1}\sqrt{\varpi^m B_8} + N^{-1}\nu_A^4\delta_1^2\varrho_1^{-2}\varpi^m B_8\\
&\quad + N_0CH^2\beta C_F^2\kappa^{-1} + N_0CH\nu_A^2N^{-1}\delta_1\sqrt{\varpi^m B_8}(\varrho_1^{-1}+\alpha^{-1})\beta C_F^2\kappa^{-1}\\
&\quad + N_0\varrho_1^{-1}\alpha^{-1}\varpi^m B_8\nu_A^2N^{-2}\delta_1^2\beta C_F^2\kappa^{-1}.
\end{aligned}
\tag{53}
$$

Applying the discrete inf-sup condition to (52), Lemma 1, (53), Theorem 3, 5, 8, (28) and (51), we have

$$
\begin{aligned}
\widetilde{\beta}\|\eta_h^m\|_0 &\leq c_A\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}} + N\|(\mathbf{e}_h^m,\boldsymbol{\xi}_h^m)\|_{\boldsymbol{D}}\|(\mathbf{u}_H^m,\mathbf{H}_H^m)\|_{\boldsymbol{D}} + N\|(\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H)\|_{\boldsymbol{D}}^2\\
&\quad + 2N\|(\mathbf{u}_h-\mathbf{u}_H,\mathbf{H}_h-\mathbf{H}_H)\|_{\boldsymbol{D}}\|(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m)\|_{\boldsymbol{D}} + N\|(\mathbf{e}_H^m,\boldsymbol{\xi}_H^m)\|_{\boldsymbol{D}}^2 + \beta C_F^2\|\nabla\theta_h^m\|_0\\
&\leq A_8^{-1}\big(NCH^2 + 2CH\nu_A^2\delta_1\varrho_1^{-1}\sqrt{\varpi^m B_8} + N^{-1}\nu_A^4\delta_1^2\varrho_1^{-2}\varpi^m B_8\\
&\quad + N_0CH^2\beta C_F^2\kappa^{-1} + N_0CH\nu_A^2N^{-1}\delta_1\sqrt{\varpi^m B_8}(\varrho_1^{-1}+\alpha^{-1})\beta C_F^2\kappa^{-1}\\
&\quad + N_0\varrho_1^{-1}\alpha^{-1}\varpi^m B_8\nu_A^2N^{-2}\delta_1^2\beta C_F^2\kappa^{-1}\big)(1+c_A).
\end{aligned}
\tag{54}
$$

Then, by the aid of the triangle inequality, we combines with (51), (53), (54) and Theorem 3 to complete the proof.

□

From Theorem 11, 13 and 15, we can see that Algorithm 3 needs more conditions than those of Algorithm 1 and 2. Hence, Algorithm 3 is more restrictive than Algorithm 1 and 2 on parameters.

5. Numerical experiments

In this section, we represent some numerical experiments to claim the accuracy and performance of the proposed algorithms.

5.1. **Experiment 1.**

5.1.1. *2D case.* In this subsection, we discuss the numerical performance of two-level Arrow-Hurwicz algorithms for the steady-state thermally coupled incompressible magnetohydrodynamics equations. Here, let the domain $\Omega = [0,1]\times[0,1]$. The right-hand side of functions $\mathbf{f},\mathbf{g}$ and γ are selected such that the exact solutions are given by:

$$
\begin{aligned}
u_1(x,y) &= x^2(x-1)^2y(y-1)(2y-1), \quad u_2(x,y) = -y^2(y-1)^2x(x-1)(2x-1),\\
H_1(x,y) &= \sin(\pi x)\cos(\pi y), \quad H_2(x,y) = -\cos(\pi x)\sin(\pi y),\\
p(x,y) &= \cos(\pi x)\cos(\pi y), \quad T(x,y) = u_1(x,y) + u_2(x,y).
\end{aligned}
$$

We choose the parameters as $s = Re = Rm = \kappa = \beta = 1$, and $\rho = 3/4, \alpha = 1/3$. Here, we set $h = 1/3^2, 1/4^2, 1/5^2, 1/6^2, 1/7^2, 1/8^2, 1/9^2, 1/10^2, 1/11^2$ and $1/12^2$. The H^1-norm of the velocity, magnetic, temperature, and L^2-norm of the pressure obtained by Algorithm 1-3 and the Arrow-Hurwicz algorithm under different grids are tabulated in Table 1-4. From these tables, we can find that the relative error estimations of all algorithms become smaller as the mesh size

h decreases. However, Algorithm 3 has the best accuracy among all algorithm, especially the error of pressure.

Table 1. Calculation results of the Arrow-Hurwicz algorithm for 2D case.

$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$	$\frac{\|p-p_h^m\|_0}{\|p\|_0}$
9	3.350e-1	1.723e-1	2.369e-1	3.718e-2
16	1.723e-1	9.788e-2	1.345e-1	1.210e-2
25	1.123e-1	6.536e-2	8.984e-2	5.557e-3
36	7.403e-2	4.361e-2	5.996e-2	2.618e-3
49	5.423e-2	3.205e-2	4.407e-2	1.568e-3
64	4.142e-2	2.543e-2	3.374e-2	1.021e-3
81	3.269e-2	1.834e-2	2.667e-2	7.242e-4
100	2.766e-2	1.587e-2	2.182e-2	1.015e-3
121	2.290e-2	1.310e-2	1.800e-2	8.602e-4
144	1.916e-2	1.091e-2	1.500e-2	7.408e-4

Table 2. Calculation results of Algorithm 1 for 2D case.

$1/h$	$1/H$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$	$\frac{\|p-p_h^m\|_0}{\|p\|_0}$
9	3	3.943e-1	1.723e-1	2.370e-1	1.624e-1
16	4	2.120e-1	9.788e-2	1.345e-1	9.586e-2
25	5	1.373e-1	6.536e-2	8.984e-2	6.345e-2
36	6	9.240e-2	4.361e-2	5.966e-2	4.457e-2
49	7	6.769e-2	3.204e-2	4.407e-2	3.308e-2
64	8	5.174e-2	2.453e-2	3.374e-2	2.551e-2
81	9	4.084e-2	1.939e-2	2.667e-2	2.025e-2
100	10	3.327e-2	1.586e-2	2.181e-2	1.647e-2
121	11	2.746e-2	1.308e-2	1.800e-2	1.364e-2
144	12	2.394e-2	1.091e-2	1.500e-2	1.147e-2

Table 3. Calculation results of Algorithm 2 for 2D case.

$1/h$	$1/H$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$	$\frac{\|p-p_h^m\|_0}{\|p\|_0}$
9	3	5.425e-1	1.730e-1	2.370e-1	2.449e-1
16	4	3.532e-1	9.788e-2	1.345e-1	1.524e-1
25	5	2.586e-1	6.536e-2	8.984e-2	1.028e-1
36	6	1.888e-1	4.361e-2	5.966e-2	7.300e-2
49	7	1.459e-1	3.205e-2	4.407e-2	5.435e-2
64	8	1.159e-1	2.453e-2	3.375e-2	4.190e-2
81	9	9.441e-2	1.939e-2	2.667e-2	3.319e-2
100	10	7.856e-2	1.586e-2	2.182e-2	2.713e-2
121	11	6.678e-2	1.309e-2	1.800e-2	2.225e-2
144	12	5.744e-2	1.091e-2	1.500e-2	1.858e-2

Table 4. Calculation results of Algorithm 3 for 2D case.

$1/h$	$1/H$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$	$\frac{\|p-p_h^m\|_0}{\|p\|_0}$
9	3	3.547e-1	1.639e-1	2.369e-1	5.284e-2
16	4	1.794e-1	9.284e-2	1.345e-1	1.857e-2
25	5	1.145e-1	6.200e-2	8.984e-2	8.406e-3
36	6	7.483e-2	4.137e-2	5.996e-2	4.037e-3
49	7	5.439e-2	3.040e-2	4.407e-2	2.242e-3
64	8	4.139e-2	2.328e-2	3.375e-2	1.346e-3
81	9	3.258e-2	1.839e-2	2.667e-2	8.587e-4
100	10	2.659e-2	1.505e-2	2.182e-2	5.816e-4
121	11	2.191e-2	1.242e-2	1.800e-2	4.044e-4
144	12	1.824e-2	1.035e-2	1.500e-2	2.886e-4

Further, we collect computational time of all algorithms in Table 5. We can see that all two-level algorithms save much more CPU time than the Arrow-Hurwicz algorithm. Particularly, Algorithm 2 costs the least computational time.

Table 5. CPU(s) time of different algorithms with different grid scale for 2D case.

$1/h$	$1/H$	Arrow-Hurwicz algorithm	Algorithm 1	Algorithm 2	Algorithm 3
36	6	27.716	3.484	2.359	4.456
49	7	51.383	5.797	4.046	7.241
64	8	87.792	9.546	6.343	12.259
81	9	141.471	16.780	9.874	17.769
100	10	185.715	22.452	14.654	27.318
121	11	279.786	30.309	21.018	42.832
144	12	441.173	43.276	30.343	62.411

5.1.2. *3D case.* In this subsection, we discuss the numerical performance of two-level Arrow-Hurwicz algorithms for the considered equations with 3D case. Here, let the domain $\Omega = [0,1]^3$. The exact solutions for the system (1) are given by:

$$u_1(x,y,z) = y^4 + z, \quad u_2(x,y,z) = x + z^3, \quad u_3(x,y,z) = x^2 + y^2,$$

$$H_1(x,y,z) = \sin(yz), \quad H_2(x,y,z) = \sin(x+z), \quad H_3(x,y,z) = y\sin(x^2),$$

$$p(x,y,z) = (2x-1)(2y-1)(2z-1), \quad T(x,y,z) = (y^4+z) + (x+z^3) + (x^2+y^2).$$

We set the parameters as $s = Re = Rm = \kappa = \beta = 1$, and $\rho = 3/4, \alpha = 1/3$. Here, we choose $h = 1/8, 1/9, 1/10, 1/11$ and $1/12$. The H^1-norm of the velocity, magnetic, temperature, and L^2-norm of the pressure obtained by Algorithm 1-3 and the Arrow-Hurwicz algorithm under different grids are tabulated in Table 6-9. From these tables, we can find that the relative error estimations of all algorithms become smaller as the mesh size h decreases.

Table 6. Calculation results of the Arrow-Hurwicz algorithm for 3D case.

$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$
8	7.501e-2	1.452e-1	6.943e-2
9	6.670e-2	1.406e-1	6.352e-2
10	6.004e-2	1.372e-1	5.893e-2
11	5.460e-2	1.347e-1	5.529e-2
12	5.001e-2	1.327e-1	5.235e-2

Table 7. Calculation results of Algorithm 1 for 3D case.

$1/h$	$1/H$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$
8	2.828	7.589e-2	9.452e-2	8.729e-2
9	3.000	6.774e-2	8.621e-2	8.272e-2
10	3.162	6.125e-2	7.974e-2	7.929e-2
11	3.317	5.596e-2	7.460e-2	7.655e-2
12	3.464	5.157e-2	7.043e-2	7.458e-2

Table 8. Calculation results of Algorithm 2 for 3D case.

$1/h$	$1/H$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$
8	2.828	7.623e-2	1.193e-1	8.729e-2
9	3.000	6.814e-2	1.132e-1	8.272e-2
10	3.162	6.170e-2	1.087e-1	7.929e-2
11	3.317	5.647e-2	1.052e-1	7.655e-2
12	3.464	5.213e-2	1.024e-1	7.458e-2

Table 9. Calculation results of Algorithm 3 for 3D case.

$1/h$	$1/H$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^m)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{H}-\mathbf{H}_h^m)\|_0}{\|\nabla\mathbf{H}\|_0}$	$\frac{\|\nabla(T-T_h^m)\|_0}{\|\nabla T\|_0}$
8	2.828	7.590e-2	1.142e-1	8.728e-2
9	3.000	6.777e-2	1.077e-1	8.272e-2
10	3.162	6.128e-2	1.028e-1	7.929e-2
11	3.317	5.600e-2	9.911e-2	7.655e-2
12	3.464	5.161e-2	9.617e-2	7.458e-2

5.2. **Thermal driven cavity problem.** In this experiment, we will test Algorithm 1-3 and the Arrow-Hurwicz algorithm by using the thermal driven cavity problem, which is investigated in [29]. The computational domain consists of a square cavity with differentially heated vertical walls where left and right walls are kept at $T = 1$ and $T = 0$, respectively. The remaining walls are insulated and there is no heat transfer through them. No-slip boundary conditions are imposed for the velocity at all walls. For the magnetic field, we set $H_1 = 1$, $\frac{\partial H_2}{\partial \mathbf{n}} = 0$ at the horizontal walls and $H_2 = 0$, $\frac{\partial H_1}{\partial \mathbf{n}} = 0$ at the vertical walls.

For implementing Algorithm 1-3, we choose the model parameters as $s = Re = 1$, $Rm = 0.1$ and take $\mathbf{f} = \mathbf{0}$, $\mathbf{g} = \mathbf{0}$, $\gamma = 0$ with the mesh sizes $1/H = 8$, $h = H^2$. The performances of the presented algorithms are compared with the Arrow-Hurwicz algorithm. In Table 10, we show that the maximum velocity at $y = 0.5$ with different thermal expansion coefficient β. From this table, we can see that the two-level algorithms spend the less computational time than that of the Arrow-Hurwicz algorithm to get almost the same maximum velocity values.

Table 10. Comparisons of the maximum velocity values obtained by different algorithms.

	$\beta = 1$	$\beta = 10$	$\beta = 100$	CPU time ($\beta = 100$)
Arrow-Hurwicz algorithm	0.188	0.223	0.576	172.255
Algorithm 1	0.190	0.225	0.578	14.910
Algorithm 2	0.190	0.225	0.578	10.545
Algorithm 3	0.189	0.224	0.578	18.591

Furthermore, in Figs.1-3, we give the numerical velocity streamlines, magnetic field lines and isotherms of the thermally coupled incompressible magnetohydrodynamics problem by the Arrow-Hurwicz algorithm and Algorithm 1-3 with different thermal expansion coefficients. From these figures, we observe that the presented two-level algorithms perform well and produce nearly identical simulation results as the Arrow-Hurwicz algorithm. Hence, the presented two-level algorithms can accurately and efficiently simulate the considered model.

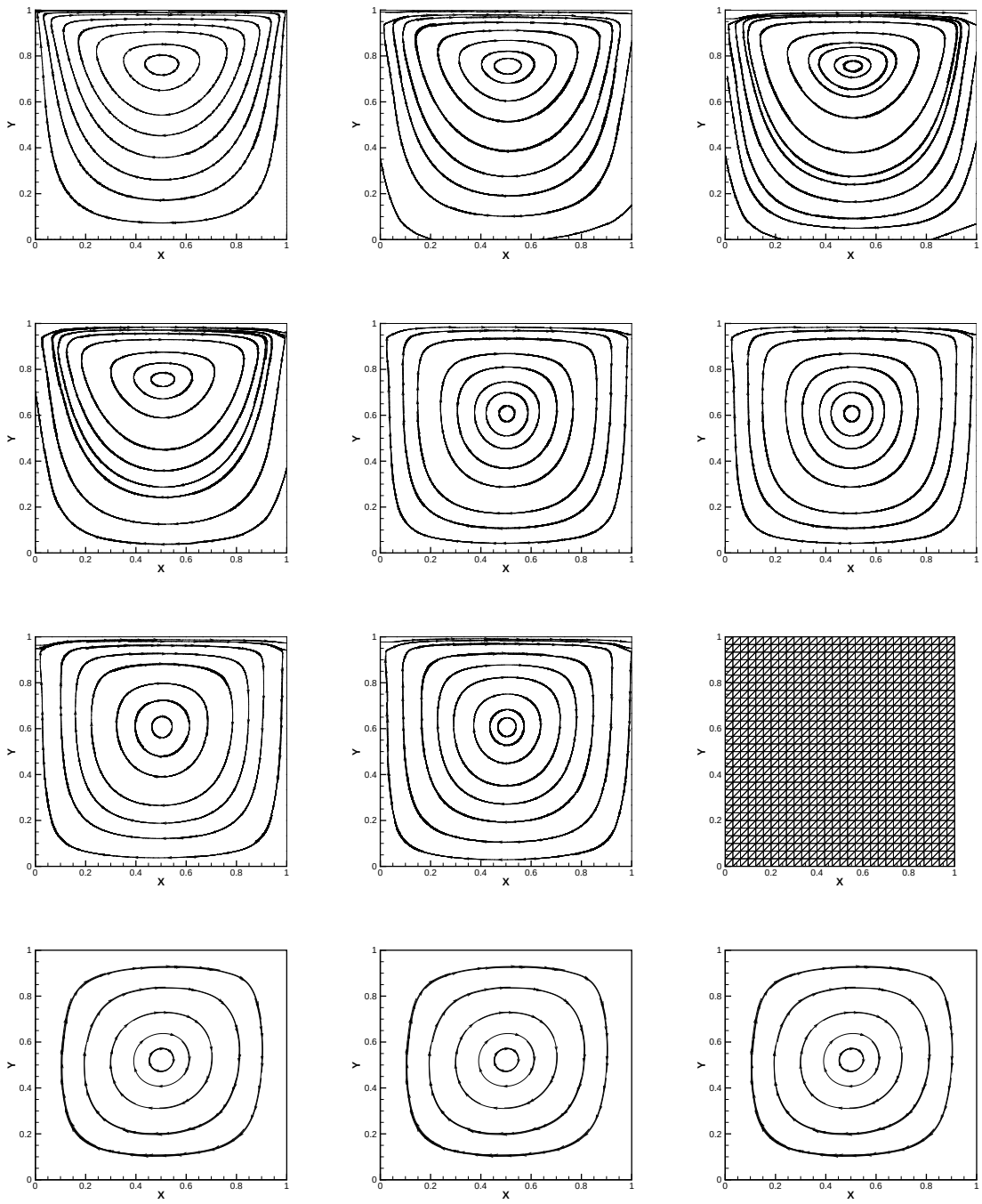


FIGURE 1. The numerical velocity streamlines of the Arrow-Hurwicz algorithm (the first column), Algorithm 1 (the second column), Algorithm 2 (the third column) and Algorithm 3 (the fourth column) at different $\beta = 1$, 100 and 500 from up to down.

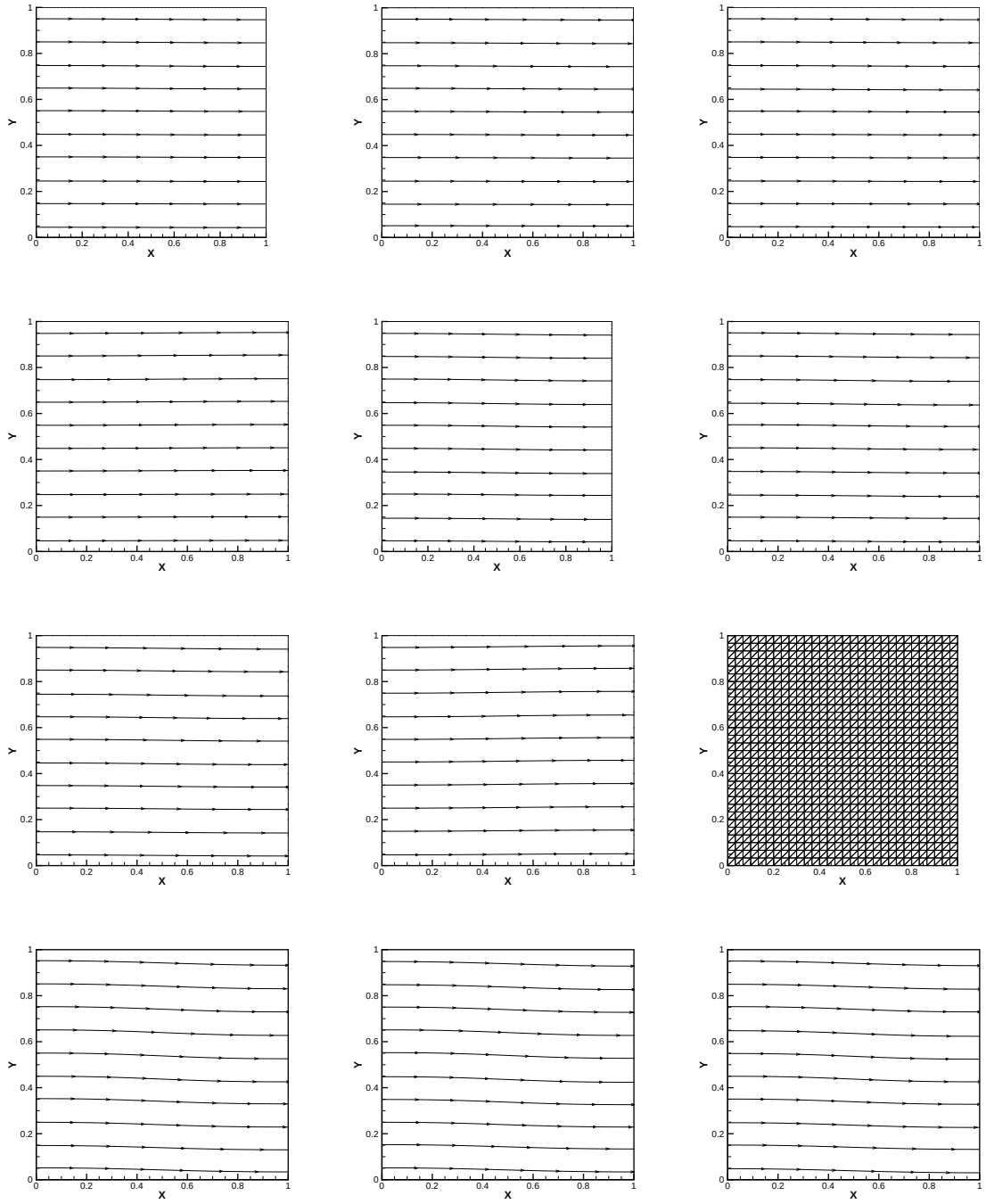


FIGURE 2. The magnetic field lines of the Arrow-Hurwicz algorithm (the first column), Algorithm 1 (the second column), Algorithm 2 (the third column) and Algorithm 3 (the fourth column) at different $\beta = 1$, 100 and 500 from up to down.

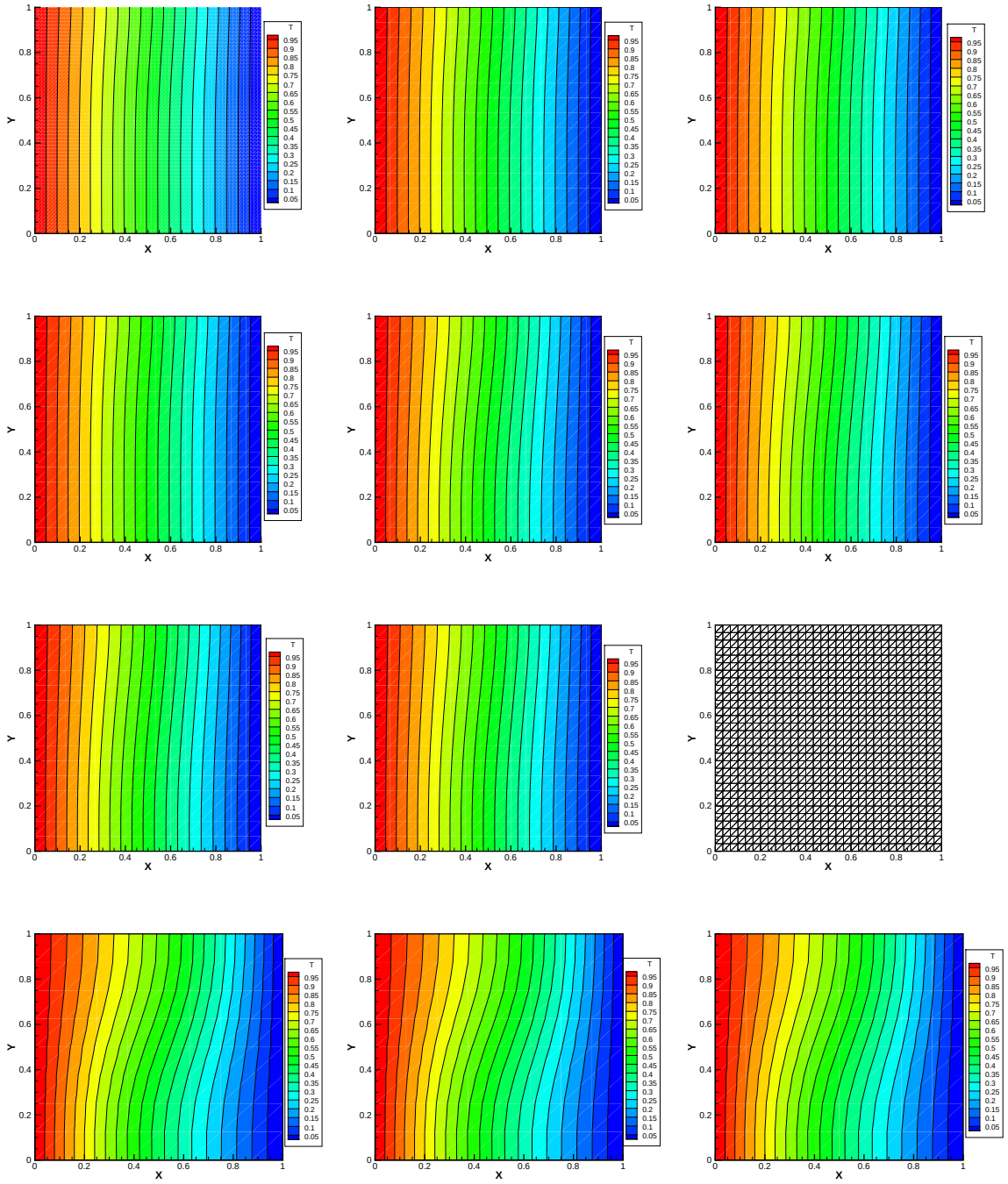


FIGURE 3. Isotherms of the Arrow-Hurwicz algorithm (the first column), Algorithm 1 (the second column), Algorithm 2 (the third column) and Algorithm 3 (the fourth column) at different $\beta = 1$, 100 and 500 from up to down.

6. Conclusion

In this paper, we propose and analyze two-level Arrow-Hurwicz algorithms for the steady-state thermally coupled incompressible magnetohydrodynamics equations. The methodology

consists of first computing a coarse-mesh approximation via the classical Arrow-Hurwicz algorithm, followed by refinement on a fine mesh using one of three iterative linearization strategies (namely, Oseen-type, Stokes-type, or Newton-type schemes). We rigorously establish stability bounds and error estimates for the resulting two-level algorithms under some assumptions. Numerical experiments are conducted to test both the accuracy and computational efficiency of the proposed approaches.

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